(2) Abstract of results Makiko Mase

We have been motivated by mirror symmetry for families of projective $K3$ surfaces, and automorphism action on algebraic $K3$ surfaces.

1. Dualities of families of $K3$ surfaces associated to Berglund-Hübsch mirror
Let $(f = 0)$ and $(f' = 0)$ be isolated hypersurface singularites in $\mathbb{C}^3$ at the origin, defining transpose-dual bimodal singularities in the sense that the matrices of exponents of f and f' are transpose to each other. Such a relation is called Berglund-Hübsch mirror. It is known that there exists a projectivization F (resp. F') of f (resp. f'), which is quasi-homogeneous of degree d (resp. d') with $K3$ weight system $\boldsymbol{a} = (a_i)_{i=0,\dots,3}$ (resp. $\boldsymbol{b} = (b_i)_{i=0,\dots,3}$), that is, the associated 3-dimensional polytope $\Delta_{\boldsymbol{a}}$ (resp. $\Delta_{\boldsymbol{b}}$) is reflexive.

In general, a 3-dimensional reflexive polytope Δ determines a toric projective Fano 3-fold $\mathbb{P}_\Delta$ and Gorenstein $K3$ (hyper)surfaces in $\mathbb{P}_\Delta$ that are parametrized by the complete anticanonical linear system of $\mathbb{P}_\Delta$ (due to Batyrev). Therefore, one obtains a family $\mathcal{F}_\Delta$ of $K3$ surfaces, and the Picard lattice Pic_Δ of $\mathcal{F}_\Delta$ as the Picard lattice of the minimal model of any generic anticanonical memeber of $\mathbb{P}_\Delta$. Also, denote by $(\mathrm{Pic}_\Delta)_{tor}$ the sublattice of Pic_Δ generated by the restrictions of the toric divisors on $\mathbb{P}_\Delta$ to the anticanonical divisor.

In particular, if $\Delta = \Delta_{\boldsymbol{a}}$ (resp. $\Delta_{\boldsymbol{b}}$) with a $K3$ weight system $\boldsymbol{a}$ (resp. $\boldsymbol{b}$), we get a family $\mathcal{F}_{\boldsymbol{a}}$ (resp. $\mathcal{F}_{\boldsymbol{b}}$) of $K3$ surfaces and the Picard lattice denoted by $\mathrm{Pic}_{\boldsymbol{a}}$ (resp. $\mathrm{Pic}_{\boldsymbol{b}}$). Denote by Δ_P the Newton polytope of a polynomial P, and by Δ^* the polar dual of Δ. In a series of studies by the applicant and her corraborators, we have obtained :

Theorem. *Under this setting,* (1) *Families $\mathcal{F}_a$ and $\mathcal{F}_b$ are polytope-dual, in the sense that there exists a pair (Δ, Δ') of reflexive polytopes such that*

$$\Delta_F \subseteq \Delta \subseteq \Delta_a, \quad \Delta_{F'} \subseteq \Delta' \subseteq \Delta_b, \quad \text{and} \quad \Delta^* \simeq \Delta'.$$

(2) *For any pair (Δ, Δ') as in part (1), the isometry holds:*

$$(\mathrm{Pic}_\Delta)^\perp_{\Lambda_{K3}} \simeq U \oplus (\mathrm{Pic}_{\Delta'})_{tor}. \quad \square$$

2. Symplectic actions by a finite group G s.t. $Q \simeq C_n$ with $n = 2, 3$
Let X be a $K3$ surface admitting a symplectic automorphism action by a finite group G, of which the abelization group $G/[G, G]$ is denoted by Q. The quotient variety X/G consists of at most rational singularities, so that there exists a minimal crepant resolution $Y \to X/G$ of singularities, where Y is a smooth variety. Moreover, it is known that Y is again a $K3$ surface, and the Picard lattice $\mathrm{Pic}\,(Y)$ of Y contains the classes of (-2)-curves that are irreducible components of the exceptional divisor of the resolution. In fact, the configuration of these classes of (-2)-curves is of the direct sum of Dynkin diagrams of types ADE. Thus, we can regard the union of classes of (-2)-curves as a lattice, which we call L_G. Therefore, the Picard lattice of Y has a sublattice L_G. It is well-known that the Picard lattice of Y is a primitive sublattice of the $K3$ lattice $\Lambda_{K3} := U^{\oplus 3} \oplus E_8^{\oplus 2}$. However, L_G is not necessarily primitive in Λ_{K3}, and there is a primitive closure $\widetilde{L_G} := L_G \otimes \mathbb{Q} \cap \mathrm{Pic}\,(Y)$.

Suppose the group Q is isomorphic to the cyclic group C_n of order $n = 2, 3$. Then, we have so far shown that the primitive closure $\widetilde{L_G}$ of L_G in the $K3$ lattice Λ_{K3} is uniquely determined.