Proceedings of The 23rd International Differential Geometry Workshop on
Submanifolds in Homogeneous Spaces & Related Topics 23(2021)

On the converse problem for austere orbits of path group actions induced by Hermann actions

Masahiro Morimoto
Osaka City University Advanced Mathematical Institute. 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan.
e-mail : mmasahiro0408@gmail.com

(2020 Mathematics Subject Classification : 53C40.)

Abstract. It is known that if an orbit of a Hermann action is an austere submanifold then the corresponding orbit of the path group action is also an austere submanifold. In this article we will discuss the converse of this property.

1 Introduction

Let $M = G/K$ be a symmetric space of compact type and H a closed subgroup of G. An isometric action of H on M is defined by

$$b \cdot (aK) := (ba)K \qquad \text{for } \ aK \in M \text{ and } b \in H.$$

Then an isometric action of $H \times K$ on G is defined by

$$(b, c) \cdot a := bac^{-1} \qquad \text{for } \ a \in G \text{ and } (b, c) \in H \times K.$$

Moreover there is an isometric action of a path group on a path space. More precisely we consider the path group $\mathcal{G} := H^1([0,1], G)$ of all Sobolev-H^1-paths from $[0,1]$ to G and the path space $V_{\mathfrak{g}} := L^2([0,1], \mathfrak{g})$ of all L^2-paths from $[0,1]$ to

Key words and phrases: Hermann action, principal curvature, austere submanifold, path group action, proper Fredholm action on Hilbert space, proper Fredholm submanifold of Hilbert space
The author was partly supported by the Grant-in-Aid for Research Activity Start-up (No.20K22309) and by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JP-MXP0619217849).

$\mathfrak{g}$. Then $\mathcal{G}$ acts on $V_{\mathfrak{g}}$ isometrically via the gauge transformations:

$$g * u := gug^{-1} - g'g^{-1} \qquad \text{for} \quad u \in V_{\mathfrak{g}} \text{ and } g \in \mathcal{G}.$$

Then the subgroup

$$P(G, H \times K) := \{g \in \mathcal{G} \mid (g(0), g(1)) \in H \times K\}$$

of $\mathcal{G}$ acts on $V_{\mathfrak{g}}$ by restriction. It is known that every orbit of the $P(G, H\times K)$-action is a proper Fredholm submanifold of the Hilbert space $V_{\mathfrak{g}}$ ([11]). It is an interesting problem to study submanifold geometry of orbits of $P(G, H \times K)$-actions.

Recall that a submanifold of a Riemannian manifold is called *austere* ([2]) if for each normal vector ξ the set of eigenvalues with multiplicities of the shape operator A_ξ is invariant under the multiplication by (-1). By definition austere submanifolds are minimal submanifolds. The concept of austere submanifolds was extended to the class of PF submanifolds in Hilbert spaces by the author ([6], [7], [8]). Recently the author [9] studied the austere property of orbits of $P(G, H \times K)$-actions under the assumption that H is a *symmetric* subgroup of G, that is, there exists an involutive automorphism τ of G such that H lies between the fixed point subgroup G^τ and its identity component G^τ_0. Such an H-action is called a *Hermann action* ([3]). To explain that result, for simplicity here we suppose that τ commutes with the involutive automorphism σ of G corresponding to K. Denoting by $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ the canonical decompositions associated to σ and τ respectively we take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{m} \cap \mathfrak{p}$ so that every H-orbit can be written in the form $H \cdot (\exp w)K$ for some $w \in \mathfrak{t}$. Then the constant path with value w is denoted by $\hat{w}$. The author showed ([9, Theorems 6.1 and 7.5, Section 8]):

(1) Suppose that an orbit $H \cdot (\exp w)K$ is an austere submanifold of M. Then the orbit $P(G, H \times K) * \hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$.

(2) Conversely suppose that the orbit $P(G, H\times K) * \hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$. If the root system Δ of $\mathfrak{t}$ is reduced then the orbit $H \cdot (\exp w)K$ is an austere submanifold of M. If the root system Δ of $\mathfrak{t}$ is non-reduced then the orbit $H \cdot (\exp w)K$ is not an austere submanifold of M in general.

In fact the author gave an example of an H-orbit which is not austere but the corresponding $P(G, H \times K)$-orbit is austere. However, in the non-reduced case, it is not clear whether such a non-austere H-orbit always exists or not. The purpose of this article is to discuss this problem more precisely. As a consequence we will show that even if the root system Δ is non-reduced, under some assumptions the austere property of $P(G, H \times K)$-orbits conversely implies the austere property of H-orbits (Theorem 3.3).

