Research plan

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To obtain a reduction formula for abelian functions, when a morphism can be defined between two algebraic curves, it is necessary to be able to write specific defining equations for these algebraic curves and morphisms. In [Katsura, Takashima 2024], for a positive integer g, the defining equation for a bielliptic hyperelliptic curve of genus $2g$ was determined. By applying this result, I will describe abelian functions associated with the bielliptic hyperelliptic curve of genus $2g$ in terms of abelian functions associated with hyperelliptic curves of genus g. This allows us to obtain a sequence of abelian functions that can be expressed in terms of elliptic functions, the simplest abelian functions. In [Kuhn 1988], the defining equation for a hyperelliptic curve of genus 2 for which there is a degree 3 morphism to an elliptic curve was determined. In [Belokolos, Enolskii 2001], the defining equation for a hyperelliptic curve of genus 2 for which there is a degree 4 morphism to an elliptic curve was determined. By applying these results, I will describe the abelian functions associated with the two genus 2 hyperelliptic curves mentioned above in terms of elliptic functions.

Let V_g be a hyperelliptic curve of genus g with one point at infinity defined by

$$y^2 = x^{2g+1} + \lambda_2 x^{2g} + \cdots + \lambda_{4g} x + \lambda_{4g+2}, \quad \lambda_i \in \mathbb{C}.$$

Let $\sigma(u_1, u_3, \ldots, u_{2g-1})$ be the sigma function associated with V_g. Let $\sigma_i = \partial_{u_i}\sigma$ and $\wp_{i,j} = -\partial_{u_i}\partial_{u_j} \log \sigma$. Let F be the function where σ_3/σ_1 is considered on the zero set of σ. I have shown that if $\lambda_{4g} = \lambda_{4g+2} = 0$, then F coincides with $\wp_{1,1}$ associated with V_{g-1}. Since $\wp_{1,1}$ satisfies the KdV hierarchy, I think that F satisfies a system of differential equations obtained by modifying the KdV hierarchy with λ_{4g} and λ_{4g+2}. We explicitly write the system of differential equations satisfied by F.

In [Tian et al. 2013], solutions to the breaking soliton equations, which are important in mathematical physics, were constructed using abelian functions associated with hyperelliptic curves with one point at infinity for genus 2 and 3. This result was shown using differential relations for abelian functions for genus 2 and 3 obtained in [Baker 1903]. On the other hand, in [Buchstaber et al. 1997], these differential relations were extended to the case of general genus. I will construct solutions to the breaking soliton equations for general genus using abelian functions.

The abelian functions $\wp_{k,l}(k, l = 1, 3, \ldots, 2g-1)$ associated with a hyperelliptic curve of genus g with one point at infinity have been extensively studied. I will consider abelian functions $\mathcal{P}_{i,j}(i, j = 2, 4, \ldots, 2g)$ associated with a hyperelliptic curves of genus g with two points at infinity

$$y^2 = \nu_0 x^{2g+2} + \nu_2 x^{2g+1} + \cdots + \nu_{4g+2} x + \nu_{4g+4}, \quad \nu_i \in \mathbb{C}, \quad \nu_0 \neq 0,$$

which were constructed by Baker. I clarified the relationship between $\mathcal{P}_{i,j}$ and $\wp_{k,l}$. I will derive the differential relations for $\mathcal{P}_{i,j}$. I showed that if $\nu_{4g+4} = 0$ and $\nu_{4g+2} = 1$, then $\mathcal{P}_{2g+2-2m,2g+2-2n}$ is identical to $\wp_{2m-1,2n-1}$ $(1 \le m, n \le g)$. Since $\wp_{k,l}$ satisfies the KdV hierarchy, the sine-Gordon equation, and the Veselov-Novikov equation, I think that the functions $\mathcal{P}_{i,j}$ satisfy differential equations obtained by modifying these equations with two parameters ν_{4g+2} and ν_{4g+4}. I will write these differential equations explicitly.