Results of my research

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A genus two handlebody-knot is a genus two handlebody embedded in the 3-sphere S^3, denoted by H. Two handlebody-knots are equivalent if one can be transformed into the other by an isotopy of S^3.

For a handlebody-knot H and its meridian system M, let $\Delta^{(d)}_{(H,M)}(t_1, t_2)$ be the d-th Alexander polynomial of a pair (H, M). The Alexander polynomial $\Delta^{(d)}_{(H,M)}(t_1, t_2)$ is an invariant of a pair (H, M) and the replacement of the meridian system acts on the Alexander polynomial as $GL(2, \mathbb{Z})$.

Suppose that H is cut open with a separating disk and we obtain a two component knotted solid tori in S^3. We consider it as a link and call it a constituent link L of H. Since there are infinitely many such separating disks for a handlebody-knot, there are infinitely many constituent links of the handlebody-knot.

For a two component link $L = K_1 \cup K_2$, if there exist mutally disjoint Seifert surfaces for K_1 and K_2 in S^3, then L is called a boundary link. In particular, for L, if there is a 2-sphere S^2 separating K_1 and K_2 in S^3, then L is called a split link.

Let $\Gamma = L \cup c = K_1 \cup K_2 \cup c$ be a handcuff graph representing a handlebody-knot H. Here c is a chain of Γ. Let M_1 and M_2 be meridians of K_1 and K_2, respectively. Let $M = \{m_1, m_2\}$. Let $\Delta_L(t_1, t_2)$ and $\Delta_{K_i}(t)$ be the Alexander polynomials of L and K, respectively. We have the following property for constituent split links of a handlebody-knot H.

Theorem 1 [O.]
For any handcuff graph Γ such that $L = K_1 \cup K_2$ is a boundary link, there exists a Laurent polynomial $p(t_1, t_2) \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]$ satisfying $p(1, t_2) = p(t_1, 1) = 1$ such that $\Delta^{(2)}_{(\Gamma,M)}(t_1, t_2) = \Delta^{(2)}_L(t_1, t_2)p(t_1, t_2)$. For any Laurent polynomial $p(t_1, t_2) \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]$ satisfying $p(1, t_2) = p(t_1, 1) = 1$, we can construct a handcuff graph $\Gamma = L \cup c = K_1 \cup K_2 \cup c$ such that $\Delta^{(2)}_{(\Gamma,M)}(t_1, t_2) = \Delta^{(2)}_L(t_1, t_2)p(t_1, t_2)$.

Corollary 2 [O.]
$L = K_1 \cup K_2$ and $L' = K'_1 \cup K'_2$ be constituent split links of a handlebody-knot H. Then $\Delta_{K_1}(t) = \Delta_{K'_1}(t)$ and $\Delta_{K_2}(t) = \Delta_{K'_2}(t)$ or $\Delta_{K_1}(t) = \Delta_{K'_2}(t)$ and $\Delta_{K_2}(t) = \Delta_{K'_1}(t)$ holds.

These results can be generalized to genus n handlebody-knots.