Throughout this article we will suppose that the involutions σ and τ commute.

2 Principal curvatures and austere property

In this section we review the facts on the principal curvatures and the austere property of H-orbits and $P(G, H \times K)$-orbits.

Take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{m} \cap \mathfrak{p}$ and consider the root space decomposition of $\mathfrak{m}$ with respect to $\mathfrak{t}$

$$\mathfrak{m} = \mathfrak{m}_0 + \sum_{\alpha \in \Delta^+} \mathfrak{m}_\alpha,$$

$$\begin{aligned} \mathfrak{m}_0 &= \{x \in \mathfrak{m} \mid \forall \eta \in \mathfrak{t},\ \mathrm{ad}(\eta)x = 0\}, \\ \mathfrak{m}_\alpha &= \{x \in \mathfrak{m} \mid \forall \eta \in \mathfrak{t},\ \mathrm{ad}(\eta)^2 x = -\langle \alpha, \eta \rangle^2 x\}. \end{aligned}$$

Since σ and τ commute we have the decomposition

$$\mathfrak{m} = \mathfrak{m} \cap \mathfrak{h} + \mathfrak{m} \cap \mathfrak{p}.$$

Since σ commutes with $\mathrm{ad}(\eta)^2$ for all $\eta \in \mathfrak{t}$ we have

$$\mathfrak{m} \cap \mathfrak{p} = \mathfrak{m}_0 \cap \mathfrak{p} + \sum_{\alpha \in \Delta_1^+} \mathfrak{m}_\alpha \cap \mathfrak{p}, \qquad \mathfrak{m} \cap \mathfrak{h} = \mathfrak{m}_0 \cap \mathfrak{h} + \sum_{\alpha \in \Delta_{-1}^+} \mathfrak{m}_\alpha \cap \mathfrak{h}.$$

$$\Delta_1^+ = \{\alpha \in \Delta^+ \mid \mathfrak{m}_\alpha \cap \mathfrak{p} \neq \{0\}\}, \qquad \Delta_{-1}^+ = \{\alpha \in \Delta^+ \mid \mathfrak{m}_\alpha \cap \mathfrak{h} \neq \{0\}\}.$$

Take $w \in \mathfrak{t}$, set $a := \exp w$ and consider the orbit $N := H \cdot (\exp w)K$ through $(\exp w)K$. Denote by L_a the isometry of M defined by $L_a(bK) = (ab)K$. Then the tangent space and the normal space of N are decomposed as follows ([1, Proposition 5.1])

$$T_{aK}N = dL_a\Big(\mathfrak{m}_0 \cap \mathfrak{h} + \sum_{\substack{\alpha \in \Delta_1^+ \\ \langle \alpha, w \rangle \notin \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{p} + \sum_{\substack{\alpha \in \Delta_{-1}^+ \\ \langle \alpha, w \rangle + \pi/2 \notin \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{h} \Big), \tag{2.1}$$

$$T_{aK}^\perp N = dL_a\Big(\mathfrak{t} + \sum_{\substack{\alpha \in \Delta_1^+ \\ \langle \alpha, w \rangle \in \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{p} + \sum_{\substack{\alpha \in \Delta_{-1}^+ \\ \langle \alpha, w \rangle + \pi/2 \in \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{h} \Big), \tag{2.2}$$

Moreover the decomposition (2.1) is just the eigenspace decomposition of the family shape operators $\{A^N_{dL_a(\xi)}\}_{\xi \in \mathfrak{t}}$. In fact ([1, Theorem 5.3])

$$\begin{aligned} dL_a(\mathfrak{m}_0 \cap \mathfrak{h}) \quad &: \quad \text{the eigenspace of eigenvalue } 0, \\ dL_a(\mathfrak{m}_\alpha \cap \mathfrak{p}) \quad &: \quad \text{the eigenspace of eigenvalue } -\langle \alpha, \xi \rangle \cot\langle \alpha, w \rangle, \\ dL_a(\mathfrak{m}_\alpha \cap \mathfrak{h}) \quad &: \quad \text{the eigenspace of eigenvalue } \langle \alpha, \xi \rangle \tan\langle \alpha, w \rangle. \end{aligned}$$

