Results of my research

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A 2-component handlebody-link is 2 handlebodies embedded in the 3-sphere, denoted by H. Two 2-component handlebody-links are equivalent if one can be transformed into the other by an isotopy of S^3.

The linking number is a classical and fundamental invariant of a 2-component link in knot theory. A. Mizusawa introduced the linking numbers as an invariant of a (not necessarily 2-component) handlebody-link. The linking numbers is a generalization of the linking number to a handlebody-link. However, the linking numbers do not inherit even the property of the linking number.

Let $\Delta_L(t_1, t_2)$ be the Alexander polynomial of a 2-component link L. G. Torres showed that $\Delta_L(1, 1)$ is the linking number of L. We introduced the linking number of a pair of handlebody-link H and generators of the first homology group of H from the viewpoint that this property is the essential property to be satisfied by the linking number.

For a 2-component handlebody-link $H = h_1 \cup h_2$, let g_1 be the genus of h_1 and g_2 the genus of h_2. Let $c_{11}, c_{12}, \ldots, c_{1g_1}$ and $c_{21}, c_{22}, \ldots, c_{2g_2}$ be bases of the first homology groups $H_1(h_1)$ and $H_1(h_2)$ of h_1 and h_2, respectively. We regard $c_{11}, c_{12}, \ldots, c_{1g_1}$ and $c_{21}, c_{22}, \ldots, c_{2g_2}$ as embedded closed oriented circles in the 3-sphere. Let $L_i = \sum_{j=1}^{g_2} lk(c_{1i}, c_{2j})$ $(i = 1, 2, \ldots, g_1)$. We define the linking number $lk(h_1, h_2)$ of H as $lk(h_1, h_2) = \gcd\{L_i | i = 1, 2, \ldots, g_1\}$.

For a 2-component handlebody-link $H = h_1 \cup h_2$, let $m_{11}, m_{12}, \ldots, m_{1g_1}$ be a meridian system of h_1 and $m_{21}, m_{22}, \ldots, m_{2g_2}$ a meridian system of h_2. Let E be the exterior of H and $\alpha : \pi_1(E) \to \mathbb{Z} = \langle t | \ \rangle$ an onto homomorphism such that $\alpha(m_{1i}) = 1$, $\alpha(m_{2j}) = t$ $(i = 1, 2, \ldots, g_1, j = 1, 2, \ldots, g_2)$. Let $\Delta_{(H,\alpha)}(t)$ be the Alexander polynomial of a pair (H, α). Then the following theorem holds.

Theorem 1 [O.]
$\Delta_{(H,\alpha)}(1) = lk(H_1, H_2)$.

Let $\nabla_L(z)$ be the Conway polynomial of a 2-component link L. J. Hoste showed that the first coefficient of $\nabla_L(z)$ is the linking number of L. the linking number $lk(H_1, H_2)$ has this property.

Theorem 2 [O.]
The first coefficient of $\nabla_{(H,\alpha)}(z)$ is $lk(H_1, H_2)$.