

# Abstract of present research

Our study is polynomial approximation theory as an application of potential theory. On  $\mathbb{R}$ , every polynomial  $P(t)$  blows up as  $|t| \rightarrow \infty$ , we must multiply a weight function  $w(t)$ . Then, for  $1 \leq p \leq \infty$  and  $fw \in L^p(\mathbb{R})$ , is there exist a sequence of polynomials  $\{P_n\}$  such that

$$\lim_{n \rightarrow \infty} \|(f - P_n)w\|_{L^p(\mathbb{R})} = 0 \quad (\text{A})$$

holds? We assume that an exponential weight  $w$  belongs to relevant class  $\mathcal{F}(C^2+)$ . Let  $w$  be  $w(t) = \exp(-Q(t))$ . We consider a function  $T(t) := tQ'(t)/Q(t)$ , ( $t \neq 0$ ). If  $T$  is bounded, then  $w$  is called a Freud-type weight, and otherwise,  $w$  is called an Erdős-type weight. In this study, we consider Erdős-type weights. Our present results are following:

1. **Convergence of the de la Vallée Poussin mean:** The de la Vallée Poussin mean  $v_n(f)$  of  $f$  is defined by  $v_n(f)(t) := \frac{1}{n} \sum_{j=n+1}^{2n} s_j(f)(t)$ , where  $s_m(f)(x)$  is the partial sum of Fourier series of  $f$  for orthogonal polynomials with respect to  $w$ . The degree of approximation for  $f$  defined by  $E_{p,n}(w; f) := \inf_{P \in \mathcal{P}_n} \|(f - P)w\|_{L^p(\mathbb{R})}$ . Here,  $\mathcal{P}_n$  is the set of all polynomials of degree at most  $n$ . We assume that  $w \in \mathcal{F}(C^2+)$  and suppose that  $T(a_n) \leq c(n/a_n)^{2/3}$  for some  $c > 0$ . Here, the notation  $a_n$  is called MRS number. Then there exists a constant  $C \geq 1$  such that for every  $n \in \mathbb{N}$  and when  $fw \in L^p(\mathbb{R})$ ,

$$\|(f - v_n(f))w\|_{L^p(\mathbb{R})} \leq CT^{1/4}(a_n)E_{p,n}(w; f). \quad (\text{B})$$

We show the conditions such that the right side of (B) converge to 0 as  $n \rightarrow \infty$ . Moreover, if  $f$  is more smoother function,  $v_n(f)$  is not only a good approximation polynomial for  $f$ , but also its derivatives give an approximation for  $f'$ .

2. **Uniform convergence of the Fourier partial sum:** By the way, we show the condition uniformly convergence of  $s_n(f)$  for a weight in a more smooth subclass  $\mathcal{F}_\lambda(C^3+)$ : Let  $w \in \mathcal{F}_\lambda(C^3+)$  with  $0 < \lambda < 3/2$ . Suppose that  $f$  is continuous and has a bounded variation on any compact interval of  $\mathbb{R}$ . If  $f$  satisfies  $\int_{\mathbb{R}} w(x)|df(x)| < \infty$ , then

$$\lim_{n \rightarrow \infty} \left\| (f - s_n(f)) \frac{w}{T^{1/4}} \right\|_{L^\infty(\mathbb{R})} = 0.$$

3. **Lagrange interpolation polynomials for Laguerre-type weights:** We assume that an exponential weight  $w(x) = \exp(-R(x))$  on  $\mathbb{R}^+ := [0, \infty)$  belongs to relevant class  $\mathcal{L}_\lambda(C^3+)$  with  $0 < \lambda < 3/2$ . Let  $w_\rho$  be  $w_\rho(x) := x^\rho w(x)$  for  $\rho > 0$ . For  $f \in C(\mathbb{R}^+)$ , we write the Lagrange interpolation polynomial  $L_{n,\rho}^*(f)(x)$  with nodes  $\{x_{j,n,\rho}\}_{j=1}^n$ , where  $\{x_{j,n,\rho}\}_{j=1}^n$  are the zeros of  $n$ -th orthogonal polynomial with respect to  $w_\rho$ . We show the condition such that Lagrange interpolation polynomial converges to  $f$  with  $w_\rho$  in  $L^p$ -norm for  $1 < p \leq 2$ : For  $p = 2$ , let  $\beta > 1/2$ . If  $(1+x)^{\beta/2+1/4}w_\rho(x)f(x) \rightarrow 0$  as  $x \rightarrow \infty$ , then we have

$$\lim_{n \rightarrow \infty} \|(L_{n,\rho}^*(f) - f)w_\rho\|_{L^2(\mathbb{R}^+)} = 0. \quad (\text{C})$$

And for  $1 < p < 2$ , let  $\beta > 1/p$  and  $(1+x)^{\beta/2+1/4}T^{*1/2}(x)\Phi^{*-1/4}(x)w_{\rho^*}(x)f(x) \rightarrow 0(x \rightarrow \infty)$ (where,  $\Phi^*(x) := (T^*(x)(1+R(x))^{2/3})^{-1}$ ). Then we have (C) in the case of  $1 < p < 2$ . Here,  $L_{n,\rho^*}^*(f)$  is the Lagrange interpolation polynomial with respect to the weight  $w_{\rho^*} := w_{\rho+1/2p-1/4}$ . Additionally, for  $2 < p$ , we showed the result (C) corresponds to a weight  $\Phi^{*(1/2-1/p)^+}(x)w_\rho(x)$ .