Thus principal curvatures of the orbit $H \cdot (\exp w)K$ in the direction of $dL_a(\xi)$ are given by

$$\begin{aligned} \{0\} \cup \{ -\langle \alpha, \xi \rangle \cot\langle \alpha, w \rangle \mid \alpha \in \Delta_1^+,\ \langle \alpha, w \rangle \notin \pi\mathbb{Z} \} \\ \{ \langle \alpha, \xi \rangle \tan\langle \alpha, w \rangle \mid \alpha \in \Delta_{-1}^+,\ \langle \alpha, w \rangle + \frac{\pi}{2} \notin \pi\mathbb{Z} \} \end{aligned}$$

where the multiplicities are respectively given by

$$\dim(\mathfrak{m}_0 \cap \mathfrak{h}), \qquad \dim(\mathfrak{m}_\alpha \cap \mathfrak{p}), \qquad \dim(\mathfrak{m}_\alpha \cap \mathfrak{h}).$$

We now consider the principal curvatures of the orbit $P(G, H\times K)*\hat{w}$. Although the eigenspace decompositions are complicated in general we can explicitly describe the principal curvatures as follows ([9, Corollary 5.2]):

Theorem 2.1 ([9]). *Let $\xi \in \mathfrak{t}$. Then the principal curvatures of the orbit $P(G, H\times K) * \hat{w}$ in the direction of $\hat{\xi}$ are given by*

$$\begin{aligned}
&\{0\} \cup \left\{ \frac{\langle \alpha, \xi\rangle}{-\langle \alpha, w\rangle + m\pi} \;\middle|\; \alpha \in \Delta_1^+,\ \langle \alpha, w\rangle \notin \pi\mathbb{Z},\ m \in \mathbb{Z}\right\} \\
&\quad \cup \left\{ \frac{\langle \alpha, \xi\rangle}{-\langle \alpha, w\rangle - \frac{\pi}{2} + m\pi} \;\middle|\; \alpha \in \Delta_{-1}^+,\ \langle \alpha, w\rangle + \frac{\pi}{2} \notin \pi\mathbb{Z},\ m \in \mathbb{Z}\right\} \\
&\quad \cup \left\{ \frac{\langle \alpha, \xi\rangle}{n\pi} \;\middle|\; \alpha \in \Delta_1^+,\ \langle \alpha, w\rangle \in \pi\mathbb{Z},\ n \in \mathbb{Z}\backslash\{0\} \right. \\
&\qquad\qquad \left. \text{or } \ \alpha \in \Delta_{-1}^+,\ \langle \alpha, w\rangle + \frac{\pi}{2} \in \pi\mathbb{Z},\ n \in \mathbb{Z}\backslash\{0\}\right\}.
\end{aligned}$$

where the multiplicities are respectively given by

$$\infty, \qquad \dim(\mathfrak{m}_\alpha \cap \mathfrak{p}), \qquad \dim(\mathfrak{m}_\alpha \cap \mathfrak{h}), \qquad \dim(\mathfrak{m}_\alpha \cap \mathfrak{p}) + \dim(\mathfrak{m}_\alpha \cap \mathfrak{h}).$$

Next we consider the austere properties of H- and $P(G, H \times K)$-orbits. The following lemma is fundamental ([4, p. 89], [9, Lemma 6.2]):

Lemma 2.2. (i) (Ikawa [4]) *the orbit $H \cdot (\exp w)K$ through $(\exp w)K$ is an austere submanifold of M if and only if the set*

$$\begin{aligned}
&\{-\alpha \cot\langle \alpha, w\rangle \mid \alpha \in \Delta_1^+,\ \langle \alpha, w\rangle \notin \frac{\pi}{2}\mathbb{Z}\} \\
&\cup\ \{\alpha \tan\langle \alpha, w\rangle \mid \alpha \in \Delta_{-1}^+,\ \langle \alpha, w\rangle \notin \frac{\pi}{2}\mathbb{Z}\}
\end{aligned}$$

with multiplicities is invariant under the multiplication by (-1).

(ii) ([9]) *the orbit $P(G, H \times K) * \hat{w}$ is an austere PF submanifold of $V_\mathfrak{g}$ if and only if the set*

$$\begin{aligned}
&\left\{ \frac{1}{-\langle \alpha, w\rangle + m\pi}\alpha \;\middle|\; \alpha \in \Delta_1^+,\ m \in \mathbb{Z},\ \langle \alpha, w\rangle \notin \frac{\pi}{2}\mathbb{Z}\right\} \\
&\cup\ \left\{ \frac{1}{-\langle \alpha, w\rangle - \frac{\pi}{2} + m\pi}\alpha \;\middle|\; \alpha \in \Delta_{-1}^+,\ m \in \mathbb{Z},\ \langle \alpha, w\rangle \notin \frac{\pi}{2}\mathbb{Z}\right\}
\end{aligned}$$

with multiplicities is invariant under the multiplication by (-1).

