Research Plan

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I will continue my research on the relationship between vertex algebras and Costello–Gwilliam factorization algebras.

■ Conformal Symmetry of Factorization Algebras

Vertex algebras of physical interest typically contain the Virasoro Lie algebra as an internal symmetry, and such vertex algebras are called conformal vertex algebras. The affine vertex algebra (at non-critical level) and the $\beta\gamma$ vertex algebra are examples of conformal vertex algebras. In the construction of vertex algebras due to Costello and Gwilliam, however, such conformal structures do not appear explicitly, which leads to the following problem.

Problem 1 : Investigate the notion of conformal symmetry for factorization algebras and construct conformal vertex algebras from factorization algebras.

A conformal vertex algebra carries an action of all generators L_n ($n \in \mathbb{Z}$) of the Virasoro Lie algebra. One may consider a weaker condition requiring only the action of L_{-1}, L_0, and L_1. Vertex algebras satisfying this condition are called Möbius vertex algebras. For example, the affine vertex algebra at the critical level is not a conformal vertex algebra but is a Möbius vertex algebra. The operators L_{-1}, L_0, and L_1 correspond respectively to infinitesimal generators of translations, rotations, and inversions. In the Costello–Gwilliam framework, vertex algebras are constructed from factorization algebras on the complex plane $\mathbb{C}$ that are equivariant with respect to the isometry group $S^1 \ltimes \mathbb{C}$, so factorization algebras with translation and rotation symmetries are considered. To incorporate inversion symmetry, it is natural to consider factorization algebras on the Riemann sphere $\mathbb{CP}^1$. As a preliminary step toward Problem 1, I will address the following more concrete problem.

Problem 1' : Construct Möbius vertex algebras from factorization algebras on the Riemann sphere $\mathbb{CP}^1$ that are equivariant under the action of $\mathrm{PSL}_2(\mathbb{C})$. Moreover, provide examples of factorization algebras to which this construction applies.

■ Zhu Algebras of Vertex Algebras and Factorization Algebras

The Zhu algebra is an associative algebra $\mathrm{Zhu}(V)$ constructed from a vertex algebra V, and it is known that there is a one-to-one correspondence between irreducible positive-energy representations of V and simple modules over $\mathrm{Zhu}(V)$ (up to isomorphism). Thus, the Zhu algebra plays an important role in the representation theory of vertex algebras. On the other hand, a satisfactory notion of the category of representations of a factorization algebra has not yet been established. To explore an appropriate definition of representations of factorization algebras, it is necessary to understand how the Zhu algebra appears in the correspondence between vertex algebras and factorization algebras on $\mathbb{C}$. As a clue to this problem, one may consider the categorical equivalence between locally constant factorization algebras on the real line $\mathbb{R}$ and associative algebras. Based on this idea, I will address the following problem.

Problem 2 : Provide a method to degenerate a factorization algebra $\mathcal{F}$ on $\mathbb{C}$ to one on $\mathbb{R}$ such that the resulting factorization algebra $\mathcal{F}|_{\mathbb{R}}$ is locally constant. Furthermore, show that the associative algebra corresponding to $\mathcal{F}|_{\mathbb{R}}$ is the Zhu algebra of the vertex algebra associated with $\mathcal{F}$.

■ Construction of Affine $\mathcal{W}$-Algebras via Factorization Envelopes

An important class of vertex algebras is given by affine $\mathcal{W}$-algebras. Most algebras in this class cannot be expressed as enveloping vertex algebras of Lie conformal algebras due to the nonlinearity appearing in the commutation relations of the Fourier modes of vertex operators. To treat such nonlinearities in a unified way, De Sole and Kac introduced the notion of nonlinear Lie conformal algebras and constructed their enveloping vertex algebras. As a result, every affine $\mathcal{W}$-algebra can be realized as the enveloping vertex algebra of some nonlinear Lie conformal algebra. Therefore, the method of [N] cannot be applied directly to affine $\mathcal{W}$-algebras, but it is expected that, with suitable modifications, one can construct factorization algebras corresponding to affine $\mathcal{W}$-algebras. I will thus address the following problem.

Problem 3 : Construct a factorization algebra on $\mathbb{C}$ from a nonlinear Lie conformal algebra L via factorization envelopes, and show that the associated vertex algebra is isomorphic to the enveloping vertex algebra of L.

[N] Y. Nishinaka, *Factorization envelopes and enveloping vertex algebras*, preprint, arXiv: 2512.07635.