Research results

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Since 19 century, elliptic operators were studied from many directions, especially in relation with physics in which the Dirichlet problem was one of the initial problem. Then following the development of the theory of the Hilbert space, the theory developed so quickly including the development of pseudo-differential operators and Fourier integral operator theory based on the revolutionary treatment of the concept of functions (that is, distribution theory where functions are being treated as observables in the sense of physics which can be understood as a dual treatment of functions). I was continuing my research in this framework on the relations of the geometric structure of manifolds and linear partial differential operators existing on manifolds in a natural way. In particular, around these twenty years I was concentrating in the study of sub-Riemannian manifolds and sub-elliptic operators and related subjects, which are defined on the manifolds based on this geometric structure. In the explanations below the number $[*]$ indicates the number of the paper(refereed) in the publication list.

The studies are classified as

(1) Study of spectral flow and Maslov index in the infinite dimension,
(2) Study of geometric quantization,
(3) Study of Laplacian and sub-Laplacian,
  (3-1) Study of heat kernel and spectrum,
  (3-2) Study of sub-Riemannian structure,
(4) Study of pseudo H-type Lie algebras(groups).

(1) Based on the studies by Floer who found a phenomenon in the transversely intersecting Lagrangian submanifolds and Yoshida's result proving the equality of Maslov index and spectral flow on the partitioned manifold, we generalized the equality "*Spectral flow = Maslov index*" in the framework of the infinite dimensional symplectic Hilbert space, where Maslov index in the infinite dimensional version was formulated as an invariant of homotopy classes of arbitrary paths with fixed end points in the Fredholm-Lagrangian-Grassmannian. This result was published in [44] and also in the papers [35], [37], [40], [42], [43], [44], [48], [50] I published related subjects. Especially the paper [36]

Kenro Furutani, *Fredholm-Lagrangian-Grassmannian and the Maslov index*, Journal of Geometry and Physics、Vol. 51, No. 3(2004), pp. 269–331.
is a review article including almost all the results obtained in the papers cited above.

(2) The purpose is to construct an operator similar to Bargmann transformation. Following the method to construct such operator in the Euclidean and sphere cases, I constructed a geodesic flow invariant Kähler structure on the punctured cotangent bundles of projective spaces (complex, quaternion projective spaces and Cayley projective plane) and by explicitly expressing an isometry group invariant and nonwhere vanishing global holomorphic section Ω of their canonical line bundles (Calabi-Yau structure). Together I determined its pairings $\Omega \wedge \overline{\Omega}$ in terms of the Liouville volume form explicitly, and constructed an operator similar to the classical Bargmann transformation. Although the original Bargmann transformation is unitary, for these cases the operators constructed were not unitary. Sphere

cases was studied by J. H. Ranwsley. These and related results were published in [2], [4], [29], [34], [39], [41], [46] and [47].

(3) Here I aimed the study to construct heat kernels for sub-Laplacians on nilmanifolds in concrete forms as a guideline of the study for this subjects. Results I had are followings:

(i) I determined the zeta-regularized determinant for Laplacians and sub-Laplacians on Heisenberg manifolds, standard sphere and some others. Also I gave a simple proof for the zeta-regularized determinant of a product type manifold which includes the case of $S^1 \times M$ in a form of infinite product which can be seen as a generalization of Kronecker's second limit formula ([6], [7], [25], [26], [27], [30], [38]).

(ii) In the paper [27], we corrected many of known facts with respect to concrete sub-Laplacians together in 100 pages +. This is the first paper starting the joint research with W. Bauer(Hannover Univ.) and C. Iwasaki(Hyogo Univ.) in this subject.

Also together with three friends (O. Calin, D-C. Chang and C. Iwasaki) we published a book on the subject of various methods for the construction of the heat kernel mainly for sub-Laplacian from Birkhäuser(now it is a part of Springer verlag).

(iii) Study on higher step Grushin operators: We constructed an action function which is one of the main constituent function of the heat kernel of the Grushin operator of two independent variables([24]). Also we constructed Green functions of higher step Grushin operators of general number of variables ([20], [21]). Here I mean that the Grushin operator is a second order sub-elliptic operator which is defined as a descended operator of an invariant sub-Laplacian on nilpotent Lie groups to a quotient space by a subgroup.

(iv) We proved that there exists a co-dimension 3 sub-Riemannian structure on Gromoll-Meyer exotic 7 sphere ([15]). In [14] we also studied several relation of the Popp's measures intrinsically defined by sub-Riemannian structures in the framework of a submersion, especially in the framework of a principal bundle.

(v) Study of nilpotent Lie groups(algebras) as examples carrying a natural sub-Riemannian structure:

We studied 2-step nilpotent Lie algebras (and groups) attached to "admissible" mudules of Clifford algebras (we call these pseudo H-type Lie algebra and group). The results obtained are their complete classification, existence of lattices and the determination of automorphism groups ([5], [8], [10], [11], [16], [18], [22]).

(vi) In addition to the above results, we studied Radon transformation from the point of the relation between S. S. Chern's notion of "incidence" relation and Fourier integral operator theory ([3]). We showed that a Radon transformation from Gromoll-Meyer exotic sphere to the standard sphere has infinite dimensional kernel and cockerel (unfortunately this is only an initial results in this direction)..