Using this lemma the following theorem was shown ([9, Theorem 7.5]):

Theorem 2.3 ([9])**.** *If the orbit $H \cdot (\exp w)K$ through $(\exp w)K$ is an austere submanifold of M then the orbit $P(G, H \times K) * \hat{w}$ through $\hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$.*

In the next section we will discuss the converse of this theorem.

3 On the converse problem

First we prepare the setting. We are supposing that the involutions σ and τ commute. In addition to this we suppose that

(i) G is simple.

From this condition it follows that the root system Δ is an irreducible root system of $\mathfrak{t}$ ([4, Lemma 4.34]). To consider the converse problem we note that if Δ is a reduced root system then the converse is true ([9, Theorem 6.1]). Therefore we suppose that

(ii) Δ is a non-reduced root system (i.e. of type BC).

We also note that if $\sigma \sim \tau$, that is, there exists an element $c \in G$ such that $\tau = \mathrm{Ad}(c) \circ \sigma \circ \mathrm{Ad}(c)^{-1}$, then the converse is true ([9, Corollary 7.4]). Hence we suppose that

(iii) $\sigma \not\sim \tau$.

Under these assumptions we consider the converse of Theorem 2.3

To consider the converse we briefly review Ikawa's work [4]. He investigated properties of the triple $(\Delta, \Delta_1, \Delta_{-1})$, formulated those properties Lie algebraically, and gave classification of such triples (without assuming that Δ is non-reduced). The following lemma concerns one of those properties ([4, Theorem 4.33 (1), see also Definition 2.2 (4)]):

Lemma 3.1 (Ikawa [4])**.** *Suppose that G is simple, $\sigma \circ \tau = \tau \circ \sigma$, and $\sigma \not\sim \tau$. Set $l := \max\{\|\alpha\| \mid \alpha \in \Delta_1 \cap \Delta_{-1}\}$. Then $\Delta_1 \cap \Delta_{-1} = \{\alpha \in \Delta \mid \|\alpha\| \leq l\}$.*

From this property it follows that if Δ is of type BC then there are three possibilities for $\Delta_1 \cap \Delta_{-1}$, namely

(I) $\Delta_1 \cap \Delta_{-1} = \{\pm e_i\}_i$,

(II) $\Delta_1 \cap \Delta_{-1} = \{\pm e_i\}_i \cup \{\pm(e_i \pm e_j)\}_{i<j}$ or

(III) $\Delta_1 \cap \Delta_{-1} = \{\pm 2e_i, \pm e_i\}_i \cup \{\pm(e_i \pm e_j)\}_{i<j}$.

where we write $\Delta^+ = \{e_i, 2e_i\}_i \cup \{e_i \pm e_j\}_{i<j}$.

We also recall his result for austere orbits of H-actions ([4, Theorem 2.18]):

Proposition 3.2 (Ikawa [4]). *Suppose that G is simple, $\sigma \circ \tau = \tau \circ \sigma$, and $\sigma \not\sim \tau$. Then the orbit $H \cdot (\exp w)K$ is an austere submanifold of M if and only if the following conditions are satisfied:*

(a) $\langle \alpha, w\rangle \in \frac{\pi}{4}\mathbb{Z}$ *holds for all* $\alpha \in \Delta$,

(b) $\langle \alpha, w\rangle \in \frac{\pi}{2}\mathbb{Z}$ *holds for all* $\alpha \in (\Delta_1 \backslash \Delta_{-1}) \cup (\Delta_{-1} \backslash \Delta_1)$,

(c) $\dim \mathfrak{m}_\alpha \cap \mathfrak{p} = \dim \mathfrak{m}_\alpha \cap \mathfrak{h}$ *holds for all* $\alpha \in \Delta_1 \cap \Delta_{-1}$ *satisfying* $\langle \alpha, w\rangle \in \frac{\pi}{4} + \frac{\pi}{2}\mathbb{Z}$.

The following theorem states that the converse is true in the case (I) if $\dim \mathfrak{t} \geq 2$.

