Future Research Plan

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I will continue my research on the existence problem of canonical metrics and stability for polarized manifolds. Below I list three concrete research themes to pursue in the future.

Criteria for uniform relative K-stability and relative K-instability

In the case of polarized toric manifolds, uniform relative K-stability characterizes extremal metrics (see, for example, [3]). Therefore, determining whether a given polarized toric manifold is uniformly relatively K-stable or relatively K-unstable is equivalent to determining the existence or non-existence of extremal metric, and it is an important problem also from the viewpoint of differential geometry. I aim to completely determine which toric Fano threefolds are uniformly relatively K-stable or not. There are 18 toric Fano threefolds in total; among them, 13 are already known to be uniformly relatively K-stable, but nothing is known for the remaining five. I would like to find general methods to decide uniform relative K-stability or relative K-instability, as well as new criteria that take into account specific geometric features such as projective bundles and blow-ups.

Study of hyperplane sections of Segre varieties

For a hypersurface X of bidegree $(1,1)$ in $\mathbf{P}^m \times \mathbf{P}^n$, the Futaki invariant is nonzero for every polarization when $m \neq n$. This naturally leads to the problem: "Determine whether X admits canonical metrics such as Kähler-Ricci solitons, Mabuchi solitons, or extremal metrics." As a first step, I will work on computing the Mabuchi constant of X. The problem "Compute the δ-invariant of X" is also algebraically interesting. When $m = n$, X admits a homogeneous Kähler–Einstein metric, and I would also like to study the question: "Determine whether constant scalar curvature Kähler metrics exist or not in a general Kähler class on X." On the other hand, when the bidegree is higher than $(1,1)$, it is not known in general whether X admits a Kähler–Einstein metric even if $m = n$ (the only cases treated are $m = n = 3$ with bidegree $(1,2)$ and $(1,3)$ in [1]), so I would like to investigate such examples from both differential- and algebro-geometric viewpoints.

Study of strong K-stability

Motivated by the perspective that the original K-stability may be insufficient to characterize the existence of constant scalar curvature Kähler metrics on general polarized manifolds, various strengthened notions of K-stability have been studied. I am interested in the relationships among these notions, and in particular I would like to understand how the strong K-stability introduced by Mabuchi [2] compares with other strengthened notions. As a first step, I will work on explicitly describing the Donaldson–Futaki invariant for sequences of test configurations in the toric setting.

References

[1] J. Cable, Kähler-Einstein metrics on symmetric general arrangement varieties. *Manuscripta Math.* **168** (2022), pp.119–135.

[2] T. Mabuchi, The Donaldson-Futaki invariant for sequences of test configurations. *Geometry and Analysis on Manifolds*, Birkhäuser, Cham, 2015, pp.395–403.

[3] Y. Nitta and S. Saito, A uniform version of the Yau-Tian-Donaldson correspondence for extremal Kähler metrics on polarized toric manifolds. arXiv:2110.10386.