4 Future research plan

I would like to consider the following two types of hierarchical structes (and their mixture) in algebraic geometry:

1. Assuming the base filed k to be perfect, consider the following truly naive hierarchical structure which everyone would come up (here $\overset{\text{bir}}{\sim}$ indicates the birational equivalence):

$$\begin{cases} X \text{ is } \textit{(-i)-rational} \text{ or } \textit{(n-i)-ruled} \quad \text{for } 0 \leq i \leq n-1, \\ \text{if } \exists \text{ an } i\text{-dimensional variety } Z^i \text{ s.t. } \mathbb{P}^{n-i} \times Z^i \overset{\text{bir}}{\sim} X. \end{cases} \tag{2}$$

   And upgrade the traditional hierarchical structure interpolating rationality and rational connectedness, investigated by many mathematicians ever since Lüroth, to the following hierarchical structure among analogous hierarchical structures :

$$\begin{aligned} (-i)\text{-rational} &\implies \text{stable } (-i)\text{-rational} \implies \text{ratract } (-i)\text{-rational} \\ &\implies \text{separably } (-i)\text{-unirational} \implies \text{separably } (-i)\text{-rationally connected} \\ &\implies (-i)\text{-rationally connected} \end{aligned}$$

   Then our principal goal here is this investigation. Although this point of view have not been taken seriously up to now, because of its extreme naiveness, natual problems emerge on and on.

   In fact, I have already obtained the following result in this regard:

   > Norihiko Minami, Generalized Lüroth problems, hierarchized I: SBNR - stably birationalized unramified sheaves and lower retract rationality, **arXiv:2210.12225**

   Although this is an application to classical algebraic geometry, the concept of "unramified Zariski sheaf,"introduced by Morel in his prominent work: $\mathbb{A}^1$-Algebraic Topology over a Field (SpringerLNM 2012).

2. Let us call two equi-dimensional complex projective varietes X, Y codimension $> c$ birational equivalent (or isomorphism in codimension c), if there exist closed subsets $E_X \subset X$, $E_Y \subset Y$ enjoying the following two properties:

   - $X \setminus E_X \xrightarrow[f]{\sim} Y \setminus E_Y$.
   - $\operatorname{codim}_X E_X > c, \quad \operatorname{codim}_Y E_Y > c$.

   Then, this is nothing but the birational equivalence when $c = 0$, nothing but the biregular equivalence when $c = \dim X = \dim Y$, and the hierarchical structure obtained by varying c is a very natural one interpolating the birational algebraic geometry and the biregular algebraic geometry.

   Concerning invariants associated with the two extreme setting of algebraic geometry, namely birational invariants and biregular invariants, substantial research have been done because of their importance. What I would like to do is to investigate codimension $> c$ birational invariants interpolating them.

   Recently, I have started to obtain some very satisfactory results in this regard, and I am now writing it in

   Norihiko Minami, *Higher codimensional birational invariants* (tentative)

   and I am going to talk about some part of it in MSJ annual meeting on March 20th, 2025 in the general session of Topology, under the title:

   > Norihiko Minami, Algebrro-geometric invariants defined purely in the real of topology

   By the way, many codimension $> c$ birational invariants obtained here turn out to be the kernal or the cokernel of the cycle maps, used to formulate the Hodge conjecture or the Tate conjecture, and their relatives.

More recently, I have started to investigate categorical applications to automated theorem proving.