Theorem 3.3. *Suppose that $\Delta_1 \cap \Delta_{-1} = \{\pm e_i\}_i$ and $\dim \mathfrak{t} \geq 2$. If the orbit $P(G, H \times K) * \hat{w}$ is austere then the orbit $H \cdot (\exp w)K$ is austere.*

Proof. From Lemma 2.2 the set

(2.1)
$$\left\{ \frac{1}{-\langle \alpha, w\rangle + m\pi}\alpha \;\middle|\; \alpha \in \Delta_1^+ \backslash \Delta_{-1}^+,\ m \in \mathbb{Z},\ \langle \alpha, w\rangle \notin \frac{\pi}{2}\mathbb{Z} \right\}$$
$$\cup \left\{ \frac{1}{-\langle \alpha, w\rangle - \frac{\pi}{2} + m\pi}\alpha \;\middle|\; \alpha \in \Delta_{-1}^+ \backslash \Delta_1^+,\ m \in \mathbb{Z},\ \langle \alpha, w\rangle \notin \frac{\pi}{2}\mathbb{Z} \right\}$$
$$\cup \left\{ \frac{1}{-\langle \alpha, w\rangle + m\pi}\alpha,\ \frac{1}{-\langle \alpha, w\rangle - \frac{\pi}{2} + m\pi}\alpha \;\middle|\; \alpha \in \Delta_1^+ \cap \Delta_{-1}^+,\ m \in \mathbb{Z},\ \langle \alpha, w\rangle \notin \frac{\pi}{2}\mathbb{Z} \right\}$$

with multiplicities is invariant under the multiplication by (-1).

By the assumption we have $\{\pm(e_i \pm e_j)\}_{i<j} \subset \Delta_1 \backslash \Delta_{-1} \cup \Delta_{-1} \backslash \Delta_1$. Suppose $e_i + e_j \in \Delta_1 \backslash \Delta_{-1}$. If $\langle e_i + e_j, w\rangle \notin \frac{\pi}{2}\mathbb{Z}$ then by the austere property of $P(G, H \times K) * \hat{w}$ the set $\{\frac{1}{-\langle e_i + e_j, w\rangle + m\pi}(e_i + e_j)\}_{m \in \mathbb{Z}}$ with multiplicities is invariant under the multiplication by (-1). However this implies $\langle e_i + e_j, w\rangle \in \frac{\pi}{2}\mathbb{Z}$. Thus consequently $\langle e_i + e_j, w\rangle \in \frac{\pi}{2}\mathbb{Z}$ holds. Similarly supposing $e_i + e_j \in \Delta_{-1} \backslash \Delta_1$ we obtain $\langle e_i + e_j, w\rangle \in \frac{\pi}{2}\mathbb{Z}$. Thus $\langle e_i + e_j, w\rangle \in \frac{\pi}{2}\mathbb{Z}$ holds for all $i < j$. By the similar arguments it follows that $\langle e_i - e_j, w\rangle \in \frac{\pi}{2}\mathbb{Z}$ holds for all $i < j$. Hence we obtain $\langle 2e_i, w\rangle \in \frac{\pi}{2}\mathbb{Z}$ and thus $\langle e_i, w\rangle \in \frac{\pi}{4}\mathbb{Z}$. From these the conditions (a) and (b) hold.

To verify the condition (c) we suppose that $\langle e_i, w\rangle = \frac{\pi}{4} + \frac{\pi}{2}\mathbb{Z}$. Then $\langle 2e_i, w\rangle = \frac{\pi}{2} + \pi\mathbb{Z}$. Thus the vector $\frac{1}{-\langle 2e_i, w\rangle + m\pi}2e_i$ or $\frac{1}{-\langle 2e_i, w\rangle - \pi/2 + m\pi}2e_i$ does not appear in the set (2.1). Hence the set

$$\left\{ \frac{1}{-\frac{1}{4}\pi + m\pi}e_i,\ \frac{1}{-\frac{3}{4}\pi + m\pi}e_i \right\}_{m \in \mathbb{Z}}$$

with multiplicities is invariant under the multiplication by (-1). More precisely

$$\frac{1}{-\frac{1}{4}\pi + m\pi} = (-1) \times \frac{1}{-\frac{3}{4}\pi + (1-m)\pi}$$

holds for all $m \in \mathbb{Z}$. Hence $\dim(\mathfrak{m}_{e_i} \cap \mathfrak{p}) = \dim(\mathfrak{m}_{e_i} \cap \mathfrak{h})$ holds. If $\langle e_i, w\rangle = \frac{3}{4}\pi + \pi\mathbb{Z}$ then it follows similarly that $\dim(\mathfrak{m}_{e_i} \cap \mathfrak{p}) = \dim(\mathfrak{m}_{e_i} \cap \mathfrak{h})$. Hence (c) holds. This proves the theorem. $\square$

