Overview of Research Contributions

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In the paper *"Klyachko's theorem in semi-finite von Neumann algebras"*, we show that a part of Klyachko's theorem can be extended to positive operators on von Neumann algebras. Here, Klyachko's theorem describes the relationship between the lists of eigenvalues of two $n \times n$ Hermitian matrices A and B, and their sum $A + B$, through a system of extensive inequalities. Although the result itself is not necessarily easy to apply, a problem that had existed for more than thirty years is essentially resolved. Later, we partially extend a multiplicative version of Klyachko's theorem, that is, the relation between the eigenvalues of A, B, and AB, to von Neumann algebras in *"Multiplicative versions of Klyachko's Theorem in finite factors"*.

Subsequently, we study operator inequalities involving convex and concave functions. For example, if f is a monotonically increasing convex function and $x, y \geq 0$ are real numbers, then in general,

$$f(x) + f(y) \leq f(x + y).$$

Whether this inequality (or its analogue) holds for matrices or operators is naturally of interest. In the finite-dimensional case, Ando and Zhan prove that it holds in the sense of majorization. We extend this majorization inequality to τ-measurable operators.

In the study of inequalities involving convex functions, the following form of Jensen's inequality is very important. Suppose x, y are real numbers and f is a convex function with $f(0) \leq 0$. If a real number a satisfies $|a| \leq 1$, then

$$f(ax) \leq af(x).$$

Various matrix and operator versions of this inequality are known. One of the classical forms is

$$\mathrm{Tr}(f(a^*xa)) \leq \mathrm{Tr}(a^*f(x)a),$$

where a is a matrix satisfying $\|a\| \leq 1$ and x is Hermitian. In *"On equality condition for trace Jensen inequality in semi-finite von Neumann algebras"*, we show that the equality condition for this inequality is given by

$$(a^*xa)^2 = a^*x^2a.$$