

# Plan of the study (Yosuke Saito)

For complex numbers  $q, p$  satisfying  $|q| < 1, |p| < 1$ , we define the theta function  $\theta_p(x)$  and the elliptic gamma function  $\Gamma_{q,p}(x)$  by

$$\bullet \theta_p(x) := \prod_{n \geq 0} (1 - xp^n)(1 - x^{-1}p^{n+1}), \quad \bullet \Gamma_{q,p}(x) := \prod_{m,n \geq 0} \frac{1 - x^{-1}q^{m+1}p^{n+1}}{1 - xq^m p^n} \quad (x \in \mathbb{C} \setminus \{0\}).$$

By setting  $D_x = x \frac{\partial}{\partial x}$  (Euler derivative), we define  $E_k(x; p) := -D_x^k \log \theta_p(x)$  ( $k \in \mathbb{Z}_{>0}$ ).

Let  $N$  be a positive integer,  $\beta$  be a complex number, and  $p$  be a complex number satisfying  $|p| < 1$ . The Hamiltonian of the elliptic Calogero-Moser system  $H_N^{\text{CM}}(\beta, p)$  is defined by

$$\bullet H_N^{\text{CM}}(\beta, p) := \sum_{i=1}^N D_{x_i}^2 - \beta(\beta-1) \sum_{1 \leq i \neq j \leq N} E_2(x_i/x_j; p).$$

Then the following fact is known:  $\Psi_N(x; \beta, p) := \prod_{1 \leq i \neq j \leq N} \theta_p(x_i/x_j)^{\beta/2}$  satisfies

$$\bullet H_N^{\text{CM}}(\beta, p) \Psi_N(x; \beta, p) = \{2N\beta D_p + C_N(\beta, p)\} \Psi_N(x; \beta, p), \quad \dots (*)$$

where  $C_N(\beta, p)$  is a complex number. It is remarkable that the derivative  $D_p = p \frac{\partial}{\partial p}$  is in the right hand side of (\*). This means that the elliptic Calogero-Moser system has a solution which involves the infinitesimal deformation of the elliptic modulus  $p$ .

Let  $N$  be a positive integer,  $q, p$  be complex numbers satisfying  $|q| < 1, |p| < 1$ , and  $t$  be a complex number satisfying  $t \in \mathbb{C} \setminus \{0\}$ . The Hamiltonian of the elliptic Ruijsenaars system  $H_N^{\text{R}}(q, t, p)$  is defined by

$$\bullet H_N^{\text{R}}(q, t, p) := \sum_{i=1}^N \prod_{j \neq i} \left\{ \frac{\theta_p(tx_i/x_j) \theta_p(qt^{-1}x_i/x_j)}{\theta_p(x_i/x_j) \theta_p(qx_i/x_j)} \right\}^{\frac{1}{2}} T_{q,x_i},$$

where  $T_{q,x}$  is the  $q$ -shift operator which is defined by  $T_{q,x} f(x) = f(qx)$ . Then the function

$$\Psi_N(x; q, t, p) := \prod_{1 \leq i \neq j \leq N} \left\{ \frac{\Gamma_{q,p}(tx_i/x_j)}{\Gamma_{q,p}(x_i/x_j)} \right\}^{1/2} \text{ satisfies}$$

$$\bullet H_N^{\text{R}}(q, t, p) \Psi_N(x; q, t, p) = t^{\frac{-N+1}{2}} \sum_{i=1}^N \prod_{j \neq i} \frac{\theta_p(tx_i/x_j)}{\theta_p(x_i/x_j)} \Psi_N(x; q, t, p). \quad \dots (**)$$

It is known that by setting  $t = q^\beta$  and by taking the limit  $q \rightarrow 1$  appropriately, the equation (\*\*) degenerates to the equation (\*). Thus it is probable that the equation (\*\*) contains a certain difference deformation of the elliptic modulus  $p$ . By standing the point of view, the author will study the problem with referring to the  $q$ -KZB heat equation introduced by Felder-Varchenko, the non-stationary Ruijsenaars function defined by Shiraishi, the quantum double-elliptic system (DELL system) introduced by Koroteev-Shakirov.