Future Research Plans

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Reinterpretation of the homotopy theory of dg categories

Using the proarrow equipment $\mathbb{DB}\mathsf{imod}^{\mathrm{rqr}}$ on the 2-category $\mathsf{DBimod}^{\mathrm{rqr}}$ constructed in my previous work, I aim to reinterpret various results in the homotopy theory of dg categories, including Toën's derived Morita theory, from the perspective of formal category theory. The derived Morita theorem is a dg analogue of the classical Morita theorem in module theory over rings, and it asserts that for dg categories $\mathcal{A}$ and $\mathcal{B}$ there is an isomorphism in $\mathsf{Ho}(\mathsf{dgCat})$

$$\mathbb{R}\mathrm{Hom}_{\mathrm{cc}}\big(D_{\mathrm{dg}}(\mathcal{A}), D_{\mathrm{dg}}(\mathcal{B})\big) \cong D_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathcal{B}),$$

where D_{dg} denotes the derived dg category of dg modules, and the left-hand side is the dg category of coproduct-preserving quasi-functors. Just as the classical Morita theorem can be obtained as a consequence of an adjoint functor theorem, this isomorphism is also expected to arise, as its dg analogue, from a suitable adjoint functor theorem. From this viewpoint, the present project seeks to establish an adjoint functor theorem within the proarrow equipment $\mathbb{DB}\mathsf{imod}^{\mathrm{rqr}}$, thereby providing a theoretical foundation that conceptually explains the validity of the derived Morita theorem.

As a further application of adjoint functor theorems for quasi-functors, I plan to prove a homotopical version of a Gabriel–Popescu-type theorem for dg categories. By comparing this result with Porta's Gabriel–Popescu-type theorem for triangulated categories, I aim to achieve a unified understanding of both results from the viewpoint of formal category theory.

Global properties of the proarrow equipment $\mathbb{DB}\mathsf{imod}^{\mathrm{rqr}}$

The main reason for regarding the 2-category of quasi-functors $\mathsf{DBimod}^{\mathrm{rqr}}$ as a 2-categorical refinement of $\mathsf{Ho}(\mathsf{dgCat})$ is the existence of the following equivalence of categories (here $\pi_0^{\cong}(-)$ denotes the operation of forming a category by taking isomorphism classes of 1-morphisms in a 2-category):

$$\pi_0^{\cong}(\mathsf{DBimod}^{\mathrm{rqr}}) \simeq \mathsf{Ho}(\mathsf{dgCat}).$$

Since $\mathsf{Ho}(\mathsf{dgCat})$ is obtained as a localization of dgCat, it is natural to expect that its higher structure, the 2-category $\mathsf{DBimod}^{\mathrm{rqr}}$, can be regarded as a "localization as a 2-category."

In this project, as a framework that generalizes this expectation, I aim to show that the proarrow equipment $\mathbb{DB}\mathsf{imod}^{\mathrm{rqr}}$ on $\mathsf{DBimod}^{\mathrm{rqr}}$ is a localization of the natural proarrow equipment $\mathbb{DG}\mathsf{Cat}$ on dgCat. The notion of localization for proarrow equipments, as well as its relationship with localization of the underlying 2-categories, has not been discussed explicitly so far. By analyzing the universal property of $\mathbb{DB}\mathsf{imod}^{\mathrm{rqr}}$, I aim to formulate an appropriate definition of localization for proarrow equipments. Furthermore, by clarifying its relationship with the underlying 2-category, I will establish the above expectation.