(2) これまでの研究成果のまとめと今後の研究計画 (英訳板)

Summary of Research Achievements

1. Hitoshi Furuhata, **Ryu Ueno**, A Variation Problem for Mappings between Statistical Manifolds, Results in Mathematics, **80**(57), 2025.
2. Ryu Ueno, Statistical Biharmonicity of Identity Maps, Information Geometry, 2025, https://doi.org/10.1007/s41884-025-00184-1.
3. Ryu Ueno, Geodesic Connectedness on Statistical Manifolds with Divisible Cubic Forms, Information Geometry, 2025, https://doi.org/10.1007/s41884-025-00185-0.
4. Ryu Ueno, The Second Variational Formula for Statistical Biharmonic Maps, arXiv.2509.04807.

Overview of Statistical Manifolds

Statistical manifolds has its origin in affine differential geometry. In particular, since their structure was applied in information geometry in the work of Amari and Nagaoka [AN], the study of statistical manifolds has developed significantly.

Statistical biharmonic maps are a class of mappings between statistical manifolds arising from a variational problem associated with the statistical bienergy functional. This is the first variational problem formulated in the geometry of statistical manifolds. The statistical bienergy is an extension of the bienergy functional in Riemannian geometry to mappings between statistical manifolds. Although harmonic maps between statistical manifolds have been introduced by Uohashi (2017) and Şimşir (2019), their definitions do not depend on the statistical structure of the target manifold. Since defining harmonic maps that fully reflect statistical structures seems difficult, in **Paper 1**, we defined statistical biharmonic maps via the Euler–Lagrange equation of the statistical bienergy. In particular, we proved that improper affine hyperspheres in affine differential geometry provide examples of statistical biharmonic maps whose tension field does not vanish. In **Paper 4**, the second variational formula of the statistical bienergy is derived. This yields an operator H on $u^{-1}TN$ associated with a map $u : M \to N$. If the target manifold is a Hessian manifold, that is, a statistical manifold with a flat affine connection, the operator H is determined solely by the Hessian curvature. The Hessian curvature on a Hessian manifold is a tensor field which plays a role analogous to sectional curvature.

In **Paper 2**, the statistical biharmonicity of the identity map between statistical manifolds with the same Riemannian metric but different affine connections is studied. It is shown that the tension field is proportional to the Tchevychev vector field, a vector field that is unique to statistical manifolds. Statistical manifolds with vanishing Tchevychev vector field are said to satisfy the equiaffine condition, which is an important condition in affine differential geometry. We obtain a partial differential equation of the Tchevychev vector field form the biharmonicity of the identity map. If this equation is satisfied, we defined that the statistical manifold is said to satisfy the semi-equiaffine condition. Under certain assumptions, we determined statistical manifolds satisfying this condition, and in particular, such structures arise on minimal surfaces with vanishing Tchevychev operator in centro-affine differential geometry.

Statistical Manifolds with Divisible Cubic Forms (Paper 3)

Statistical manifolds with cubic forms divisible by the metric arise naturally in affine differential geometry. In this work, global properties of such manifolds are obtained. In particular, a sufficient condition geodesic connectedness for the affine connection, that is, the existence of geodesics between any two points is established for the first time in the geometry of statistical manifolds. In information geometry, geodesics are used to compute the Kullback–Leibler divergence via the generalized Pythagorean theorem. Moreover, divergence functions on general statistical manifolds are defined using geodesic connectedness. However, there has been no prior work on investigating the geodesic connectedness of affine connections of the statistical manifold[AF, AN, K].

Furthermore, a new pseudo-distance structure is induced from the statistical structure. Analogues of the Hopf–Rinow theorem and the Cartan–Hadamard theorem are established for this class of statistical manifolds.