Fumichika Takamizo

Research Plan

Relation of Pisot finiteness properties

Let $\beta > 1$. Frougny and Solomyak introduced the following conditions:

$$(\mathrm{F}_1)\ \ \mathbb{Z} \subset \mathrm{Fin}(\beta) \qquad (\mathrm{PF})\ \ \mathbb{Z}_{\geq 0}[\beta^{-1}] \subset \mathrm{Fin}(\beta) \qquad (\mathrm{F})\ \ \mathbb{Z}[\beta^{-1}] \subset \mathrm{Fim}(\beta)$$

where $\mathrm{Fin}(\beta)$ is the set of real number x such that $|x|$ has finite β-expansion. We know some interesting results on these finiteness properties. The following table shows previous research results:

	Class	Algebraic structure of $\mathrm{Fin}(\beta)$	Sufficiency for (F)	Sufficiency for (PF)
(F_1)	Pisot	?	?	?
(PF)	Pisot	Closed under addition & multiple	$d_\beta(1)$ is finite	—
(F)	Pisot	Ring	—	—

The property (F_1) is not well-known. However, I recently found a Pisot number β which has property (F_1) without (PF). So I will work on the following projects.

Necessity and Sufficient condition for property (F_1)

Let $\beta > 1$ be an algebraic integer with minimal polynomial $x^d - a_{d-1}x^{d-1} - \cdots - a_1 x - a_0$ and define τ_β, which is a transformation on $\mathbb{Z}^{d-1}$, by

$$\tau_\beta(l_1, l_2, \cdots, l_{d-1}) \coloneqq \left(l_2, \cdots, l_{d-1}, -\lfloor l_1 a_0 \beta^{-1} + l_2(a_1\beta^{-1} + a_0\beta^{-2}) + \cdots + l_{d-1}(a_{d-2}\beta^{-1} + \cdots + a_0\beta^{-d+1})\rfloor\right).$$

τ_β is a kind of generalization of β-transformation T when β is an algebraic integer with degree d. Now define $\tau_\beta^*(\boldsymbol{l}) = -\tau_\beta(-\boldsymbol{l})$ and

$$Q_\beta = \left\{\boldsymbol{l} = (l_1, l_2, \cdots, l_{d-1}) \in \mathbb{Z}^{d-1} \middle| \exists \{\boldsymbol{l}_n\}_{n=1}^N\ s.t.\ \boldsymbol{l}_N = \boldsymbol{l},\ \ \boldsymbol{l}_{n+1} \in \{\tau_\beta(\boldsymbol{l}_n), \tau_\beta^*(\boldsymbol{l}_n)\}\ and\ \boldsymbol{l}_1 = (0, \cdots, 0, 1)\right\}.$$

When β is a Pisot number, Q_β is a finite set and so

$$P_\beta \coloneqq \{\boldsymbol{l} \in Q_\beta \mid \exists k \in \mathbb{N}; \tau_\beta^k(\boldsymbol{l}) = \boldsymbol{l}\}$$

is also finite set. The condition $\tau_\beta^{-1}(P_\beta) \subset P_\beta$ plays an important role in the proof of a new sufficient condition for property (F_1) that I found. So I expect that $\tau_\beta^{-1}(P_\beta) \subset P_\beta$ if and only if β has property (F_1) without (PF). Hence I aim to solve this problem for a future work.

An algebraic structure of $\mathrm{Fin}(\boldsymbol{\beta})$ under property (F_1)

In the condition (PF), the opposite inclusion is obviously true. Similarly, (F) is also so on. Thus an algebraic structure of $\mathrm{Fin}(\beta)$ is already known when β has property (PF) or (F). On the other hand, when β has property (F_1), an algebraic structure of $\mathrm{Fin}(\beta)$ still remains as an unsolved problem. So I try to study an algebraic structure of $\mathrm{Fin}(\beta)$ when β has property (F_1).