Proceedings of The 23rd International Differential Geometry Workshop on
Submanifolds in Homogeneous Spaces & Related Topics 23(2021)

Canonical connections of a Sasakian manifold and invariant submanifolds with parallel second fundamental form

Yoshihiro Ohnita
Osaka City University Advanced Mathematical Institute (OCAMI), *3-3-138 Sugimito, Sumiyoshi-ku, Osaka, 558-8585, Japan*
e-mail : ohnita@sci.osaka-cu.ac.jp

(2020 Mathematics Subject Classification : 53C25, 53C40, 53B25)

Abstract. In this paper we observe that M-connections of a Sasakian manifold introduced first by Okumura ([8]) are canonical connections in the sense of Olmos-Sánchez ([9]) and we show that the parallelism of the second fundamental form of an invariant submanifold in a regular Sasakian manifold relative to an M-connection is equivalent to the parallelism of the second fundamental form of the corresponding complex submanifold in the quotient Kähler manifold. This result generalizes a main result in [7] which was essential to give a new proof of Nakagawa-Takagi's theorem classifying complex submanifolds of complex projective spaces with parallel second fundamental form.

Introduction

Let N be a Riemannian manifold with a Riemannian metric g_N and ∇^N denotes the Levi-Civita connection of g_N. The concept of the *canonical connections* of a Riemannian manifold was used by Olmos-Sánchez in [9]. An affine connection ∇^c of N is called a *canonical connection* of (N, g_N) if

(1) ∇^c is a metric connection:

$$\nabla^c g = 0. \tag{0.1}$$

Key words and phrases: Sasakian manifold, invariant submanifold, canonical connection, parallel second fundamental form.

* This work is partly supported by JSPS KAKENHI Grant Numbers JP17H06127, JP18K03307, JP18H03668, JP21K03252 and by Osaka City University Advanced Mathematical Institute: MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849

(2) The tensor field D on N of type $(1,2)$ defined by $D := \nabla^N - \nabla^c$ satisfies the equation

$$\nabla^c D = 0. \tag{0.2}$$

The Levi-Civita connection ∇^N is a *trivial* canonical connection of a Riemannian manifold ($D = 0$).

Let $\varphi : N \to \tilde{N}$ be a smooth immersion of N into a Riemannian manifold $\tilde{N}$. Then the *usual* covariant derivative of the second fundamental form α^N of φ is defined in terms of the normal connection $\nabla^\perp$ and the Levi-Civita connection ∇^N as

$$(\nabla^*_X \alpha^N)(Y,Z) := \nabla^\perp_X(\alpha^N(Y,Z)) - \alpha^N(\nabla^N_X Y, Z) - \alpha^N(Y, \nabla^N_X Z). \tag{0.3}$$

When N satisfies $\nabla^* \alpha^N = 0$, we say that a submanifold N of $\tilde{N}$ has *parallel second fundamental form* in the usual sense. Symmetric R-spaces standardly imbedded in a Euclidean space are well-known to be characterized by Ferus ([2]) in 1974 as submanifold immersed in Euclidean space satisfying $\nabla^* \alpha^N = 0$. As a wider concept the covariant derivative of the second fundamental form α^N in terms of the normal connection $\nabla^\perp$ and a canonical connection ∇^c is defined by

$$(\nabla^c_X \alpha^N)(Y,Z) := \nabla^\perp_X(\alpha^N(Y,Z)) - \alpha^N(\nabla^c_X Y, Z) - \alpha^N(Y, \nabla^c_X Z). \tag{0.4}$$

General R-spaces standardly imbedded in a Euclidean space were characterized by Olmos-Sánchez ([9]) in 1991 as submanifold immersed in Euclidean space satisfying $\nabla^c \alpha^N = 0$, namely *parallel second fundamental form relative to a canonical connection* ∇^c, as follows:

Theorem 0.1 (Olmos and Sánchez [9]). *Let N be a connected compact submanifold fully embedded in the Euclidean space $\mathbb{R}^l$. Then the following three conditions are equivalent each other:*

(1) *There is a canonical connection ∇^c on N such that*

$$\nabla^c \alpha^N = 0. \tag{0.5}$$

(2) *N is a homogeneous submanifold with constant principal curvatures.*

(3) *N is an orbit of an s-representation, that is, an R-space standardly embedded in the Euclidean space.*

