

# Research Results

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I am working on  **$(\mathfrak{g}, K)$ -modules over commutative rings and related geometry**. The concept of  $(\mathfrak{g}, K)$ -modules is an algebraic model of representations of real reductive Lie groups.

My first research is on **cohomologically induced modules over commutative rings**. In the theory of  $(\mathfrak{g}, K)$ -modules over the field  $\mathbb{C}$  of complex numbers, the cohomological induction for a morphism  $(\mathfrak{q}, M) \rightarrow (\mathfrak{g}, K)$  of Harish-Chandra pairs is the right derived functor of the right adjoint functor  $I_{\mathfrak{q}, M}^{\mathfrak{g}, K}$  of the restriction functor from the category of  $(\mathfrak{g}, K)$ -modules to that of  $(\mathfrak{q}, M)$ -modules. This derived functor supplies us important representations like principal series representations and cohomologically induced modules.

**I introduced a general definition of  $(\mathfrak{g}, K)$ -modules over commutative rings and established their basic theory**, based on categorical insights to the theory over  $\mathbb{C}$ . In particular, I constructed the right adjoint functor  $I_{\mathfrak{q}, M}^{\mathfrak{g}, K}$  and its right derived functor (Paper 9, Preprint 3). In Papers 8 and 10, **I worked on the flat base change properties of the right derived functor**. In Paper 10, I gave affirmative results under certain conditions. In Paper 8, I gave a counterexample when the ground ring is the ring  $\mathbb{Z}$  of integers. In this counterexample, I found that a certain integral model of the induction which should give principal series representations provides us an integral model of a discrete series representation. This is interesting because **this cannot happen over  $\mathbb{C}$** .

Incidentally, some models have smaller rings of definition which are not obtained from the cohomological induction over commutative rings. **Such smaller rings are expected to be important in applications to number theory**. Together with Fabian Januszewski, I turn my eyes to **the geometric construction**: The cohomologically induced module is isomorphic to the global section module of the twisted D-module theoretic direct image of the equivariant line bundle on the “corresponding” closed  $K$ -orbit on the partial flag variety of  $\mathfrak{g}$ . We focused on the fact that this construction relies on geometric operations. We thought that **the smaller ring the orbit and the line bundles are defined over, the smaller ring the resulting module is defined over**. We ran this idea. Firstly, **I worked on the descent problems on the rings of definition of partial flag schemes and equivariant line bundles on them** (Papers 7, 2). Then we introduced the notion of **stable parabolic subgroups**, and solved the problem of the ring of definition of the orbit decomposition of their moduli space. In particular, **we solved the descent problem of the rings of definition of the closed orbits corresponding to cohomologically induced modules**. We also established the theory of **twisted D-modules over general base schemes**. As a result, **we obtain  $\mathbb{Z}[1/2]$ -forms of cohomologically induced modules** (Preprint 2). **The I proved that they are free as  $\mathbb{Z}[1/2]$ -modules** (Paper 6). As an application of its proof, **we proved finiteness of their  $(\mathfrak{g}, K)$ -cohomology** (Preprint 2).

In Paper 2, I worked on the descent problem of rings of definition of non-closed orbits. I found the standard  $\mathbb{Z}[1/2]$ -forms of all  $\mathrm{SO}(3, \mathbb{C})$ -orbits on the flag variety of  $\mathrm{SL}_3(\mathbb{C})$ . I proved that **they are imbedded affinely over  $\mathbb{Z}[1/2]$** , and that **these  $\mathbb{Z}[1/2]$ -forms exhibit a set theoretic decomposition of the flag scheme of  $\mathrm{SL}_3$  over  $\mathbb{Z}[1/2]$  (the  $\mathbb{Z}[1/2]$ -form of the  $K$ -orbit decomposition of the flag scheme)**. I gave its generalization for general reductive group schemes in Paper 5.

Towards the problem of fields of definition of infinite-dimensional representations, I established generalities on classification of irreducible representations and fields of definition in Paper 1 as a generalization of the works on rationality of finite-dimensional irreducible representations by Lowy, Borel–Tits, and Tits.

In Preprint 1, I started to work on **basic study of contractions families** from the perspectives of abstract algebraic geometry. Here a contraction family is a certain one-parameter family obtained by replacing a structure constant of Lie groups or Lie algebra. I also introduced the quotient of contraction group schemes (e.g., of contraction families of symmetric pairs), and studied basic structures of the quotient spaces. The quotient attaches **new varieties (manifolds) which connect different symmetric varieties (spaces)**. They are expected to give new insights into studies on symmetric spaces and special functions on them in the future.

In Paper 4, I **classified irreducible representations of quasi-reductive algebraic supergroups, and determined the division superalgebras of their endomorphisms**. I examined them for fundamental examples of real quasi-reductive algebraic supergroups.