

Future research plan (Atsushi Katsuda)

In the recent preprint, *An extension of the Floquet-Bloch theory to nilpotent groups and its applications*, arXiv:2509.16848, we outlined future research directions in Chapter 11. Among them, we would like to highlight several that seem relatively feasible or hold promise for further development.

- (1) **Nilpotent Chebotarev density theorem for compact negatively curved manifolds** It is well known that the prime geodesic theorem holds for geodesic flows on negatively curved manifolds, as a geometric analogue of the prime number theorem in number theory. Furthermore, geometric analogues of Dirichlet's theorem on arithmetic progressions and its generalization, the Chebotarev density theorem, have been considered even in the context of extensions by infinite discrete groups, going beyond formal analogies with number theory.

In the aforementioned preprint, several results were obtained for nilpotent extensions in the case of compact hyperbolic Riemann surfaces. We aim to extend these results to more general variable negatively curved manifolds and more general hyperbolic dynamical systems such as Anosov systems. In these cases, instead of harmonic analysis tools like the Laplacian used for hyperbolic Riemann surfaces, dynamical systems methods will be employed. We believe the goal can be achieved by discussing the problem in contrast with the abelian extension case.

- (2) **From abelian to nilpotent extensions: Several examples** The standard Floquet-Bloch theory applies to abelian extensions and is one of the fundamental tools in condensed matter physics. Applications include studies of the Martin boundary of periodic elliptic operators, horocycle flows and frame flows in abelian extensions, and results such as the Louville-Riemann-Roch theorem for abelian extensions. We would like to explore extensions of these results to the nilpotent extension case.
- (3) **Toward more general discrete groups** Furthermore, we aim to consider Floquet-Bloch theory for solvable or amenable groups more complex than nilpotent groups, using methods such as orbit theory and relative iterated integrals. The initial goal is the case of lattices in 3-dimensional solvable Lie groups.
- (4) **Quasimorphisms, Modified Riemann-Hilbert Problems, and Opers** So far, the Chebotarev-type theorems have assumed the existence of a surjective homomorphism from the fundamental group of a compact manifold to a group Γ . Sarnak, however, considered a density theorem under the assumption of a quasi-homomorphism from the fundamental group of a manifold with ends to $\mathbb{Z}$, and investigated the relationship between prime closed orbits and the linking number with knots determined by the ends.

This result was inspired by Ghys's ICM lecture, where trefoil knots, linking numbers, closed orbits in dynamical systems, and number-theoretic objects such as the Rademacher symbol and the Dedekind eta function were beautifully connected.

In the original problem, the Hodge-de Rham theorem and its extension to nilpotent cases play an important role. This provides a solution to the Riemann-Hilbert problem in the C^∞ category (i.e., finding a flat connection whose monodromy representation corresponds to a given representation of the fundamental group).

Here, we propose a new perspective: interpreting quasimorphisms as a kind of discrete curvature and seeking a connection (not necessarily flat) that realizes it as holonomy—a modified Riemann-Hilbert problem. In fact, the Rademacher symbol can be interpreted as discrete curvature, and the logarithm of the Dedekind eta function corresponds to the connection.

These ideas were extended to torus knots by Matsuzaka and Ueki, and we hope to explore further developments. This research may also connect to broader topics such as the quantum Riemann-Hilbert problem, geometric Langlands correspondence, opers, non-abelian Hodge theory, and integrable systems.