Future Research Plan

1. Geodesic Connectedness of Statistical Manifolds

Geodesic connectedness has been scarcely studied in statistical geometry, and the **first result was obtained in Paper 3**. We will investigate this property for statistical manifolds of constant curvature.

For a statistical manifold (M, ∇, g), the sectional curvature is defined using the curvature tensor of ∇ and the metric g. **Statistical manifolds with constant sectional curvature is an important class of statistical manifolds** appearing in information geometry, affine differential geometry, and projective differential geometry. Simply connected statistical manifolds of constant curvature admit geometric divergences, which coincide with the Kullback–Leibler divergence in information geometry[AN, K]. These divergences can be computed using geodesic connectedness of ∇ and its dual connection.

We aim to establish sufficient conditions for geodesic connectedness of affine connections on such manifolds. By the fundamental theorem of affine differential geometry, it is known that a simply connected statistical manifold can be realized as a hypersurface in the affine space. Therefore, the universal covering of any constant curvature statistical manifold can be embedded into $\mathbb{R}^{n+1}$. If the lifted connection is geodesically connected, then the affine connection of the base space is also geodesically connected.

The geodesic connectedness of affine connections on statistical manifolds possesses a wide range of potential applications. The global properties of affine connections derived from this geodesic connectedness are expected to initiate further studies, including symmetric spaces endowed with statistical structures, analytical investigations of geometric divergences, and the role of completeness of affine connections on statistical manifolds. Even restricting these studies to statistical manifolds of constant curvature is expected to lead to applications in fields such as information geometry and projective differential geometry, where such statistical manifolds naturally arise.

2. Hypersurface Theory in the Probability Simplex

We study the hypersurface theory in the probability simplex, which is one of the most fundamental examples of statistical manifolds. The n-dimensional probability simplex S_n is the manifold consisting of all discrete probability distributions with $n + 1$ labels. Any statistical manifold can be embedded into this probability simplex as a statistical submanifold [L]. The probability simplex S_n is equipped with a statistical structure $(\nabla^{((m))}, g^F)$ induced from the Amari-Chenstov theorem, where $\nabla^{(m)}$ is called the mixture connection, g^F is the Fisher metric, and the dual connection $\nabla^{(e)}$ is called the exponential connection. Information geometry can be regarded as the **theory of submanifolds of the probability simplex** S_n given by $\nabla^{(m)}$-autoparallel submanifolds and $\nabla^{(e)}$-autoparallel submanifolds of S_n. These submanifolds have been extensively studied due to their statistical significance; however, other submanifolds of the probability simplex S_n remain largely unexplored.

In this study, we initiate the hypersurface theory in the probability simplex. This is because, as in Study 1., techniques from affine differential geometry are expected to be applicable. Since the probability simplex S_n is a statistical manifold of constant curvature, there exists an affine immersion $f : S_n \to \mathbb{R}^{n+1}$ inducing $(\nabla^{(m)}, g^F)$ on S_n. For an $(n-1)$-dimensional submanifold M^{n-1} of the probability simplex S_n, letting $\iota : M \to S_n$ denote the inclusion map, we obtain an affine immersion $f \circ \iota : M^{n-1} \to \mathbb{R}^{n+1}$ of codimension 2.

In the geometry of statistical manifolds, there exist various important classes. We aim to determine necessary and sufficient conditions under which an $(n-1)$-dimensional statistical submanifold M^{n-1} of the probability simplex S_n belongs to these classes. Furthermore, we aim to express these conditions in probabilistic terms and **provide an interpretation within information geometry**.

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