My future research plan

Toshihiro Nakanishi

My research subjects are Teichmüller spaces and mapping class groups. The mapping class group is an important research subject in many fields such as topology, hyperbolic geometry, theory of low-dimensional manifolds, combinatorial group theory, dynamical systems and complex analysis. The study of the mapping class group produced the beautiful Nielsen-Thurston theory and has spurred the advance of mathematics in the 21st century. We obtain a deep insight of the mapping class group when it is paired with an action on some geometrical objects. One of them is the Teichmüller space, a deformation space of Riemann surfaces on which the mapping class group acts as the group of holomorphic automorphisms. The mapping class group is also called the Teichmüller modular group.

A Teichmüller space admits several coordinate systems. Coordinate system induced by holomorphic quadratic differentials on a Riemann surface employed by Teichmüller and Bers, and Fenchel-Nielsen coordinates based on hyperbolic geometry are among them. Let S be an orientable surface of genus g with boundary curves C_1,..., C_m and $\mathcal{T}_{g,m}(L_1, ..., L_m)$ the Teichmüller space of hyperbolic structures on S with C_k totally geodesic of length L_k. Lengths of suitably chosen $d+1$ closed geodesic curves embed $\mathcal{T}_{g,m}(L_1, ..., L_m)$ into $\mathbb{R}^{d+1}$, where $d = 6g - 4 + 3m$ is the dimension of Teichmüller space (P. Schmutz et al.) (It is known that no tuple of lengths of d closed geodesic curves can embed the Teichmüller space into $\mathbb{R}^d$.) I am interested in how a mapping class is described in terms of geodesic length parameters and proved that it is a rational transformation in the parameter space. This enables us to treat the mapping class group and the Teichmüller space and also related subjects in 3-dimensional manifolds, dynamical systems and number theory in quantative and computational methods. My reseach plan is to tackle the following problems to find

(1) A fundamental region for the action of mapping class group and also the moduli space.
(2) Methods to calculate the Weil-Petersson volume of a moduli space.
(3) Integer solutions of Diopahntine equations analogous to that of the Markoff equation.
(4) Group representations of the mapping class group.
(5) Action of the mapping class group on the $SL(2,\mathbb{C})$-representation space of $\pi_1(S)$.
(6) Examples of hyperbolic 3-manifolds which fiber over the circle.

We have tried to apply the rational representation of the mapping class group to find Kleinian groups and hyperbolic 3-manifolds derived from the dynamics of the mapping classes but not have been successful so far, due to the difficulty to determine discreteness of a subgroup of $SL(2,\mathbb{C})$. Therefore our research plan is to find effective computational methods for discreteness criterion for subgroups of $SL(2,\mathbb{C})$ and then make full use of our rational representation of the mapping class group to find many examples of Kleinian groups and then hyperbolic 3-manifolds which fiber over the circle. Our research is supported by JSPS KAKENHI Grant in Aid for Scientific Research (C) for the period 2024-2026.