Research plan (April 2026 $\sim$)

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The study of elliptic operators in the framework of global setting brought deep understandings of the manifold theory and analysis. In this year, mainly I am going to study global phenomena in relation with geometric structure and analysis.

Laplace-Beltrami operator and Dirac operator are defined by means of geometric structures on manifolds, the first one is a second order elliptic differential operator and the second one is a first order elliptic differential operator, respectively. Sub-Riemannian structures and associated sub-Laplacians are one of my main subject. A sub-Laplacian is a sub-elliptic second order differential operator and is defined based on a sub-Riemannian structure.

This geometric structure (= sub-Riemannian structure) assumes the existence of a bracket generating sub-bundle in the tangent bundle.

The opposite structure, that is the foliation structure, was studied since many years ago. On the other hand until recently it has not been so much studied of such structure from global analytic point of view, so that I am expecting that the research of this subject has an enough meaning even by comparing with Riemannian manifolds theory and analysis on them.

Contact manifolds and nilpotent Lie groups are basic examples of manifolds carrying a sub-Riemannian structure and it happens often that the total space of a Riemannian submersion has both structures, foliation and sub-Riemannian. Hence there are ample examples of such manifolds to be studied.

Although we can define a transversely elliptic operator on foliated manifolds, there are no natural differential operator associated to the foliation structure. On the other hand there is an intrinsically defined second order differential operator (we call it a sub-Laplacian) on sub-Riemannian manifolds reflecting the sub-Riemannian structure. Hence, again there must be enough meaning to study the sub-Laplacian in contrast with the Laplacian in the Riemannian case.

Since this operator satisfies the "*sub-elliptic estimate*" (= sub-ellipticity), which was proved by Hörmander and the basic structure of the spectrum is similar to elliptic operator cases (on compact manifolds). However, there is non trivial characteristic variety so that from the geometric point of view, we must be more careful to study of this type operators and this fact may include difficulties in the study, and at the same time we can expect the possibility of new phenomena other than obtained by the property of ellipticity. So, I am going to continue the research on these topics under the direction to find possible new phenomena which will not be included in the elliptic operator theory, and together including the problems whether named classical manifolds have this structure and analytically similar properties of this type operator with the elliptic cases, like Weyl low or an explicit construction of the heat kernel.

With these in mind, I am planing the research on the following explicit problems (a) $\sim$ (d) in coming several years:

(a) Nilpotent Lie groups are examples of sub-Riemannian manifolds carrying a "good" sub-Riemannian structure (such is called equi-regular sub-Riemannian structure). Among them I was studying a class of algebras (and groups) attached to Clifford algebras, which we call pseudo H-type algebras(groups) in these years. As a continuation, I am going to construct and classify "*orthonormal invariant lattices*" for higher dimensions. Such lattices (=uniform discrete subgroups) are a special type among integral lattices, even so their classification are still unclear, some follows Bott periodicities, but in the higher dimension it appears new types of such integral lattices.

Also as a continuation from the last years, I am going to deal with a problem on the geodesic orbit manifold theory, that is we study which cases of pseudo H-type Lie groups equipped with the natural left invariant pseudo-Riemannian metric are geodesic orbit manifolds or not together with two collaborators in complete form, and together we will deal with general 2 step nilpotent Lie groups on this geodesic orbit property.

(b) At the same time I continue a study of Radon transformation from the point of the relation of incidence relation (by S.S. Chern) and composition of Fourier integral operator theory (transversal and clean intersection theorems) in relation with the problem (d) below, in which I am expecting to find a Fredholm Radon transformaion.

(c) Since the autumn 2020, I had started a problem on the construction of the Green kernel and heat kernel of a sub-Laplacian on a manifold with conic singularity with two collaborators by the method of symbolic calculus. However there remains still unclear problem. In coming year we will restrict the study to the low (2, and 3)dimensional cases, since where the operator is not essentially selfadjoint and more complicated.

(d) Although Lie groups have always several invariant sub-Riemannian structures, it is not so simple to classify such structures on their symmetric spaces (more generally homogeneous). So if I have a time I will try to construct explicitly such a structure or classify them for some cases or a particular symmetric space.

The spaces and fibrations appearing here are highly relating with the problem (b), and inccidentaly I try to find a good case of "incidence relation" in the sense of S.S. Chern which may give us Fredhom Rradon transformation.