Abstract of present research

Our study is polynomial approximation theory as an application of potential theory. On $\mathbb{R}$, every polynomial $P(t)$ blows up as $|t| \to \infty$, we must multiply a weight function $w(t)$. Then, for $1 \le p \le \infty$ and $fw \in L^p(\mathbb{R})$, is there exist a sequence of polynomials $\{P_n\}$ such that

$$\lim_{n\to\infty} \|(f-P_n)w\|_{L^p(\mathbb{R})} = 0 \tag{A}$$

holds? We assume that an exponential weight w belongs to relevant class $\mathcal{F}(C^2+)$. Let w be $w(t) = \exp(-Q(t))$. We consider a function $T(t) := tQ'(t)/Q(t)$, $(t \neq 0)$. If T is bounded, then w is called a Freud-type weight, and otherwise, w is called an Erdős-type weight. In this study, we consider Erdős-type weights. Our present results are following:

1. **Convergence of the de la Vallée Poussin mean**: The de la Vallée Poussin mean $v_n(f)$ of f is defined by $v_n(f)(t) := \frac{1}{n}\sum_{j=n+1}^{2n} s_j(f)(t)$, where $s_m(f)(x)$ is the partial sum of Fourier series of f for orthogonal polynomials with respect to w. The degree of approximation for f defined by $E_{p,n}(w;f) := \inf_{P\in\mathcal{P}_n} \|(f-P)w\|_{L^p(\mathbb{R})}$. Here, $\mathcal{P}_n$ is the set of all polynomials of degree at most n. We assume that $w \in \mathcal{F}(C^2+)$ and suppose that $T(a_n) \le c\,(n/a_n)^{2/3}$ for some $c > 0$. Here, the notation a_n is called MRS number. Then there exists a constant $C \ge 1$ such that for every $n \in \mathbb{N}$ and when $fw \in L^p(\mathbb{R})$,

   $$\|(f - v_n(f))w\|_{L^p(\mathbb{R})} \le CT^{1/4}(a_n)E_{p,n}(w;f). \tag{B}$$

   We show the conditions such that the right side of (B) converge to 0 as $n \to \infty$. Moreover, if f is more smoother function, $v_n(f)$ is not only a good approximation polynomial for f, but also its derivatives give an approximation for f'.
2. **Uniform convergence of the Fourier partial sum**: By the way, we show the condition uniformly convergence of $s_n(f)$ for a weight in a more smooth subclass $\mathcal{F}_\lambda(C^3+)$: Let $w \in \mathcal{F}_\lambda(C^3+)$ with $0 < \lambda < 3/2$. Suppose that f is continuous and has a bounded variation on any compact interval of $\mathbb{R}$. If f satisfies $\int_{\mathbb{R}} w(x)|df(x)| < \infty$, then

   $$\lim_{n\to\infty} \left\|(f - s_n(f))\frac{w}{T^{1/4}}\right\|_{L^\infty(\mathbb{R})} = 0.$$
3. **Lagrange interpolation polynomials for Laguerre-type weights**: We assume that an exponential weight $w(x) = \exp(-R(x))$ on $\mathbb{R}^+ := [0,\infty)$ belongs to relevant class $\mathcal{L}_\lambda(C^3+)$ with $0 < \lambda < 3/2$. Let w_ρ be $w_\rho(x) := x^\rho w(x)$ for $\rho > 0$. For $f \in C(\mathbb{R}^+)$, we write the Lagrange interpolation polynomial $L^*_{n,\rho}(f)(x)$ with nodes $\{x_{j,n,\rho}\}_{j=1}^n$, where $\{x_{j,n,\rho}\}_{j=1}^n$ are the zeros of n-th orthogonal polynomial with respect to w_ρ. We show the condition such that Lagrange interpolation polynomial converges to f with w_ρ in L^p-norm for $1 < p \le 2$: For $p = 2$, let $\beta > 1/2$. If $(1+x)^{\beta/2+1/4}w_\rho(x)f(x) \to 0$ as $x \to \infty$, then we have

   $$\lim_{n\to\infty} \|(L^*_{n,\rho^*}(f) - f)w_\rho\|_{L^2(\mathbb{R}^+)} = 0. \tag{C}$$

   And for $1 < p < 2$, let $\beta > 1/p$ and $(1+x)^{\beta/2+1/4}T^{*1/2}(x)\Phi^{*-1/4}(x)w_{\rho^*}(x)f(x) \to 0 (x \to \infty)$(where, $\Phi^*(x) := (T^*(x)(1+R(x))^{2/3})^{-1}$). Then we have (C) in the case of $1 < p < 2$. Here, $L^*_{n,\rho^*}(f)$ is the Lagrange interpolation polynomial with respect to the weight $w_{\rho*} := w_{\rho+1/2p-1/4}$. Additionally, for $2 < p$, we showed the result (C) corresponds to a weight $\Phi^{*(1/2-1/p)^+}(x)w_\rho(x)$.

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