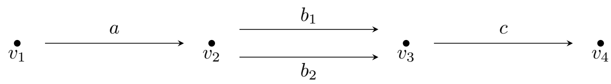
これまでの研究結果のまとめ (英訳) Fumika Mizoguchi

In geometry, one of the fundamental problems is to study whether a given Lie group admits special left-invariant geometric structures. In particular, an important problem is to study the relationship between the existence of geometric structures on Lie algebras and the algebraic structures of Lie algebras. In this regard, an important theorem has recently been proven by Böhm, C. and Lafuente, R. A.. If a homogeneous manifold M admits an expanding Ricci soliton g, then (M, g) is isometric to a solvable Lie group with a left-invariant metric ([BL]). A Ricci soliton metric is a special solution to the Ricci flow equation that is preserved under diffeomorphisms and scaling; it generalizes Einstein metrics. The statement of this theorem is known as the generalized Alekseevskii conjecture and suggests that Ricci soliton homogeneous manifolds are reducible to solvable Lie groups. Therefore, it is necessary to study which solvable Lie groups admit left-invariant Ricci solitons. Moreover, nilpotent Lie groups are considered particularly important because they provide many examples of Ricci solitons that are not Einstein metrics. For Ricci soliton nilpotent Lie groups, the two-step case has been studied actively. On the other hand, there are many unexplained problems about higher step nilpotent Lie groups, and not many examples are known.

Thus, we constructed nilpotent Lie algebras from quivers. A quiver is a directed graph where loops and multiple arrows between two vertices are allowed. A path in a quiver is a sequence of arrows that are connected in the correct direction, and the number of arrows in the path is called the length of the path. For a quiver Q, a set $\mathfrak{n}_Q$ is defined as a vector space having all paths in Q as basis with the product that is given by the concatenation of paths. It is called the path algebra. We defined the Lie algebra $\mathfrak{n}_Q$ by giving the Lie bracket as the commutator product of this product in the path algebra. Here, a path whose source and target coincide is called a cycle. The Lie algebra obtained by a quiver with cycles is infinite-dimensional, as there are infinitely many paths in such a quiver. For a finite quiver without cycles, there exists the maximum of length of paths which is called the length of a quiver. We consider only finite quivers without cycles. If Q is a quiver of length m, then the Lie algebra $\mathfrak{n}_Q$ is an m-step nilpotent Lie algebra. Therefore this method can construct nilpotent Lie algebras of arbitrarily high steps, and they are relatively easy to handle since they are obtained by quivers, which are combinatorial objects. Consider the following quiver Q:



Then the Lie algebra $\mathfrak{n}_Q$ is given by $\mathfrak{n}_Q = \mathrm{span}\{a, b_1, b_2, c, ab_1, ab_2, b_1c, b_2c, ab_1c, ab_2c\}$. The Lie brackets are $[a, b_1] = ab_1$, $[a, b_2] = ab_1$, $[b_1, c] = b_1c$, $[b_2, c] = b_2c$, $[a, b_1c] = [ab_1, c] = ab_1c$ and $[a, b_2c] = [ab_2, c] = ab_2c$. The above bracket is skew-symmetric and the other bracket products are zero. We proved the following theorem.

Theorem (M.-Tamaru, [MT])**.** Let Q be a finite quiver without cycles and $\mathfrak{n}_Q$ be the nilpotent Lie algebra obtained by Q. Then, the simply-connected Lie group corresponding to $\mathfrak{n}_Q$ always admits a left-invariant Ricci soliton.

The method of constructing nilpotent Lie algebras from quivers is valuable for studying geometric structures, and thus it is expected to have applications to other geometric structures as well. Additionally, this method raises new problems about how the existence of geometric structures is related to the combinatorial properties of quivers.

References

[BL] Böhm, C. and Lafuente, R. A., "Non-compact Einstein manifolds with symmetry", J. Amer. Math. Soc. **36** (2023), no. 3, 591–651.

[MT] Mizoguchi, F. and Tamaru, H., "Nilpotent Lie algebras obtained by quivers and Ricci solitons", Adv. Math. **480** (2025), 110464 [22 pp].