Research results so far

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To date, many approximation theorems for fixed points have been established in Hilbert spaces and Banach spaces. Our research has focused on examining whether analogous results hold in CAT spaces under assumptions that are considered reasonable in that setting. Since Hadamard spaces (i.e., complete CAT(0) spaces) are a generalization of Hilbert spaces in a direction different from that of Banach spaces, results obtained there naturally yield a certain type of generalization at the level of theorems. A typical example is given in [1].

In [1], for a finite family of mappings (under appropriate assumptions) in a complete CAT(1) space, a W-mapping is constructed using convex combinations and compositions of the mappings. By introducing a multi-step iterative scheme, an approximation theorem for common fixed points is obtained. This work can be viewed as an extension of [3], which establishes an approximation theorem for common fixed points of finitely many mappings in Banach spaces, and [4], which derives an approximation theorem for a single mapping in a complete CAT(1) space. The assumptions imposed on each mapping are broad enough to include important concrete examples such as resolvents and nonexpansive mappings.

Moreover, in [2], the multi-step iteration introduced in [1] is applied, in complete CAT(1) spaces, to fixed point approximation methods known as the contraction projection method and the CQ projection method. As a result, well-known theorems in Hilbert spaces are extended to the setting of CAT(1) spaces. In [2], a nonexpansive assumption is imposed on finitely many mappings. While this is a natural and reasonable property in existing results for Hilbert and Banach spaces, important examples of nonexpansive mappings are not well known in CAT(1) spaces. Consequently, it is desirable to further develop the theory under assumptions on mappings that include many concrete examples, such as being strongly quasi-nonexpansive and Δ-demiclosed.

Although not discussed in [2] and remaining unpublished, it has been shown by Kimura and Takahashi that, for resolvents in CAT(1) spaces as defined by Kimura and Kosaka (which are strongly quasi-nonexpansive and Δ-demiclosed), the results of [2] can indeed be proved by exploiting their intrinsic properties.

Another result that is directly relevant to future research is [5]. In [5], for a family of mappings on a Hadamard space satisfying a property called uniformly continuous total asymptotically quasi-nonexpansive, an approximation theorem for common fixed points is obtained by considering a certain multi-step iterative process. Building on this result, we have investigated new developments by

(1) considering CAT(1) spaces instead of Hadamard spaces, and (2) replacing the multi-step iteration with a contraction projection method.

For both approaches (1) and (2), discussions with other specialists have given us confidence in the validity of the problems themselves. On the other hand, the proof in [5] offers few algebraic manipulations or inequality estimates that can be directly applied to (1) and (2). In particular, the assumption that mappings are uniformly continuous total asymptotically quasi-nonexpansive is not appropriate in the CAT(1) setting, in the sense that no concrete examples are known.

Although this differs somewhat from the original problem, in the current fiscal year we have focused on developing the results of [5] in the sense of (1) and (2) above, while making use of the techniques we have accumulated in our previous studies on CAT(1) spaces and contraction projection methods. Although we have not yet reached final results, specific obstacles have been identified, and we have found several results in the literature that may serve as candidate tools for overcoming these difficulties. In parallel, we are also preparing to compile these findings into a research paper.

References

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