Recently in [7] we used Olmos-Sánchez's Theorem 0.1 in order to give a new proof of Nakagawa-Takagi's theorem classifying complex submanifolds M in complex projective spaces $\mathbb{C}P^n$ satisfying $\nabla^* \alpha^M = 0$. Let $\pi : S^{2n+1}(1) \to \mathbb{C}P^n$ be the Hopf fibration over the n-dimensional complex projective space $\mathbb{C}P^n$, which is regarded as a Riemannian submersion with totally geodesic fibers. An essential result in [7] was as follows:

Theorem 0.2 ([7])**.** *Suppose that M^m be an m-dimensional complex submanifold of $\mathbb{C}P^n$ and $\hat{M}^{2m+1} = \pi^{-1}(M^m)$ be the inverse image of M under the Hopf fibration π. Then there exists a non-trivial canonical connection ∇^c on the inverse image $\hat{M} = \pi^{-1}(M)$ such that the following two conditions are equivalent each other:*

(A) *M^m has parallel second fundamental form: $\nabla^* \alpha^M = 0$.*

(B) *$\hat{M}^{2m+1} = \pi^{-1}(M^m)$ has parallel second fundamental form relative to the canonical connection ∇^c: $\nabla^c \alpha^{\hat{M}} = 0$.*

In [7] this non-trivial canonical connection ∇^c was explicitly given by defining the tensor field D of type $(1,2)$ on $\hat{M}^{2m+1} = \pi^{-1}(M^m) \subset S^{2n+1}(1) \subset \mathbb{C}^{n+1}$ as

$$\begin{cases} D_{\tilde{X}}(\tilde{Y}) := -\langle \sqrt{-1}\tilde{X}, \tilde{Y} \rangle \sqrt{-1}\mathbf{x} \in \mathcal{V}_{\mathbf{x}}\hat{M}, \\ D_{\tilde{X}}(V) := \sqrt{-1}\tilde{X} = \widetilde{JX} \in \mathcal{H}_{\mathbf{x}}\hat{M}, \\ D_V(\tilde{X}) := \dfrac{1}{2}\sqrt{-1}\tilde{X} = \dfrac{1}{2}\widetilde{JX} \in \mathcal{H}_{\mathbf{x}}\hat{M}, \\ D_V(V) := 0 \end{cases} \tag{0.6}$$

for each horizontal vectors $\tilde{X}$, $\tilde{Y}$ and the vertical vector $V = \sqrt{-1}\mathbf{x}$ on $\hat{M}^{2m+1}$.

The purpose of this paper is to generalize Theorem 0.2 from odd-dimensional standard spheres $S^{2n+1}(1)$ to general Sasakian manifolds and invariant submanifolds. We use the notion of *M-connections* of a Sasakian manifold as canonical connections. In 1960s it was introduced first by Masafumi Okumura and investigated by Kanji Motomiya, Touru Kato, Toshio Takahashi ([8], [6], [4], [11]), as a one-parameter family of affine connections on a Sasakian manifold which make all structure tensor fields to be parallel. We observe that each M-connection of a Sasakian manifold is a canonical connection in the above sense of Olmos-Sánchez. The inverse image $\hat{M}^{2m+1} = \pi^{-1}(M^m)$ here is an *invariant submanifold* of $S^{2n+1}(1)$ as a Sasakian manifold. We will give attentions to invariant submanifolds of a Sasakian manifold. Main results are Theorems 2.2 and 2.4 in Section 2. Though M-connections are parametrized by real numbers $r \in \mathbb{R}$, an M-connection with $r = -\frac{1}{2}$ is crucial for our problem. We show that the parallelism of the second fundamental form of an invariant submanifold in a regular Sasakian manifold relative to the M-connection with $r = -\frac{1}{2}$ is equivalent to the usual parallelism of the second fundamental form of the corresponding complex submanifold in the quotient Kähler manifold.

This paper is organized as follows: In Section 1 we explain the definition of M-connections on a Sasakian manifold and discuss a relation of M-connections with canonical connections in the sense of Olmos-Sánchez. In Section 2 we discuss the covariant derivatives and the parallelism of the second fundamental form of invariant submanifolds in a general Sasakian manifold with respect to M-connections. Main results and formulas are described precisely. Finally we illustrate our result in the Riemannian submersion setting of invariant submanifolds of regular Sasakian manifolds and corresponding complex submanifolds of its quotient Kähler manifolds, which generalizes Theorem 0.2.

