Research Plan

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Let $\beta > 1$. Frougny and Solomyak introduced the following conditions:

$$(\mathrm{F}_1)\quad \mathbb{N} \subset \mathrm{Fin}(\beta)$$

$$(\mathrm{PF})\quad \mathbb{Z}_{\geq 0}[\beta^{-1}] \subset \mathrm{Fin}(\beta)\ where\ \mathbb{Z}_{\geq 0}[\beta^{-1}] = \{\sum_{k=1}^{n} a_k\beta^{-k} \mid a_k \in \mathbb{Z}_{\geq 0}\}$$

$$(\mathrm{F})\quad \mathbb{Z}[\beta^{-1}]_{\geq 0} \subset \mathrm{Fin}(\beta)\ where\ \mathbb{Z}[\beta^{-1}]_{\geq 0} = \{\sum_{k=1}^{n} a_k\beta^{-k} \mid a_k \in \mathbb{Z}\} \cap [0,\infty)$$

where $\mathrm{Fin}(\beta)$ is the set of nonnegative number x such that x has a finite β-expansion. (F_1) includes the other properties and it is known that β is an algebraic integer if $\beta \in (\mathrm{F}_1)$. So $\beta \in (\mathrm{PF})$ is equivalent to $\mathbb{Z}_{\geq 0}[\beta^{-1}] = \mathrm{Fin}(\beta)$, and $\beta \in (\mathrm{F})$ is also equivalent to $\mathbb{Z}[\beta^{-1}]_{\geq 0} = \mathrm{Fin}(\beta)$. In addition, if $\beta \in (\mathrm{F}_1)$, then β is also a Pisot number. Summarizing these results, we have the following table.

	Class	Algebraic structure of $\mathrm{Fin}(\beta)$	Sufficiency for (F)	Sufficiency for (PF)
(F_1)	Pisot	?	?	?
(PF)	Pisot	Closed under addition & multiplication	$d_\beta(1)$ is finite	—
(F)	Pisot	Closed under addition, multiplication & subtraction	—	—

Here closed under subtraction means that if $x, y \in \mathrm{Fin}(\beta)$ $(x < y)$, then $y - x \in \mathrm{Fin}(\beta)$.

(F_1) is not yet known. However, I recently found a $\beta \in (\mathrm{F}_1) \setminus (\mathrm{PF})$. So I will work on the following projects related to (F_1).

1. Necessity and Sufficient condition for property (F_1)

Let $\beta > 1$ be an algebraic integer with minimal polynomial $x^d - a_{d-1}x^{d-1} - \cdots - a_1x - a_0$ and for $\boldsymbol{l} = (l_1, l_2, \cdots, l_{d-1}) \in \mathbb{Z}^{d-1}$, define τ by

$$\tau(\boldsymbol{l}) \coloneqq (l_2, \cdots, l_{d-1}, -\lfloor \lambda(\boldsymbol{l}) \rfloor)$$

$$where\ \lambda(\boldsymbol{l}) = \boldsymbol{l} \cdot (a_0\beta^{-1}, a_1\beta^{-1} + a_0\beta^{-2}, \cdots, a_{d-2}\beta^{-1} + \cdots + a_0\beta^{-d+1})$$

where $\cdot$ is inner product. Then τ is the transformation on $\mathbb{Z}^{d-1}$, corresponding to β-transformation T. In addition, letting $\{\lambda\}(\boldsymbol{l}) \coloneqq \{\lambda(\boldsymbol{l})\}$, we have the following commutative diagram.

$$\begin{array}{ccc} \mathbb{Z}^{d-1} & \xrightarrow{\tau} & \mathbb{Z}^{d-1} \\ \{\lambda\}\downarrow & & \downarrow\{\lambda\} \\ \mathrm{Fin}(\beta)\cap[0,1) & \xrightarrow[T]{} & \mathrm{Fin}(\beta)\cap[0,1) \end{array}$$

Thus $\{\lambda\}(F_\beta) \subset \mathrm{Fin}(\beta) \cap [0,1)$ where $F_\beta := \{\boldsymbol{l} \in \mathbb{Z}^{d-1} \mid \exists k \geq 0;\ \tau^k(\boldsymbol{l}) = \boldsymbol{0}\}$. Now define

$$Q_\beta \coloneqq \{\boldsymbol{l} = (l_1, \cdots, l_{d-1}) \in \mathbb{Z}^{d-1} | \exists \{\boldsymbol{l}_n\}_{n=1}^{N}\ s.t.\ \boldsymbol{l}_N = \boldsymbol{l},\ \boldsymbol{l}_{n+1} \in \{\tau(\boldsymbol{l}_n), \tau^*(\boldsymbol{l}_n)\}\ \&\ \boldsymbol{l}_1 = \boldsymbol{e}\}$$

$$where\ \boldsymbol{e} := (0, \cdots, 0, 1) \in \mathbb{Z}^{d-1}\ \&\ \tau^*(\boldsymbol{l}) := -\tau(-\boldsymbol{l}).$$

It is known that Q_β is a finite set when β is a Pisot number. By my research,

$$\tau_\beta^{-1}(P_\beta) \subset P_\beta\ \ \&\ \{\sum_{n}^{r} a_k\tau^k(\boldsymbol{e}) \,|a_k \in \mathbb{Z}_{\geq 0}\} \cap [-\delta, \delta]^{d-1} \subset F_\beta$$

$$where\ P_\beta := \{\boldsymbol{l} \in Q_\beta \big| \exists k > 0;\ \tau_\beta^k(\boldsymbol{l}) = \boldsymbol{l}\}\ \ \&\ \delta \coloneqq \max\{|l_j|\ |\ (l_1, l_2, \cdots, l_{d-1}) \in P_\beta\}$$

is a sufficient condition for (F_1). Currently, I expect that above conditions are equivalent to $\beta \in (\mathrm{F}_1) \setminus (\mathrm{PF})$. So I aim to solve this problem as my future work.

2. An algebraic structure of $\mathrm{Fin}(\beta)$ under property (F_1)

For $\beta \in (\mathrm{F}_1)$, the algebraic structure of $\mathrm{Fin}(\beta)$ still remains as an unsolved problem. However, I expect that if $\beta \in (\mathrm{F}_1)$, then $\mathrm{Fin}(\beta)$ is closed under multiplication. So I will try this problem as one of my future work. In addition, I plan to consider the equivalent condition when $\mathrm{Fin}(\beta)$ is closed under multiplication.