Research plans (Hitoshi Konno)

We will work on generalizing the results obtained so far and solving conjectures, and also aim to combine algebraic analysis methods based on representations of elliptic quantum groups with the geometric representation theory proposed by Okounkov et al.

(1) **Representations of elliptic quantum groups and geometry of quiver varieties**

We aim to establish the equivalence of the elliptic quantum groups $U_{q,p}(\widehat{\mathfrak{g}})$ and $U_{t_1,t_2,p}(\mathfrak{g}_{tor})$ with the "quantum group structures on $E_T(X)$" geometrically formulated by Okounkov et al. We will extend the correspondence between the integration kernels of the vertex operators of elliptic quantum groups and elliptic Stabs to any representations, and also clarify the direct relationship between the Gauss decomposition of the L-operators and Stabs. On the other hand, we will also work on directly deriving representations of elliptic quantum groups using elliptic Stab, and clarify the representation theoretical structure of the equivariant elliptic cohomology. We will also extend the construction of K-theoretic vertex functions in terms of the vertex operators to the general case of quivers, and provide a representation theoretical derivation of the p-KZ equation and quantum difference equations for the vertex functions. Furthermore, it is expected that the three-dimensional Mirror symmetry (or more generally, symplectic duality) between quiver varieties predicted in supersymmetric gauge theories can be understood as a duality between vertex functions. We will investigate this conjecture based on representation theory of elliptic quantum groups.

(2) **Reformulation of the deformed W-algebras based on elliptic quantum groups**

The elliptic quantum groups $U_{q,p}(\widehat{\mathfrak{g}})$ and $U_{t_1,t_2,p}(\mathfrak{g}_{tor})$ have the aspect of being algebras formed by screening currents of the deformed W-algebras. By using this, we can give an alternative formulation of the deformed W-algebra. In the conventional formulation, the deformed W-algebra itself does not have a coalgebra structure, making it impossible to define and systematically derive vertex operators, which play an important role in representation theory and mathematical physics. This problem can be solved by using the coalgebra structure of the elliptic quantum groups. We will establish this by investigating derivation of vertex operators systematically and clarify their exchange relations for various deformed W-algebras, including those of affine quiver type. We will also establish a derivation of difference equations for K-theoretic vertex functions and their elliptic extensions as correlation functions (expectation values and traces) and investigate their solutions.

(3) **Representation theoretical formulation of quantum equivariant K-theory and quantum equivariant elliptic cohomology for quiver varieties**

As an algebraic complement to the geometric formulation of quantum equivariant K-theory $QK_T(X)$ for quiver varieties X by Okoukov et al., we aim to provide a representation theoretical formulation of the ring structure of $QK_T(X)$, in particular the quantum product. To do this, we use the three-way relationship between representations of quantum groups, quantum integrable systems and quantum equivariant K-theory. The class of equivariant K-theory corresponds to the conserved quantities of quantum integrable systems, which in turn correspond to the Gelfand-Tsetlin (GT) subalgebras of the corresponding quantum groups. In the localized situation, any K-theory class can be diagonalized on the fixed point class, which can be recast as the diagonalization problem for the quantum integrable systems, which can be solved by the Bethe ansatz method derived from the quantum group structure. We apply this three-way relationship to the elliptic case and investigate the structure of quantum products of the elliptic cohomology classes as well as the K-theory classes from the diagonalization of the elliptic GT subalgebras. We carry out this for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, where X corresponds to the cotangent bundle to the Grassmannian and the (partial) flag variety, respectively, and verify the results by comparing them with the explicit formulas for the quantum products obtained in the previous studies by Smirnov and Mihalcea et al. Furthermore, we extend this to the case of the elliptic quantum toroidal algebras $U_{t_1,t_2,p}(\mathfrak{g}_{tor})$, which corresponds to the affine quiver varieties.