Research Summary

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Research on a Gabriel–Popescu-type theorem for dg categories

Among abelian categories, Grothendieck categories form a particularly important class. A Grothendieck category is defined by intrinsic categorical properties, but it is also characterized externally as an appropriate subcategory of a module category; this extrinsic characterization is known as the Gabriel–Popescu theorem. I investigated a dg-categorical analogue of this theorem.

Both abelian categories and dg categories can be viewed as enriched categories: the former are enriched over the category of abelian groups Ab, while the latter are enriched over the category of chain complexes Ch. Based on this observation, I adopted the strategy of generalizing the Gabriel–Popescu theorem from the viewpoint of enriched category theory. Following this approach, I proved an analogous result for enriched categories over a Grothendieck monoidal category satisfying suitable conditions.

Research on higher structures in the homotopy theory of dg categories

For dg categories, there is a weak notion of equivalence called quasi-equivalence, and it is natural to regard quasi-equivalent dg categories as representing the same object. In this sense, the theory that studies dg categories up to quasi-equivalence is called the homotopy theory of dg categories. Since the 2000s, the localized category $\mathsf{Ho}(\mathsf{dgCat})$ of dg categories has been studied as a framework for this homotopy-theoretic viewpoint. However, the categorical structure of $\mathsf{Ho}(\mathsf{dgCat})$ alone seems not to fully capture the intrinsically 2-categorical information carried by dg categories.

Unlike objects such as topological spaces, dg categories are themselves categories, and therefore their homotopy theory should naturally possess higher-dimensional structure. From this perspective, I introduced a 2-category $\mathsf{DBimod}^{\mathrm{rqr}}$, which extends $\mathsf{Ho}(\mathsf{dgCat})$, and investigated its 2-categorical properties. In particular, I proved that quasi-equivalences behave as equivalences in the 2-category $\mathsf{DBimod}^{\mathrm{rqr}}$. Equivalences are the natural notion of "sameness" of objects in a 2-category, and this observation suggests that $\mathsf{DBimod}^{\mathrm{rqr}}$ provides a natural 2-categorical refinement of $\mathsf{Ho}(\mathsf{dgCat})$.

Furthermore, I proved that $\mathsf{DBimod}^{\mathrm{rqr}}$ admits a structure called a proarrow equipment. A proarrow equipment is a structure defined on a 2-category and was introduced in formal category theory. Formal category theory aims to axiomatize category theory itself by abstracting the two-dimensional information inherent in categories. Just as homological algebra in module theory is abstracted within the framework of abelian categories, the notion of a proarrow equipment allows one to develop category theory axiomatically in a general 2-categorical setting. Using the proarrow equipment structure on $\mathsf{DBimod}^{\mathrm{rqr}}$, I studied the homotopy theory of dg categories from the perspective of formal category theory. In particular, I introduced a notion of homotopy limits for dg categories and showed that pretriangulatedness of dg categories can be understood as a form of completeness.