Abstract of future research

For the details of some notations, refer to abstract of present research.

1. **Weighted polynomial approximation of two variables**: We study approximation problems on $\mathbb{R}^2$. Currently, we are studying on the problem of approximating a function f that satisfies $fW \in L^p(\mathbb{R}^2)$ when the weights are separated into products of one variable, that is $W(x,y) = w_1(x)w_2(y)$. The de la Vallée Poussin mean of two variables:

   $$V_n(f)(x_1,x_2) := \frac{1}{n^2} \sum_{m_2=n+1}^{2n} \sum_{m_1=n+1}^{2n} s_{m_1,m_2}(f;x_1,x_2)$$

   (where, $s_{m_1,m_2}(f;x_1,x_2)$ is Fourier sum) H.N. Mhaskar presented it as a concrete example of a two-variable polynomial, but there has been no research on its use in approximating functions. We have been shown the boundedness of L^p, the convergence condition by the modulus of contiuity, estimates of the degree of approximation, and the estimate of the convergence property when f is a continuous and bounded variation function. Also, as another topic, we study iterated Lagrange interpolation polynomials:

   $$L_n(W;(x_1,x_2)) := L_{n,2}(w_2^2, L_{n,1}(w_1^2, f_{x_2}; x_1); x_2)$$

   (where, Ln, i is the Lagrange interpolation polynomial of degree n for a weight w_i). We investigate convergence conditions, etc. Furthermore, expanding the scope of our investigation to more general two variable weights is also a future challenge.

2. **Refinement of previous results**
   (a) **Lagrange interpolation polynomials for Laguerre-type weights**: We are studying weighted convergence condition of the Lagrange interpolation polynomial on $\mathbb{R}^+$. For a continuous function f on $\mathbb{R}^+$, we need to find the condition such that

   $$\lim_{n\to\infty} \|(L^*_{n,\rho^*}(f) - f)w_\rho\|_{L^p(\mathbb{R}^+)} = 0 \tag{A}$$

   for $1 < p < \infty$. We already showed (A) in the case of $p = 2$ and $1 < p < 2$. We have also shown the similarities for the cases $2 < p < \infty$ for the weight $\Phi^{*(1/2-1/p)^+}(x)w_\rho(x)$. But these conditions are very complicated and not continuous for p. Moreover, we don't know error estimates for fixed n. These problem are future tasks.

   (b) **de la Vallée Poussin mean**: At this stage, we show L^p boundedness of derivatives of the de la Vallée Poussin mean. One of these is the following: Suppose that w belongs to $\mathcal{F}_\lambda(C^4+)$ which is a smooth subclass of $\mathcal{F}(C^2+)$. If $T^{(2j+1)/4}fw \in L^p(\mathbb{R})$, then for $2 \le p \le \infty$,

   $$\|v_n(f)^{(j)}w\|_{L^p(\mathbb{R})} \le C\left(\frac{n}{a_n}\right)^j \|T^{(2j+1)/4}fw\|_{L^p(\mathbb{R})} \tag{B}$$

   for all $1 \le j \le k$ and $n \in \mathbb{N}$. But, we could know (B) holds true or not for $1 \le p \le 2$. We use duality of L^1-norm and Riesz-Thorin interpolation theorem to prove L^p boundedness of the de la Vallée Poussin mean. But, unfortunately, we cannot use duality of L^1-norm because T remains in the proof and it is unbounded. We would like to find the way to break through obstructions by unboundedness of T. The solution to the problem in the Erdős-type case is to construct a method for evaluating unbounded T for each problem.

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