Summary of Research Results (Hitoshi Konno)

The following results were obtained in the research on formulation and representation of elliptic quantum groups, elucidation of their correspondence with geometry associated with the K-theory of quiver varieties and their elliptic cohomology versions, and applications to the algebraic analysis of solvable lattice models and supersymmetric gauge theories.

(1) Formulation of elliptic quantum groups

The elliptic quantum group $U_{q,p}(\widehat{\mathfrak{g}})$ associated with an affine Lie algebra $\widehat{\mathfrak{g}}$ is formulated as a p-deformation of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ in the Drinfeld realization. The algebra $U_{q,p}(\widehat{\mathfrak{g}})$ can also be regarded as a q-deformation of the W-algebra associated with the GKO coset of $\widehat{\mathfrak{g}}$. Hence $U_{q,p}(\widehat{\mathfrak{g}})$ is a unique elliptic deformation of $U_q(\widehat{\mathfrak{g}})$. Furthermore, we have shown that $U_{q,p}(\widehat{\mathfrak{g}})$ possesses the coalgebra structure given by the Hopf algebroid structure introduced by Etingof-Varchenko with an extension to the elliptic case and including the central extension. Furthermore, we have extended this formulation to the elliptic quantum toroidal algebra $U_{t_1,t_2,p}(\mathfrak{g}_{tor})$ associated with the toroidal algebra $\mathfrak{g}_{tor}$ and conjectured that this also provides an alternative formulation of the affine quiver-type deformed W-algebra introduced by Kimura-Pestun. On the other hand, corresponding to the three equivalent formulations of $U_q(\widehat{\mathfrak{g}})$, we have also given other two equivalent formulations for elliptic quantum groups. These formulations are quasi-Hopf deformations of $U_q(\widehat{\mathfrak{g}})$ realized in terms of the Chevalley generators (the Drinfeld-Jimbo realization), which yield the vertex-type elliptic quantum group $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ and the face-type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{g}})$, and a formulation of the elliptic quantum group $E_{q,p}(\widehat{\mathfrak{sl}}_N)$ whose generators are L-operator operators satisfying the $RLL = LLR^*$ type relation with R being an elliptic solution of the face-type Yang-Baxter equation.

(2) Representations of elliptic quantum groups and their relation to equivariant K-theory and equivariant elliptic cohomology of quiver varieties

Okounkov and his collaborators have shown that there exist "good classes" called stable envelopes (Stab) on the equivariant cohomology $H_T^*(X)$, the equivariant K-theory $K_T(X)$, and even the equivariant elliptic cohomology $E_T(X)$ of a quiver variety X, and that quantum group structures can be formulated geometrically using Stab. In particular, in the elliptic case, the "quantum group structure on $E_T(X)$" is expected to be the same as our $U_{q,p}(\widehat{\mathfrak{g}})$ or $U_{t_1,t_2,p}(\mathfrak{g}_{tor})$ depending on X. To establish this conjecture, we have shown with several examples that the integration kernels appearing in the vertex operators of the elliptic quantum groups can be identified with the elliptic Stabs for the equivariant elliptic cohomology $E_T(X)$ of the corresponding quiver variety X. The key to this is the consistency between the structure of the shuffle product of the integral kernels obtained by algebraic manipulations in the composition of vertex operators and the structure of the geometric shuffle product of the elliptic Stab. This consistency also leads us to predict that the tensor products of vector representations of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ or the tensor products of the level (0,-1) representations of $U_{t_1,t_2,p}(\mathfrak{gl}_{1,tor})$ or $U_{t_1,t_2,p}(\mathfrak{gl}_{N,tor})$ are equivalent to the corresponding geometric representations of the same algebra on the corresponding equivariant elliptic cohomology. In fact, in the case of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, we found a bijective relation between the Gelfand-Tsetlin (GT) basis in the former and the fixed point classes in $E_T(X)$, and showed that the finite-dimensional representations on the GT basis can be lifted to geometric representations on the fixed point classes. On the other hand, Okounkov et al. have introduced K-theoretic vertex functions as generating functions of counting the quasi-maps from $\mathbb{P}^1$ to X, and formulated the quantum equivariant K-theory $QK_T(X)$. In the cases of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, $U_{t_1,t_2,p}(\mathfrak{gl}_{1,tor})$ and $U_{t_1,t_2,p}(\mathfrak{gl}_{N,tor})$, we found that the corresponding K-theoretic vertex functions can be derived representation theoretically as the expectation values of the compositions of the vertex operators of elliptic quantum groups. This indicates a new relationship between $QK_T(X)$ and elliptic quantum groups.

(3) **Algebraic analysis of solvable lattice models and supersymmetric gauge theories**

Based on the representation of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{g}})$, we extended the algebraic analysis formulation of solvable lattice models developed by Kashiwara-Jimbo-Miwa to the case of elliptic models, and gave algebraic analysis formulations for the ABF model, 8-vertex model, their fusion model, and dilute A_L model, and also derived their correlation functions. On the other hand, as mentioned in (1), the elliptic quantum toroidal algebra $U_{t_1,t_2,p}(\mathfrak{gl}_{1,tor})$ gives a formulation of the Jordan quiver variety $\mathcal{M}(n,r)$ ($\cong$ instanton moduli space) type deformation W-algebra. We showed that its generating function $T(u)$ can be obtained as a composition of the vertex operators of $U_{t_1,t_2,p}(\mathfrak{gl}_{1,tor})$, and that the Hirzebruch χ_p-genus and elliptic genus of $\mathcal{M}(n,r)$ can be obtained as the expectation value and trace of the composition of $T(u)$, respectively. These genus give the instanton partition functions in the five and six dimensional extension of the four-dimensional $\mathrm{N} = 2^*$ supersymmetric gauge theory, respectively. This result indicates that algebraic analysis methods are also useful for studying supersymmetric gauge theories.