M. Khovanov constructed a homological link invariant which is a refinement of the Jones polynomial. As well-known, the Jones polynomial is a quantum link invariant defined using a quantum group $U_q(\mathfrak{sl}_2)$ and its vector representation. This fact led me to ask the following questions:

Can we construct homological link invariants which refine other quantum link invariants?

$$\begin{tikzcd}
 & \bigoplus \mathrm{Hom}_{U_q(\mathfrak{sl}_n)}(\wedge^{\underline{i}}, \wedge^{\underline{j}}) \arrow[d, squiggly, "\text{Yonezawa [Doctoral thesis]}"'] & U_q(\mathfrak{gl}_m) \arrow[l, "\gamma_m^n"'] \arrow[r, "\gamma_m"] \arrow[d, squiggly, "\text{Khovanov-Lauda, Rouquier}"] & \bigoplus \mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(S^{\underline{i}}, S^{\underline{j}}) \arrow[d, squiggly, dotted, "\text{Khovanov-Lauda-Sussan-Yonezawa}"] \\
\text{Link invariant } CKh(q,t,s) & \mathrm{HMF}_m^n \arrow[l, Rightarrow] & \mathcal{U}(\mathfrak{gl}_m) \arrow[l, "\Gamma_m^n"', "\text{Mackaay-Yonezawa}"] \arrow[r, "\Gamma_m", "\text{Khovanov-Lauda-Sussan-Yonezawa}"'] & \mathrm{Bim}(m,k)
\end{tikzcd}$$

(1) Summary of the thesis "Quantum $(\mathfrak{sl}_n, \wedge V_n)$ link invariant and matrix factorizations": Khovanov and Rozansky introduced matrix factorizations defining a homological link invariant which refines the quantum link invariant associated with $U_q(\mathfrak{sl}_n)$ and its vector representation. In this paper, we generalize Khovanov–Rozansky's matrix factorizations and define a new link invariant $CKh(q,t,s)$ which refines the quantum link invariant $CJ(q)$ associated with $U_q(\mathfrak{sl}_n)$ and its fundamental representations (Research in blue on the above figure). The link invariant $CJ(q)$ is recovered as $CKh(q,-1,1)$.

(2) Summary of the paper "$\mathfrak{sl}_N$-Web categories and categorified skew Howe duality": On the antisymmetric tensor product $\wedge^k(\mathbb{C}^n \otimes \mathbb{C}^m)$, we have a left $U_q(\mathfrak{sl}_n)$ action and a right $U_q(\mathfrak{gl}_m)$ action such that these two actions commute. Hence, we have a $U_q(\mathfrak{gl}_m)$ representation (top-left morphism γ_m^n on the above figure).

$$\gamma_m^n : U_q(\mathfrak{gl}_m) \to \bigoplus_{\sum_{\alpha=1}^m i_\alpha = k,\, \sum_{\alpha=1}^m j_\alpha = k} \mathrm{Hom}_{U_q(\mathfrak{sl}_n)}(\wedge^{i_1} \otimes \cdots \otimes \wedge^{i_m}, \wedge^{j_1} \otimes \cdots \otimes \wedge^{j_m}),$$

where $\wedge^i$ is the i-th fundamental representation of $U_q(\mathfrak{sl}_n)$ $(i = 1, \ldots, n-1)$ and the trivial representation $(i = 0, n)$. The following two privious research facts are known: (A) The quantum group $U_q(\mathfrak{gl}_m)$ is categorified by the category $\mathcal{U}(\mathfrak{gl}_m)$ introduced by Khovanov–Lauda and Rouquier and (B) $\bigoplus \mathrm{Hom}_{U_q(\mathfrak{sl}_n)}(\wedge^{\underline{i}}, \wedge^{\underline{j}})$ is categorified by the category of matrix factorizations HMF_n^m in my thesis (left wavy arrow in blue on the above figure). From these facts, we expected that we have a functor $\Gamma_m^n : \mathcal{U}(\mathfrak{gl}_m) \to \mathrm{HMF}_m^n$ (bottom-left functor Γ_m^n in green on the above figure) and we constructed a functor in this paper.

(3) Summary of the paper "Braid group actions from categorical symmetric Howe duality on deformed Webster algebras": On the symmetric product $S^k(\mathbb{C}^2 \otimes \mathbb{C}^m)$, we have a $U_q(\mathfrak{gl}_m)$ representation (top-right morphism γ_m on the above figure). From the fact that the tensor representation $S^{\underline{i}} = V_{i_1\varpi} \otimes \cdots \otimes V_{i_m\varpi}$ is categorified by the projective module category of the Webster algebra, we expected that we have a functor from the category $\mathcal{U}(\mathfrak{gl}_m)$ to the bimodule category of the Webster algebra. However, there are obstacles when we use the original Webster algebra. In this paper, we defined a deformed Webster algebra $W(\mathbf{s}, k)$ and constructed a functor Γ_m from $\mathcal{U}(\mathfrak{gl}_m)$ to the bimodule category $\mathrm{Bim}(m,k)$ of $W(\mathbf{s}, k)$ (bottom-right functor Γ_m in orange). Subsequently, we defined a braid group action on the homotopy category $K^b(\mathrm{Bim}(m,k))$ using the functor.