

I have been studying moduli spaces of left-invariant (pseudo-)Riemannian metrics on a Lie group G . Here, the moduli space is defined as the orbit space of the action of the group $\mathbb{R}_{>0} \times \text{Aut}(G)$ on the space of all left-invariant metrics $\mathcal{M}_{p,q}(G)$ with signature (p, q) on G . There are three backgrounds to study the moduli spaces:

- (B1) Since there are too many left-invariant metrics on a Lie group G , it is hard to determine whether G admits nice left-invariant metrics (*e.g.* left-invariant Ricci solitons) or not. If one describes the moduli space of left-invariant metrics on G , one may be able to determine the existence of nice metrics on G .
- (B2) The space $\mathcal{M}_{p,q}(G)$ is a noncompact symmetric space, and the action of $\mathbb{R}_{>0} \times \text{Aut}(G)$ on $\mathcal{M}_{p,q}(G)$ is an isometric action. Isometric actions on noncompact symmetric spaces have been studied actively. The $\mathbb{R}_{>0} \times \text{Aut}(G)$ -actions may provide nice examples.
- (B3) If G is a three dimensional solvable Lie group, it has been known that a Riemannian metric $\langle \cdot, \cdot \rangle \in \mathcal{M}_{3,0}(G)$ is a Ricci soliton if and only if the orbit $\mathbb{R}_{>0} \times \text{Aut}(G) \cdot \langle \cdot, \cdot \rangle \subset \mathcal{M}_{3,0}(G)$ is a minimal submanifold. The fact implies that one can study left-invariant metrics on a Lie group G in terms of the geometry of the $\mathbb{R}_{>0} \times \text{Aut}(G)$ -actions.

For the background (B1), we have determined the moduli spaces of pseudo left-invariant metrics for each signatures on a particular solvable Lie group G . Also, by applying the description of the moduli spaces, we have shown that any left-invariant metrics on G have constant sectional curvature. These results have been published as Paper 5 in the List.

For the background (B2), I have given some examples of n -dimensional Lie groups G whose $\mathbb{R}_{>0} \times \text{Aut}(G)$ -actions on $\mathcal{M}_{n,0}(G)$ are hyperpolar actions with singular orbits. Little is known about hyperpolar actions with singular orbits, and nontrivial examples have been required. These results have been published as Paper 7 in the List.

For the background (B3), I have shown that if the orbit $\mathbb{R}_{>0} \times \text{Aut}(G) \cdot \langle \cdot, \cdot \rangle \subset \mathcal{M}_{n,0}(G)$ is an isolated orbit through a Riemannian metric $\langle \cdot, \cdot \rangle \in \mathcal{M}_{n,0}(G)$, then the metric $\langle \cdot, \cdot \rangle$ is a Ricci soliton. An isolated orbit is a minimal submanifold. The result asserts that the previous result “minimal \Rightarrow Ricci soliton” in three dimensional solvable cases holds for general settings if one strengthens the assumption “minimal” to “isolated”. These results have been published as Paper 1 in the List.

In Paper 1, I also have obtained a simple characterization for isolated orbits of the proper isometric actions. For a proper isometric action of a Lie group H on a Riemannian manifold X , an orbit $H \cdot p$ is an isolated orbit if and only if there are no nonzero H -invariant normal vector fields on $H \cdot p$. Now I have generalized the essence of isolated orbit to arbitrary submanifolds as follows:

- A Riemannian submanifold Y in a Riemannian manifold X is called an arid submanifold if there exists some proper isometric action on X such that the action preserves Y , and Y does not admit nonzero invariant normal vector fields under the action.