Research Result

Let $\beta > 1$ and denote by $\lfloor y \rfloor$ the integer part of y. By using the transformation $T: [0,1] \to [0,1)$ defined by $T(x) = \beta x - \lfloor \beta x \rfloor$, we get the expansion of $x \in [0,1)$ as shown below:

$$x = c_1\beta^{-1} + c_2\beta^{-2} + \cdots + c_n\beta^{-n} + \cdots = \sum_{n=1}^{\infty} c_n\beta^{-n}$$

$$where\ c_n := \lfloor \beta T^{n-1}(x) \rfloor.$$

Then we write $d_\beta(x) = c_1c_2\cdots$. For $x \geq 0$, define a β-*expansion* of x by

$$x = c_1\beta^{L-1} + c_2\beta^{L-2} + \cdots + c_{L-1}\beta + c_L + c_{L+1}\beta^{-1} + \cdots = \sum_{n=1}^{\infty} c_n\beta^{L-n}$$

$$where\ L = L(x) := \min\{n \in \mathbb{Z}_{\geq 0} \mid x\beta^{-n} < 1\}, \qquad d_\beta(\beta^{-L}x) = c_1c_2\cdots.$$

Here $c_1c_2\cdots c_n$ can be the empty word. We say that x has a finite β-expansion if its tail is only zero.

It is known that any positive integer has a finite decimal expansion. As a generalization of this finiteness property, Frougny and Solomyak introduced the following conditions:

$$(\mathrm{F}_1)\quad \mathbb{N} \subset \mathrm{Fin}(\beta)$$

$$(\mathrm{PF})\quad \mathbb{Z}_{\geq 0}[\beta^{-1}] \subset \mathrm{Fin}(\beta)\ where\ \mathbb{Z}_{\geq 0}[\beta^{-1}] = \{\sum_{k=1}^{n} a_k\beta^{-k} \mid a_k \in \mathbb{Z}_{\geq 0}\}$$

$$(\mathrm{F})\quad \mathbb{Z}[\beta^{-1}]_{\geq 0} \subset \mathrm{Fin}(\beta)\ where\ \mathbb{Z}[\beta^{-1}]_{\geq 0} = \{\sum_{k=1}^{n} a_k\beta^{-k} \mid a_k \in \mathbb{Z}\} \cap [0,\infty)$$

where $\mathrm{Fin}(\beta)$ is the set of nonnegative number x such that x has a finite β-expansion.

(1) **Finite β-expansion and Odometers** ([1] in Peer-reviewed papers, joint work with M. Yoshida)

In this study, we introduced the odometer $H_{\beta^{-1}}$ associated with β-numeration system and proved the followings: β has property (F) if and only if $H_{\beta^{-1}}$ is surjective, and β has property (PF) if and only if $H_{\beta^{-1}}$ is injective. Furthermore, when β is an algebraic integer, we represent a procedure of carry operation in $H_{\beta^{-1}}$ by a transducer. As a result, we also proved that β has property (F) if and only if $H_{\beta^{-1}}$ is computable.

(2) **Some class of cubic Pisot numbers with finiteness property** ([2] in Peer-reviewed papers, joint work with M. Yoshida)

Akiyama characterized cubic Pisot units with property (F). Also, Akiyama et al. found some classes of cubic Pisot numbers with property (F) by using a set of witnesses. In this paper, we obtained a generalization of Akiyama's cubic Pisot units theorem by hand computing. Moreover, in that proof, we found some classes of cubic Pisot numbers with property (F) by using the set smaller than set of witnesses. As a result, we found a new class of cubic Pisot numbers with property (F).

(3) **Finite beta-expansions of natural numbers** ([3] in Peer-reviewed papers)

It is known that β is a Pisot number if β has property (F_1). However, other results had not been known. In this study, we give a sufficient condition for property (F_1) and find the β which has property (F_1) without (PF).