Throughout this paper any manifold is smooth, connected and second-countable.

1 Canonical connections of Sasakian manifolds

Let M^{2m+1} be a $(2m+1)$-dimensional Sasakian manifold with structure tensor fields ϕ,ξ,η and g ([1] and its References). Then the fundamental equations for those structure tensor fields are described as follows: For each $X, Y \in TM^{2m+1}$

$$\phi^2 = \phi \circ \phi = -\mathrm{I} + \eta \otimes \xi, \tag{1.1}$$

$$\eta(\xi) = 1, \tag{1.2}$$

$$g(X, \xi) = \eta(X), \tag{1.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.4}$$

$$(d\eta)(X, Y) = g(\phi(X), Y), \tag{1.5}$$

$$(\nabla_X \phi)(Y) = \eta(Y)X - g(X, Y)\xi \tag{1.6}$$

Here ϕ is a tensor field of type $(1,1)$, ξ is a vector field on M^{2m+1} which is called a *characteristic vector field* or *Reeb vector field*, η is a 1-form on M^{2m+1} and g is a Riemannian metric on M^{2m+1}. We denote by $\nabla = \nabla^M$ the Levi-Civita connection of g. For each $X \in TM^{2m+1}$, we have

$$\nabla_X \xi = \phi X. \tag{1.7}$$

If a vector field X is invariant under the flow of ξ, then we have

$$\nabla_\xi X = \nabla_X \xi + [\xi, X] = \nabla_X \xi = \phi X. \tag{1.8}$$

For each point $x \in M$ set

$$\begin{aligned} &\mathcal{V}_x M^{2m+1} := \mathbb{R}\xi_x, \\ &\mathcal{H}_x M^{2m+1} := \{ v \in T_x M \mid g_x(\xi_x, v) = 0 \} = \phi_x(T_x M^{2m+1}) \end{aligned}$$

and we have an orthogonal direct sum decomposition of TM^{2m+1} into vector subbundles as

$$TM^{2m+1} = \mathcal{V}M^{2m+1} \oplus \mathcal{H}M^{2m+1}. \tag{1.9}$$

Any vector $X \in TM^{2m+1}$ can be decomposed as $X = \mathcal{V}X + \mathcal{H}Y$, where $\mathcal{V}X$ and $\mathcal{H}X$ denote the $\mathcal{V}M^{2m+1}$-component and the $\mathcal{H}M^{2m+1}$-component of X, respectively.

Now, following the definition (0.6) of the canonical connection $\nabla^c = \nabla - D$ on $\hat{M} \subset S^{2n+1}(1) \subset \mathbb{C}^{n+1}$, we can define a tensor field $\tilde{D}$ of type $(1,2)$ on a general Sasakian manifold M^{2m+1} by

$$\begin{cases} \widetilde{D}_{\tilde{X}} \tilde{Y} := -g(\phi\tilde{X}, \tilde{Y})\xi \in \mathcal{V}_x M^{2m+1}, \\ \widetilde{D}_{\tilde{X}} \xi := \phi\tilde{X} \in \mathcal{H}_x M^{2m+1}, \\ \widetilde{D}_\xi \tilde{X} := \dfrac{1}{2}\phi\tilde{X} \in \mathcal{H}_x M^{2m+1}, \\ \widetilde{D}_\xi \xi := 0 \end{cases} \tag{1.10}$$

for each $\tilde{X}, \tilde{Y} \in \mathcal{H}_x M^{2m+1}$ and the characteristic vector $\xi \in \mathcal{V}_x M^{2m+1}$ at a point $x \in M^{2m+1}$ on M^{2m+1}, and define an affine connection $\widetilde{\nabla} := \nabla - \widetilde{D}$ on a Sasakian manifold M^{2m+1}.

