Plan of the study (Yosuke Saito)

For a complex number q satisfying $|q|<1$, set

$$(x;q)_\infty := \prod_{n\geq 0}(1-xq^n) \quad (x\in\mathbb{C}), \quad \theta_q(x) := (x;q)_\infty (qx^{-1};q)_\infty \quad (x\in\mathbb{C}\backslash\{0\}).$$

By setting $D_x = x\dfrac{\partial}{\partial x}$ (Euler derivative), we define $E_k(x;q) := -{D_x}^k \log\theta_q(x)$ $(k\in\mathbb{Z}_{>0})$. For complex numbers q, p satisfying $|q|<1$, $|p|<1$, set

$$(x;q,p)_\infty := \prod_{m,n\geq 0}(1-xq^m p^n) \quad (x\in\mathbb{C}), \quad \Gamma_{q,p}(x) := \frac{(qpx^{-1};q,p)_\infty}{(x;q,p)_\infty} \quad (x\in\mathbb{C}\backslash\{0\}).$$

Let N be a positive integer, β be a complex number, and p be a complex number satisfying $|p|<1$. The Hamiltonian of the elliptic Calogero-Moser system $H_N^{\mathrm{CM}}(\beta,p)$ is defined by

$$H_N^{\mathrm{CM}}(\beta,p) := \sum_{i=1}^{N} {D_{x_i}}^2 - \beta(\beta-1)\sum_{1\leq i\neq j\leq N} E_2(x_i/x_j;p).$$

Then the following fact is known: $\Psi_N(x;\beta,p) := \prod_{1\leq i\neq j\leq N}\theta_p(x_i/x_j)^{\beta/2}$ satisfies

$$H_N^{\mathrm{CM}}(\beta,p)\Psi_N(x;\beta,p) = \{2N\beta D_p + C_N(\beta,p)\}\Psi_N(x;\beta,p), \quad \cdots(*)$$

where $C_N(\beta,p)$ is a complex number. It is remarkable that the derivative $D_p = p\dfrac{\partial}{\partial p}$ is in the right hand side of $(*)$. This means that the elliptic Calogero-Moser system has a solution which involves the infinitesimal deformation of the elliptic modulus p.

Let N be a positive integer, q, p be complex numbers satisfying $|q|<1$, $|p|<1$, and t be a complex number satisfying $t\in\mathbb{C}\backslash\{0\}$. The Hamiltonian of the elliptic Ruijsenaars system $H_N^{\mathrm{R}}(q,t,p)$ is defined by

$$H_N^{\mathrm{R}}(q,t,p) := \sum_{i=1}^{N}\prod_{j\neq i}\left\{\frac{\theta_p(tx_i/x_j)\theta_p(qt^{-1}x_i/x_j)}{\theta_p(x_i/x_j)\theta_p(qx_i/x_j)}\right\}^{\frac{1}{2}} T_{q,x_i},$$

where $T_{q,x}$ is the q-shift operator which is defined by $T_{q,x}f(x) = f(qx)$. Then the function $\Psi_N(x;q,t,p) := \prod_{1\leq i\neq j\leq N}\left\{\dfrac{\Gamma_{q,p}(tx_i/x_j)}{\Gamma_{q,p}(x_i/x_j)}\right\}^{1/2}$ satisfies

$$H_N^{\mathrm{R}}(q,t,p)\Psi_N(x;q,t,p) = t^{\frac{-N+1}{2}}\sum_{i=1}^{N}\prod_{j\neq i}\frac{\theta_p(tx_i/x_j)}{\theta_p(x_i/x_j)}\Psi_N(x;q,t,p). \quad \cdots(**)$$

It is known that by setting $t=q^\beta$ and by taking the limit $q\to 1$ appropriately, the equation $(**)$ degenerates to the equation $(*)$. Thus it is probable that the equation $(**)$ contains a certain difference deformation of the elliptic modulus p. By standing the point of view, the author will study the elliptic Ruijsenaars system.