Research Plan

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In complex geometry, analytic, differential-geometric, and algebraic methods interact to describe geometric objects such as canonical metrics and notions of positivity. The aim of this project is to develop a unified framework in geometric analysis, using the solvability of nonlinear partial differential equations (PDEs) as a guiding principle, to connect these three theories. For the complex Monge–Ampère (MA) equation, solvability, curvature positivity, and algebraic-geometric positivity are known to coincide. The goal of this research is to extend this correspondence to more general geometric PDEs and to canonical metric problems for holomorphic vector bundles. As applications, I aim to reinterpret central algebraic-geometric notions such as partial ampleness of line bundles, ampleness of vector bundles, and Bridgeland stability from a differential-geometric perspective via PDEs.

The research plan consists of the following two main directions.

1. Solvability and positivity for Hessian equations:

For the MA equation, Yau's solution of the Calabi conjecture established a correspondence between solvability and curvature positivity. Moreover, through the Kodaira embedding theorem and the Nakai–Moishezon criterion, this curvature positivity is characterized by algebraic-geometric positivity, such as ampleness and positivity of intersection numbers. In complex geometry, curvature is described by Hessians, and many geometric equations can be formulated in a unified way as nonlinear PDEs of Hessian type. However, in this general setting, it remains largely open how far the correspondence among solvability, differential-geometric positivity, and algebraic-geometric positivity extends.

In this project, I aim to systematically establish this threefold correspondence for general classes of Hessian equations. While the relationship between solvability and differential-geometric positivity has been substantially clarified by works of Székelyhidi and Guo–Song, the emphasis here is on developing a weak solution theory that incorporates degenerate positivity conditions. Building on the techniques developed in my previous work, I will establish existence and uniqueness of weak solutions in degenerate settings and analyze their limiting behavior. This approach allows one to investigate regularity properties of weak solutions and to relate them to underlying algebraic-geometric features.

As an application, I will study partial ampleness arising in Andreotti–Grauert theory via solvability of appropriate Hessian-type equations. This provides a new analytic viewpoint on positivity notions in algebraic geometry, grounded in nonlinear PDE methods.

2. Canonical metrics and positivity for vector bundles:

The existence of canonical metrics on holomorphic vector bundles is another fundamental problem that should be understood through nonlinear geometric PDEs. Recently, Dervan, McCarthy, and Sektnan introduced the notion of *Z-critical metrics* as canonical metrics on holomorphic vector bundles, whose existence is formulated as the solvability of a nonlinear PDE, called the Z-critical equation. Z-critical metrics were introduced as differential-geometric counterparts of Bridgeland stability in algebraic geometry.

In this project, I aim to characterize the solvability of the Z-critical equation in terms of curvature positivity and numerical conditions, such as slope-type inequalities. Furthermore, I will investigate under which conditions this characterization corresponds to Bridgeland stability. This study seeks to deepen the fundamental principle in complex geometry that relates the existence of canonical metrics to algebraic-geometric stability, now in the context of vector bundles.

In addition, I will study the nonlinear system proposed by Demailly to capture positivity of vector bundles analytically. This system was introduced to characterize Griffiths positivity through solvability of PDEs. Starting from my results in complex dimension one, I aim to extend the correspondence among solvability, Griffiths positivity, and ampleness to higher dimensions. As a consequence, this project aims toward a resolution of the Griffiths conjecture, which asserts the equivalence between Griffiths positivity and ampleness for holomorphic vector bundles.