Past research achievements

My research crosses three fields: complex geometry, several complex variables, and algebraic geometry. I have been studying geometry and global analysis for algebraic varieties (more generally, non-algebraic Kähler varieties), motivated by birational geometry, particularly, the MMP (Minimal Model Program). The distinctive feature of my research is to employ transcendental methods arising from several complex variables and differential geometry to study algebraic geometry. This feature is an advantage of my research, and it allows us to deal with non-algebraic objects (e.g., Kähler manifolds, curvatures, singular metrics, differential equations etc.). Specifically, my achievements can be divided into the following three categories:

(1) Studies of the cohomology theory in terms of pseudo-effective line bundles;
(2) Studies of higher Chern classes for pseudo-effective vector bundles;
(3) Studies of the Minimal Model Program for compact Kähler 3-folds.

(1): The cohomology theory, represented by Kodaira's vanishing theorem, plays a decisive role in complex geometry. A "good" cohomology theory requires a certain positivity condition for line bundles. My achievement (1) is to extend some classical theories (e.g., vanishing theorems of the Kawamata-Viehweg, Nakano, and Bogomolov types) to pseudo-effective line bundles ([論文 1]). The pseudo-effectivity is the weakest non-negativity that naturally appears in algebraic geometry; therefore, my achievement (1) is the best generalization in some sense. The pseudo-effectivity can be treated analytically by using singular metrics or curvatures. In this study, I have learned the latest techniques of the pseudo-effectivity and singular metrics to study a certain non-negativity.

(2): Chern classes of vector bundles (e.g., tangent bundles) are significant for studying geometry of manifolds. The study of higher Chern classes often requires a proper definition of the product of positive curvature currents (i.e., differential forms with measure coefficients). This problem on the product corresponds to the theory of intersection numbers in algebraic geometry and to the theory of Monge-Ampère products in complex analysis. I have constructed the Monge-Ampère product representing the Segre current of pseudo-effective vector bundles([論文 2–4]). This allows us to understand geometry of pseudo-effective vector bundles. As an application, I obtained flatness criteria of tangent bundles, which generalize the result of [CCM] from algebraic varieties to (non-algebraic) Kähler manifolds. This application characterizes flat manifolds and is important in the study of non-negative curvature manifolds.

(3): I established a structure theorem of compact Kähler 3-folds with nef anti-canonical bundle in [論文 7] (joint work with S. Matsumura). This is a definitive result for compact Kähler 3-folds in this field. The history of this field originates from the results of Mori and Siu-Yau; they completely classified Kähler manifolds whose tangent bundles have positive curvature. Subsequently, Mok and Demailly-Peternell-Schneider established the structures of Kähler manifolds whose tangent bundle is nef (which means having the non-negative curvature in some sense). It is natural from the viewpoint of algebraic geometry to generalize this result from tangent bundles to anti-canonical bundles (i.e., the determinant bundle of tangent bundles). This was conjectured by Demailly-Peternell-Schneider (leading experts in the field of complex geometry) in the 90s, but has remained unsolved until now. Recently, Cao-Höring proved this conjecture for algebraic varieties in [CH, Cao], which is a breakthrough in this field. However, this conjecture was not known for (non-algebraic) Kähler manifolds at all. Achievement (3) solves this conjecture for compact K ahler 3-folds. This is the first step toward fully resolving the conjecture in all dimensions for (non-algebraic) Kähler manifolds. Our proof is based on a detailed study of the MMP for Kähler 3-folds. The MMP is an algebraic concept, but I have introduced an analytical method to clarify what non-negative curvature brings to the MMP.

[論文 1]Xiaojun Wu (single author), On the hard Lefschetz theorem for pseudoeffective line bundles, Internat. J. Math. 32 (2021), no. 6, Paper No. 2150035, 25 pp.

[論文 2]Xiaojun Wu (single author), Intersection theory and Chern classes in Bott-Chern cohomology, Arkiv för Matematik, 61 (2023), 55 pp.

[論文 3]Xiaojun Wu (single author) , A study on nefness in higher codimension, Bull. Soc. Math. France 150 (2022), no. 1, 209--249.

[論文 4]Xiaojun Wu (single author), Pseudo-effective and numerically flat reflexive sheaves, J. Geom. Anal. 32 (2022), no. 4, Paper No. 124, 61 pp.

[論文 5]Xiaojun Wu (single author), Note on holomorphic Morse inequalities tensoring with a coherent sheaf, to appear in Pure and Applied Mathematics Quarterly.

[論文 6]Xiaojun Wu (single author), Note on asymptotic behaviour of the canonical ring, Comptes Rendus Mathématiques. Volume 362 (2024), 1389--1397.

[論文 7]Shin-ichi Matsumura, Xiaojun Wu (collaboration), Compact Kähler three-folds with nef anti-canonical bundle, Mathematische Annalen, Volume 391 (2025), 1253–1289.

[論文 8]Junyan Cao, Patrick Graf, Philipp Naumann, Mihai Paun, Thomas Peternell, Xiaojun Wu (collaboration), Hermite-Einstein metrics in singular settings, accepted by Asian J. Math.

[論文 9]Philipp Naumann, Xiaojun Wu (collaboration), Albanese map for Kähler manifolds with nef anticanonical bundle, Math. Z. 311, 25 (2025).