Here we shall explain the notion of M-connections. For each real number $r \in \mathbb{R}$, define a tensor field $D^{(r)}$ of type $(1, 2)$ on a Sasakian manifold M^{2m+1} by

$$D^{(r)}_X Y := -g(\phi(X), Y)\xi - r\eta(X)\phi(Y) + \eta(Y)\phi(X) \quad (\forall X, Y \in TM^{2m+1}) \tag{1.11}$$

and define an affine connection of M^{2m+1} as

$$\nabla^{(r)} := \nabla - D^{(r)}. \tag{1.12}$$

We should notice that

Lemma 1.1. *When $r = -\dfrac{1}{2}$, it holds $D^{(r)} = \widetilde{D}$ and thus $\nabla^{(r)} = \widetilde{\nabla}$.*

Since

$$\begin{aligned} g(D^{(r)}_X Y, Z) =& - g(\phi(X), Y)g(\xi, Z) - r\eta(X)g(\phi(Y), Z) + \eta(Y)g(\phi(X), Z) \\ =& - g(\phi(X), Y)g(\xi, Z) - r\eta(X)g(\phi(Y), Z) + g(\xi, Y)g(\phi(X), Z) \end{aligned}$$

is skew-symmetric with respect to Y and Z, the affine connection $\nabla^{(r)}$ is a metric connection with respect to g, that is,

$$\nabla^{(r)} g = 0. \tag{1.13}$$

For each $X \in TM^{2m+1}$, as

$$D^{(r)}_X \xi = -g(\phi(X), \xi)\xi - r\eta(X)\phi(\xi) + \eta(\xi)\phi(X) = \phi(X)$$

we have

$$\nabla^{(r)}_X \xi = \nabla_X \xi - D^{(r)}_X \xi = \phi(X) - \phi(X) = 0. \tag{1.14}$$

Thus

$$(\nabla^{(r)}_X \eta)(Y) = g(Y, \nabla^{(r)}_X \xi) = 0. \tag{1.15}$$

By definition we compute

$$\begin{aligned} (D^{(r)}_X \phi)Y =& D^{(r)}_X (\phi(Y)) - \phi(D^{(r)}_X Y) \\ =& - g(\phi(X), \phi(Y))\xi - r\eta(X)\phi(\phi(Y)) + \eta(\phi(Y))\phi(X) \\ & - \phi(-g(\phi(X), Y)\xi - r\eta(X)\phi(Y) + \eta(Y)\phi(X)) \\ =& - (g(X, Y) - \eta(X)\eta(Y))\xi - r\eta(X)(-Y + \eta(Y)\xi) + 0 \\ & - 0 + r\eta(X)(-Y + \eta(Y)\xi) - \eta(Y)(-X + \eta(X)\xi)) \\ =& - g(X, Y)\xi + \eta(Y)X \\ =& (\nabla_X \phi)(Y). \end{aligned}$$

Thus we have

$$(\nabla^{(r)}_X\phi)(Y) = (\nabla_X\phi)(Y) - (D^{(r)}_X\phi)Y = 0. \tag{1.16}$$

It follows from (1.13), (1.14), (1.15) and (1.16) that

$$\nabla^{(r)} D^{(r)} = 0. \tag{1.17}$$

Therefore we obtain

Lemma 1.2. *For each $r \in \mathbb{R}$, the affine connection $\nabla^{(r)}$ is a canonical connection of a Sasakian manifold M^{2m+1} with respect to the structure Riemannian metric g.*

2 Parallelism of the second fundamental form of an invariant submanifold with respect to the canonical connection

Let $\hat{M}$ be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ with (ϕ, ξ, η, g), that is, by definition $\phi_x(T_x\hat{M}) \subset T_x\hat{M}$ for each $x \in \hat{M}$. For the definition and fundamental properties of invariant submanifolds refer [1], [5] and so on. Then $\hat{M}$ is also known to be a Sasakian manifold by restriction and a minimal submanifold in $(\widetilde{M}, g)$ in the sense of vanishing the mean curvature vector field. Since $\hat{M}$ is a Sasakian manifold, $\hat{M}$ is equipped with M-connections $\{\nabla^{(r)} \mid r \in \mathbb{R}\}$ explained in Section 1. The Gauss formula and the Weingarten formula are

