Research Summary

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The applicant's research focuses on the existence of canonical metrics in Kähler geometry and on elucidating the correspondence between differential-geometric positivity and algebraic-geometric positivity. Yau's solution of the Calabi conjecture established the existence of Ricci-flat Kähler–Einstein metrics, leading to profound applications in algebraic geometry and the emergence of Calabi–Yau geometry. At its core, this result demonstrates a correspondence between the solvability of the complex Monge–Ampère (MA) equation and the positivity of curvature. On the other hand, the Kodaira embedding theorem and the Nakai–Moishezon criterion show that this curvature positivity is characterized by algebraic-geometric notions such as ampleness and the positivity of intersection numbers. Thus, for the MA equation, a threefold correspondence among solvability, differential-geometric positivity, and algebraic-geometric positivity is established.

Building on this philosophy, the applicant has studied the solvability of canonical metric equations such as the J-equation and the deformed Hermitian–Yang–Mills (dHYM) equation, as well as the relationship between solvability and positivity for more general nonlinear geometric partial differential equations.

Summary of Paper 1:

In this paper, the solvability of the J-equation and the dHYM equation on holomorphic fibrations is studied from the geometry of the base and the fibers. The J-equation is closely related to constant scalar curvature Kähler metrics, while the dHYM equation plays an important role in mirror symmetry. It is shown that the solvability on the total space follows from the solvability on both the base and the fibers. In particular, for the dHYM equation, the paper provides the first concrete examples where solvability holds outside the previously assumed supercritical phase. Furthermore, in view of the established correspondence among solvability, differential-geometric positivity, and algebraic-geometric positivity for the J-equation and the supercritical dHYM equation, these results show that positivity on the base and the fibers is inherited by the total space. Conversely, it is also shown that semipositivity on the total space descends to the base and the fibers.

Summary of Paper 2:

This paper investigates weak solutions of the J-equation and the dHYM equation in situations where positivity degenerates to semipositivity. As mentioned above, the solvability of these equations is characterized by differential-geometric and algebraic-geometric positivity. In complex dimension two, it is shown that, under such degenerate positivity assumptions, the parabolic versions of these equations, namely the J-flow and the dHYM flow, converge to weak solutions of the corresponding elliptic equations. These flows arise naturally as gradient flows of associated energy functionals, and therefore possess canonical dynamical structures. This guarantees that the weak solutions obtained as limits of the flows are mathematically natural and geometrically meaningful.

Summary of Paper 3:

In this paper, a correspondence conjecture among solvability, differential-geometric positivity, and algebraic-geometric positivity is proposed and established for quotient Hessian equations, which generalize the Monge–Ampère equation and the J-equation. While the correspondence between solvability and differential-geometric positivity had been previously understood, only partial conjectures relating solvability to algebraic-geometric positivity had been proposed by Székelyhidi. The paper introduces a notion of algebraic-geometric positivity that fully characterizes solvability, and verifies the conjecture for special projective bundles and for first Chern classes of semi-ample line bundles. This provides the first examples in which Székelyhidi's conjecture is completely confirmed.

Summary of Paper 4:

In this paper, a framework is developed to characterize positivity of holomorphic vector bundles on compact Riemann surfaces via the solvability of nonlinear partial differential equations. For line bundles, the Kodaira embedding theorem asserts that the existence of a positively curved metric is equivalent to ampleness. Griffiths proposed a natural generalization to vector bundles, conjecturing that Griffiths positivity of curvature is equivalent to ampleness of the bundle (the Griffiths conjecture). Recently, Demailly introduced a system of nonlinear equations, combining Monge–Ampère-type and Hermitian–Einstein-type equations, to capture this notion of positivity analytically. In this work, the system is analyzed in complex dimension one, and it is shown that solvability follows from ampleness. As a consequence, solvability, Griffiths positivity, and ampleness are shown to be equivalent on Riemann surfaces, yielding a new analytic proof of the Griffiths conjecture in dimension one.