# Future research plans

*a._Background of proposed research plan*

One striking aspect of complex geometry is the inherent rigidity in the way complex submanifolds embed. Given a compact complex variety C embedded in two complex varieties M and M' of the same dimension, whether there exists a neighborhood of C in M that is equivalent to a neighborhood of C in M′—a problem where only the cases with "curvature" have been understood and have attracted significant work since the late 1960s (by, for example, Grauert and Griffiths), while very little is known about the "flat" case (i.e. with no curvature) because geometric methods are not well suited; it was V.I. Arnold in the 1970s who showed that techniques from dynamical systems could be crucial—in fact, by studying the embeddings of an elliptic curve (i.e. a one-dimensional complex torus) in a complex surface, he demonstrated that issues resembling the "small divisors" problems common in celestial mechanics naturally appear in this context—and L. Stolovitch and X. Gong [GS] have significantly extended Arnold's results using methods very similar to those of dynamical systems that consist in a kind of "linearization" requiring control over quantities analogous to small divisors (which theoretically arise as bounds for solutions of cohomology equations, though their asymptotic behavior is not easy to determine a priori). Beyond this neighborhood-classification problem, the existence of holomorphic foliations near C in M also involves similar issues that have recently regained interest in the community, with the case of a curve in a surface having been solved by T. Ueda in the 1980s and the work of [GS] extending the understanding to the general situation; yet, while part of the problem involves a kind of linearization that we can manage using dynamical systems methods, translating our hypotheses into accessible geometric terms remains challenging, and other aspects—for example, when linearization fails—remain completely unknown.

*b. _Propose*

The Ueda theory is also related to the following important problem where T. Koike has widely studied. A nef line bundle may admit several singular positive metrics, each characterized by distinct singularities. The analysis of the minimality of these singularities is of considerable importance. In a paper of Demailly-Peternell-Schneider, they pose the following fundamental problem.

Question: Let X be a compact Kähler variety that is rationally connected and whose anticanonical bundle is nef. One then asks whether the anticanonical is automatically hermitian semi-positive. In particular, is it always true that the blow-up of the projective plane at nine points lying on a smooth cubic curve has a semi-positive anticanonical bundle?

Note that, in this concrete example, there exists a "singular positive metric." The question concerns the regularity of this singular positive metric: is it always smooth? Brunella's work establishes that minimality is closely related to the underlying dynamical conditions. Thus, it is shown that singular positive metrics are not necessarily smooth, which answers the question in the negative. Conversely, Demailly states the following conjecture:

Conjecture: Without any particular dynamical condition and apart from certain specific examples, every singular positive metric on the nef anticanonical line bundle of a compact Kähler variety can be very singular.

However, concrete examples confirming this conjecture, even in low dimensions, remain rare.

During my PhD, my advisor Jean-Pierre Demailly hosted a visit by Professor T. Koike, who was already very interested in these problems and obtained many important results in this direction using Ueda's theory. Later, during my postdoctoral period under the direction of Professor Laurent Stolovitch, I further studied Ueda's theory and had fruitful discussions with Professor Koike at the SCV, CR Geometry, and Dynamics conference at RIMS. As an expert in analytic methods in algebraic geometry, I hope that my knowledge of algebraic varieties will be useful in a potential collaboration with Professor Koike to construct the example conjectured by Demailly.

*c. Expected results and impacts*

We hope that our approach will lead to a better understanding of the convergence of the linearization map in the presence of the "small divisors" phenomenon. In general, necessary and sufficient conditions to guarantee this convergence are rare in dynamical systems theory. For instance, after Yoccoz's groundbreaking work in complex dynamics on curves, extending the results to higher-dimensional cases has remained an open problem for more than thirty years. Following an in-depth study of the potential sources of divergence, our goal is to construct a metric on the anticanonical bundle of certain complex varieties and to address the conjecture raised by Demailly.

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[GS]X. Gong, L. Stolovitch, Equivalence of neighborhoods of embedded compact complex manifolds and higher codimension foliations, Arnold Math. J. (2021) p. 1-85.