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha^{\hat{M}}(X, Y), \tag{2.1}$$

$$\overline{\nabla}_X \nu = -A^{\hat{M}}_\nu(X) + \nabla^\perp_X \nu. \tag{2.2}$$

for any vector fields X, Y on $\hat{M}$ and any normal vector field ν to $\hat{M}$. Here $\alpha^{\hat{M}}$ and $A^{\hat{M}}$ denote the second fundamental form and the shape operator of $\hat{M}$ in $\widetilde{M}$, respectively. Then the second fundamental form and the shape operator of $\hat{M}$ satisfy the following equations ([5]):

$$\alpha^{\hat{M}}(\phi(X), Y) = \alpha^{\hat{M}}(X, \phi(Y)) = \phi(\alpha^{\hat{M}}(X, Y)), \tag{2.3}$$

$$\alpha^{\hat{M}}(X, \xi) = 0 \tag{2.4}$$

and

$$\phi(A^{\hat{M}}_\nu(X)) = -A^{\hat{M}}_\nu(\phi(X)) = A^{\hat{M}}_{\phi(\nu)}(X), \tag{2.5}$$

$$A^{\hat{M}}_\nu(\xi) = 0 \tag{2.6}$$

for each $X, Y \in T\hat{M}$ and each $\nu \in T^\perp\hat{M}$. By differentiating covariantly the equation (2.4) in terms of the Levi-Civita connection $\nabla^{\hat{M}}$, we compute

$$\begin{aligned}
0 =& \nabla^\perp_Y(\alpha^{\hat{M}}(X, \xi)) \\
=& (\nabla^*_Y\alpha^{\hat{M}})(X, \xi) + \alpha^{\hat{M}}(\nabla^{\hat{M}}_Y X, \xi) + \alpha^{\hat{M}}(X, \nabla^{\hat{M}}_Y\xi) \\
=& (\nabla^*_Y\alpha^{\hat{M}})(X, \xi) + \alpha^{\hat{M}}(X, \phi(Y)).
\end{aligned}$$

It follows from this equation and (2.4) that

Proposition 2.1. *If $\hat{M}$ has parallel second fundamental form in the sense of $\nabla^*\alpha^{\hat{M}}=0$, then $\hat{M}$ must be totally geodesic, that is, $\alpha^{\hat{M}}=0$.*

By covariantly differentiating the equation (2.4) in terms of the canonical connection $\nabla^{(r)}$, we compute

$$\begin{aligned}0 =&\nabla^{\perp}_{Y}(\alpha^{\hat{M}}(X,\xi))\\ =&(\nabla^{(r)}_{Y}\alpha^{\hat{M}})(X,\xi)+\alpha^{\hat{M}}(\nabla^{(r)}_{Y}X,\xi)+\alpha^{\hat{M}}(X,\nabla^{(r)}_{Y}\xi)\\ =&(\nabla^{(r)}_{Y}\alpha^{\hat{M}})(X,\xi).\end{aligned}$$

Hence for any $r\in\mathbb{R}$ we have

$$(\nabla^{(r)}_{X}\alpha^{\hat{M}})(\xi,Y)=0 \tag{2.7}$$

for each $X,Y\in T\hat{M}$.

Note that any vectors $\tilde{X},\tilde{Y}\in\mathcal{H}_x\hat{M}$ can be locally extended to vector fields $\tilde{X},\tilde{Y}\in\mathcal{H}\hat{M}$ invariant under the flow of ξ. Thus by (1.8) we have

$$\nabla^{(r)}_{\xi}\tilde{X}=\nabla^{\hat{M}}_{\xi}\tilde{X}-D^{(r)}_{\xi}\tilde{X}=\phi(\tilde{X})+r\phi(\tilde{X})=(1+r)\tilde{X}. \tag{2.8}$$

