RESEARCH RESULTS SO FAR

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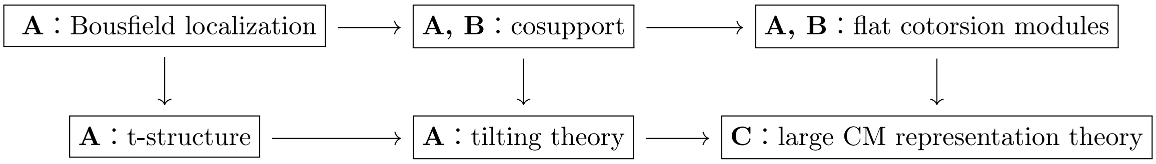
The main themes of my research so far are: **(A) Derived categories over commutative noetherian rings**, **(B) Flat cotorsion modules and cosupport**, **(C) Large (i.e., possibly infinitely generated) Cohen–Macaulay representation theory**. The following diagram shows the flow of my research.



Summaries and relationship of them are explained below.

1. Bousfield localization, cosupport, and flat cotorsion modules

Bousfield localization is a sort of localization of categories. This concept originally occurred in algebraic topology. From a topological point of view, Greenlees–May (1992) introduced the notion of local homology, which is dual to local cohomology having already been important in commutative algebra and algebraic geometry. As is well known, these functors can be interpreted as Bousfield (co)localizations.

In general, it is very difficult to explicitly describe Bousfield localizations. However, in a collaboration with Yuji Yoshino, we gave a concrete way to explicitly describe every Bousfield localization on the (unbounded) derived category of a commutative noetherian ring of finite Krull dimension (2018). The notion of cosupport, which is dual to the notion of support (for modules and complexes), was a key in that work. This duality is an analogue to those displayed on the left and right sides below:

local cohomology	support	injective modules
↕	↕	↕
local homology	cosupport	flat (cotorsion) modules

Flat cotorsion modules are flat modules additionally satisfying some homological condition. The categorical properties of them are, in some sense, as good as the projective modules and injective modules. For example, the derived category of any ring can be realized as the homotopy category of K-flat complexes of flat cotorsion modules. Partly motivated by this fact, Peder Thompson and I proved that every object in the derived category $D(R)$ of a commutative noetherian ring of finite Krull dimension can be replaced by a minimal K-flat complex of flat cotorsion modules (2020). A minimal pure-injective resolution of R is a spacial case of this replacement, and its structure is closely related to the cosupport of R. I completely determined the cosupport of every affine ring over a field. As an application, I partly and affirmatively solved Gruson's conjecture (2015), which is an analogue to Nakayama's conjecture in the context of finite-dimensional algebras.

The general importance of flat cotorsion modules have been clear sufficiently; e.g., in terms of derived categories and model categories. However, their concrete structure were not studied beyond Enochs' theorem over a commutative noetherian ring R (1984). In a collaboration with Ryo Kanda (2022), we extended Enochs' theorem about structure of flat cotorsion module over R to Noether algebras (i.e., module-finite algebras over R). As a consequence, the points represented by flat cotorsion modules in the Ziegler spectrum are classified, and this is related to the study 3 below.

2. t-structure and tilting theory

The notion of t-structures is a natural generalization of Bousfield localizations. For the derived category $D(R)$ of a commutative noetherian ring R, a classification of the compactly generated t-structures is known, and such t-structures have a good property called "homotopically smashing". In a collaboration with Michal Hrbek, we conversely proved that every homotopically smashing t-structure in $D(R)$ is compactly generated (2021). This can be interpreted as an affirmative answer to a telescope conjecture generalized to t-structures, where the original telescope conjecture was affirmatively solved by Neeman for $D(R)$ (1992).

It is know that if a compactly generated t-structure is non-degenerate, then the t-structure is induced by a cosilting object. In a collaboration with Michal Hrbek and Jan Šťovíček, we concretely constructed many cosilting objects (and (silting objects) in $D(R)$. Furthermore, we studied when they are cotilting (tilting), related this question with the condition that R is a homomorphic image of a Cohen–Macaulay (CM) ring (2022).

3. Large CM representations

The study of CM modules over an order has two aspects; a representation theoretic study of singularities and a generalization of representation theory over a finite-dimensional algebras. By Auslander–Ringel–Tachikawa theorem, a finite-dimensional algebra has infinite representation type if and only if there exists an infinitely generated indecomposable module. This naturally gives a motivation to consider representation theory of large modules.

The Ziegler spectrum is a topological space whose points are the isoclasses of indecomposable pure-injective modules, and it provides a reasonable place to discuss representation theory of large modules. Puninski (2018) and Puninski–Los (2019) studied one-dimensional hyperplanes of countable CM representation type, incorporating "infinitely generated CM modules", and using the Ziegler spectrum. As a foundation to continue their attempt theoretically, I proved an Auslander–Ringel–Tachikawa type theorem for the stable category of Gorenstein-projective modules over a complete Gorenstein order (2022). The classification of flat cotorsion points of the Ziegler spectrum mentioned in 1 above is also related to this aim. Moreover, a cotilting object mentioned in 2 has an important role to handle the class of infinitely generated CM modules in the sense of Puninski.