By using (2.8) we compute

$$\begin{aligned}0 =&\xi(g(\alpha^{\hat{M}}(\tilde{X},\tilde{Y}),\tilde{\nu}))\\ =&g(\nabla^{\perp}_{\xi}(\alpha^{\hat{M}}(\tilde{X},\tilde{Y})),\tilde{\nu})+g(\alpha^{\hat{M}}(\tilde{X},\tilde{Y}),\nabla^{\perp}_{\xi}\tilde{\nu})\\ =&g((\nabla^{(r)}_{\xi}\alpha^{\hat{M}})(\tilde{X},\tilde{Y}),\tilde{\nu})\\ &+g(\alpha^{\hat{M}}(\nabla^{(r)}_{\xi}\tilde{X},\tilde{Y}),\tilde{\nu})+g(\alpha^{\hat{M}}(\tilde{X},\nabla^{(r)}_{\xi}\tilde{Y}),\tilde{\nu})+g((\nabla^{(r)}_{\xi}\alpha^{\hat{M}})(\tilde{X},\tilde{Y}),\nabla^{\perp}_{\xi}\tilde{\nu})\\ =&g((\nabla^{(r)}_{\xi}\alpha^{\hat{M}})(\tilde{X},\tilde{Y}),\tilde{\nu})\\ &+g(\alpha^{\hat{M}}((1+r)\phi(\tilde{X}),\tilde{Y}),\tilde{\nu})+g(\alpha^{\hat{M}}(\tilde{X},(1+r)\phi(\tilde{Y})),\tilde{\nu})+g((\nabla^{(r)}_{\xi}\alpha^{\hat{M}})(\tilde{X},\tilde{Y}),\nabla^{\hat{M}}_{\xi}\tilde{\nu})\\ =&g((\nabla^{(r)}_{\xi}\alpha^{\hat{M}})(\tilde{X},\tilde{Y}),\tilde{\nu})\\ &+g(\alpha^{\hat{M}}((1+r)\phi(\tilde{X}),\tilde{Y}),\tilde{\nu})+g(\alpha^{\hat{M}}(\tilde{X},(1+r)\phi(\tilde{Y})),\tilde{\nu})+g((\nabla^{(r)}_{\xi}\alpha^{\hat{M}})(\tilde{X},\tilde{Y}),\phi(\tilde{\nu}))\\ =&g((\nabla^{(r)}_{\xi}\alpha^{\hat{M}})(\tilde{X},\tilde{Y}),\tilde{\nu})\\ &+(1+r)g(\alpha^{\hat{M}}(\phi(\tilde{X}),\tilde{Y}),\tilde{\nu})+(1+r)g(\alpha^{\hat{M}}(\tilde{X},\phi(\tilde{Y})),\tilde{\nu})-g(\phi(\nabla^{(r)}_{\xi}\alpha^{\hat{M}})(\tilde{X},\tilde{Y}),\tilde{\nu})\\ =&g((\nabla^{(r)}_{\xi}\alpha^{\hat{M}})(\tilde{X},\tilde{Y}),\tilde{\nu})+(1+2r)g(\alpha^{\hat{M}}(\phi(\tilde{X}),\tilde{Y}),\tilde{\nu}).\end{aligned}$$

Hence we obtain

$$(\nabla^{(r)}_{\xi}\alpha^{\hat{M}})(\tilde{X},\tilde{Y})+(1+2r)\alpha^{\hat{M}}(\phi(\tilde{X}),\tilde{Y})=0 \tag{2.9}$$

for each $\tilde{X}, \tilde{Y} \in \mathcal{H}\hat{M}$. In particular we find that if $r = -\dfrac{1}{2}$, then we have

$$(\nabla^{(r)}_{\xi}\alpha^{\hat{M}})(\tilde{X}, \tilde{Y}) = (\nabla^{(-\frac{1}{2})}_{\xi}\alpha^{\hat{M}})(\tilde{X}, \tilde{Y}) = 0. \tag{2.10}$$

For each $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathcal{H}\hat{M}$, by using the definition (1.11) and (2.4) we compute

$$\begin{aligned}(\nabla^{(r)}_{\tilde{X}}\alpha^{\hat{M}})(\tilde{Y}, \tilde{Z}) =&\nabla^{\perp}_{\tilde{X}}(\alpha^{\hat{M}}(\tilde{Y}, \tilde{Z})) - \alpha^{\hat{M}}(\nabla^{(r)}_{\tilde{X}}\tilde{Y}, \tilde{Z}) - \alpha^{\hat{M}}(\tilde{Y}, \nabla^{(r)}_{\tilde{X}}\tilde{Z})\\ =&\nabla^{\perp}_{\tilde{X}}(\alpha^{\hat{M}}(\tilde{Y}, \tilde{Z})) - \alpha^{M}(\nabla^{\hat{M}}_{\tilde{X}}\tilde{Y}, \tilde{Z}) - g(\phi\tilde{X}, \tilde{Y})\alpha^{\hat{M}}(\xi, \tilde{Z})\\ &- \alpha^{\hat{M}}(\tilde{Y}, \nabla^{\hat{M}}_{\tilde{X}}\tilde{Z}) - g(\phi\tilde{X}, \tilde{Z})\alpha^{\hat{M}}(\tilde{Y}, \xi)\\ =&\nabla^{\perp}_{\tilde{X}}(\alpha^{\hat{M}}(\tilde{Y}, \tilde{Z})) - \alpha^{\hat{M}}(\nabla^{\hat{M}}_{\tilde{X}}\tilde{Y}, \tilde{Z}) - \alpha^{\hat{M}}(\tilde{Y}, \nabla^{\hat{M}}_{\tilde{X}}\tilde{Z})\\ =&(\nabla^{*}_{\tilde{X}}\alpha^{\hat{M}})(\tilde{Y}, \tilde{Z}),\end{aligned} \tag{2.11}$$

where we use

$$\nabla^{(r)}_{\tilde{X}}\tilde{Y} = \nabla^{\hat{M}}_{\tilde{X}}\tilde{Y} - D^{(r)}_{\tilde{X}}\tilde{Y} = \nabla^{\hat{M}}_{\tilde{X}}\tilde{Y} + g(\phi(\tilde{X}), \tilde{Y})\xi.$$

Therefore combing (2.7), (2.9) and (2.11) we obtain

Theorem 2.2. *Let $\hat{M}$ be an invariant submanifold of a Sasakian manifold $\widetilde{M}$. Suppose that $\hat{M}$ is not totally geodesic. Then $r = -\dfrac{1}{2}$ if and only if*

$$(\nabla^{(r)}_{X}\alpha^{\hat{M}})(Y, Z) = (\nabla^{*}_{\mathcal{H}X}\alpha^{\hat{M}})(\mathcal{H}Y, \mathcal{H}Z) \tag{2.12}$$

for each $X, Y, Z \in T\hat{M}$.

Corollary 2.3. *For any invariant submanifold $\hat{M}$ of a Sasakian manifold $\widetilde{M}$, the following two conditions are equivalent each other:*

(a) *$\hat{M}$ has η-parallel second fundamental form in the sense of* [3]*:*

$$(\nabla^{*}_{\mathcal{H}X}\alpha^{\hat{M}})(\mathcal{H}Y, \mathcal{H}Z) = 0$$

for each $X, Y, Z \in T\hat{M}$.

(b) *$\hat{M}$ has parallel second fundamental form with respect to the canonical connection $\nabla^{(-\frac{1}{2})}$:*

$$\nabla^{(-\frac{1}{2})}_{X}\alpha^{\hat{M}} = 0$$

for each $X, Y, Z \in T\hat{M}$.

Furthermore we suppose that $\widetilde{M}^{2n+1}$ is a *regular* Sasakian manifold. Let $Q^n := \widetilde{M}^{2n+1}/\xi$ be its quotient Kähler manifold by the flow of the Reeb vector field ξ with the natural projection $\pi : \widetilde{M}^{2n+1} \to Q = \widetilde{M}^{2n+1}/\xi$, which is a Riemannian submersion of totally geodesic fibers. Let M^m be a complex submanifold of the quotient Kähler manifold $Q^n = \widetilde{M}^{2n+1}/\xi$. Then the inverse image $\hat{M}^{2m+1} = \pi^{-1}(M^m)$ is an invariant submanifold of $\widetilde{M}^{2n+1}$ and thus it is also a Sasakian manifold:

$$\begin{tikzcd}
\hat{M}^{2m+1} = \pi^{-1}(M^m) \arrow[r, "\hat{\varphi}"] \arrow[d, "\pi"', "S^1"] & \widetilde{M}^{2n+1} \arrow[d, "\pi"', "S^1"] \\
M^m \arrow[r, "\varphi"] & Q^n := \widetilde{M}^{2n+1}/\xi
\end{tikzcd}$$

By applying Theorem 2.2 to this setting of Riemannian submersions we obtain a formula

$$\begin{aligned}
(\nabla^{(-\frac{1}{2})}_X \alpha^{\hat{M}})_x(Y, Z) =&(\nabla^*_{\mathcal{H}X}\alpha^{\hat{M}})_x(\mathcal{H}Y, \mathcal{H}Z) \\
=&((\nabla^*_{(d\pi)_x X}\alpha^M)_{\pi(x)}((d\pi)_x Y, (d\pi)_x Z))^{\tilde{}}
\end{aligned} \tag{2.13}$$

for each $x \in \hat{M}$ and any $X, Y, Z \in T_x\hat{M}$. Here $(\ )^{\tilde{}}$ denotes the horizontal lift of a vector on Q under the Riemannian submersion $\pi : \widetilde{M}^{2n+1} \to Q^n$.

Now we obtain

Theorem 2.4. *Suppose that $\widetilde{M}^{2n+1}$ is a* regular *Sasakian manifold. Let $Q^n := \widetilde{M}^{2n+1}/\xi$ be its quotient Kähler manifold with the natural projection $\pi : \widetilde{M}^{2n+1} \to Q = \widetilde{M}^{2n+1}/\xi$. Let M^m be an m-dimensional complex submanifold of Q and $\hat{M}^{2m+1} = \pi^{-1}(M^m)$ be the inverse image of M^m under the projection π. Then there exists a non-trivial canonical connection ∇^c on the inverse image $\hat{M}^{2m+1} = \pi^{-1}(M^m)$ such that the following two conditions are equivalent each other:*

(A) *M^m has parallel second fundamental form: $\nabla^*\alpha^M = 0$.*

(B) *$\hat{M}^{2m+1} = \pi^{-1}(M^m)$ has parallel second fundamental form relative to the canonical connection ∇^c: $\nabla^c\alpha^{\hat{M}} = 0$.*

Proof. Since the inverse image $\hat{M}^{2m+1} = \pi^{-1}(M^m)$ of a complex submanifold M is an invariant submanifold of $\widetilde{M}^{2n+1}$, we take an M-connection $\nabla^{(-\frac{1}{2})}$ relative to its Sasakian structure as a canonical connection ∇^c on $\widetilde{M}^{2n+1}$. The claim of this theorem follows from the formula (2.13). □

Acknowledgements. This article is based on the author's talk at the 23rd International Differential Geometry Workshop on Submanifolds in Homogeneous Spaces and Related Topics & the 19th RIRCM-OCAMI Joint Differential Geometry Workshop (Zoom Meeting) on July 2–3 in 2021. The author sincerely would like to thank Professors Young Jin Suh, Changhwa Woo, Hyunjin Lee, Takayuki Okuda, Kei Kondo and Hiroshi Tamaru for their various efforts and excellent organization of international workshops. The author also would like to thank Professor Jong Taek Cho for his interest in this work, especially valuable suggestions of the papers [3] and [11] on invariant submanifolds of Sasakian manifolds and Okumura's M-connections.

References

[1] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds.* Second edition. Progress in Mathematics, 203. Birkhauser Boston, Ltd., Boston, MA, 2010. xvi+343 pp. ISBN: 978-0-8176-4958-6

[2] D. Ferus, *Immersions with parallel second fundamental form.* Math. Z. **140** (1974), 87–92.

[3] M. Kon, *Invariant submanifolds in Sasakian manifolds.* Math. Ann. **219** (1976), 277–290.

[4] T. Kato and K. Motomiya, *A study on certain homgeneous spaces.* Tohoku Math. J. **21** (1969), 1–20.

[5] M. Kon, *Invariant submanifolds in Sasakian manifolds.* Math. Ann. **219** (1976), 277–290.

[6] K. Motomiya, *A study on almost contact manifolds.* Tohoku Math. J. **20** (1968), 73–90.

[7] Y. Ohnita, *Parallel Kähler submanifolds and R-spaces*, to appear in Contemporary Mathematics, Differential Geometry and Global Analysis. In Honor of Tadashi Nagano.

[8] M. Okumura, *Some remarks on spaces with a certain contact structure.* Tohoku Math. J. **14** (1962), 135–145.

[9] C. Olmos and C. Sánchez, *A geometric characterization of the orbits of s-representations.* J. reine angew. Math. **420** (1991), 195–202.

[10] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.

[11] Toshio Takahashi, *Sasakian ϕ-symmetric spaces.* Tohoku Math. J. **29** (1977), 91–113.