Example 1. An example of the triple (G, K, H) satisfying the condition $\Delta_1 \cap \Delta_{-1} = \{\pm e_i\}_i$ is given by $(SU(r+s+t), S(U(r+s)\times U(t)), S(U(r)\times U(s+t)))$, $(Sp(r+s+t), Sp(r+s)\times Sp(t), Sp(r)\times Sp(s+t)$ where $r < t$, $1 \leq s$, or $(SO(4r+4), U(2r+2), U(2r+2)')$; see [4, Theorem 2.19] and [5, p. 228] for details.

The following proposition concerns the cases (II) and (III).

Proposition 3.4. *Suppose* $\Delta_1 \cap \Delta_{-1} \supset \{\pm e_i\}_i \cup \{\pm(e_i \pm e_j)\}_{i<j}$. *If the orbit* $P(G, H\times K) * \hat{w}$ *is austere then* $\langle e_i, w\rangle \in \frac{\pi}{8}\mathbb{Z}$ *holds for all* i. *Moreover*

(i) *if* $\langle e_i, w\rangle \in \frac{\pi}{4}\mathbb{Z}$ *holds for all* i *then the orbit* $H\cdot(\exp w)K$ *is austere.*

(ii) *if there exists* i *such that* $\langle e_i, w\rangle \in \frac{\pi}{8} + \frac{\pi}{4}\mathbb{Z}$ *then the orbit* $H\cdot(\exp w)K$ *is not austere.*

Proof. Let $e_i + e_j \in \Delta^+$. Suppose $\langle e_i + e_j, w\rangle \notin \frac{\pi}{2}\mathbb{Z}$. Since $e_i + e_j \in \Delta_1 \cap \Delta_{-1}$ it follows from Lemma 2.2 that the set

$$\left\{\frac{1}{-\langle e_i+e_j, w\rangle + m\pi}(e_i+e_j),\ \frac{1}{-\langle e_i+e_j, w\rangle - \frac{\pi}{2} + m\pi}(e_i+e_j)\right\}_{m\in\mathbb{Z}}$$

with multiplicities is invariant under the multiplication by (-1). Thus for each $\epsilon \in \{\pm 1\}$ there exists $\epsilon' \in \{\pm 1\}$ such that

(2.2)
$$\frac{1}{-\langle e_i+e_j, w\rangle - \frac{1}{2}\arg\epsilon + m\pi}(e_i+e_j) = (-1)\times\frac{1}{-\langle e_i+e_j, w\rangle - \frac{1}{2}\arg\epsilon' + m'\pi}(e_i+e_j).$$

From this we have

$$\langle e_i + e_j, w\rangle = -\frac{1}{4}\arg\epsilon - \frac{1}{4}\arg\epsilon', \qquad \mod \pi\mathbb{Z}.$$

Since $\epsilon, \epsilon' \in \{\pm 1\}$ it follows that $\langle e_i + e_j, w\rangle \in \frac{\pi}{4}\mathbb{Z}$. Thus we have $\langle e_i + e_j, w\rangle \in \frac{\pi}{4}\mathbb{Z}$. Similarly we have $\langle e_i - e_j, w\rangle \in \frac{\pi}{4}\mathbb{Z}$. Hence we obtain $\langle 2e_i, w\rangle \in \frac{\pi}{4}\mathbb{Z}$ and thus $\langle e_i, w\rangle \in \frac{\pi}{8}\mathbb{Z}$.

Suppose that $\langle e_i, w\rangle \in \frac{\pi}{4}\mathbb{Z}$ holds for all i. Then $\langle e_i \pm e_j, w\rangle \in \frac{\pi}{4}\mathbb{Z}$ and $\langle 2e_i, w\rangle \in \frac{\pi}{2}\mathbb{Z}$. Thus clearly the conditions (a) and (b) hold. To verify the condition (c) we take $\alpha \in \Delta_1^+ \cap \Delta_{-1}^+$ satisfying $\langle \alpha, w\rangle \in \frac{\pi}{4} + \frac{\pi}{2}\mathbb{Z}$. Then $\alpha = e_i$ or $e_i \pm e_j$. Thus in both cases the set

$$\left\{ \frac{1}{-\frac{1}{4}\pi + m\pi}\alpha,\ \frac{1}{-\frac{3}{4}\pi + m\pi}\alpha \right\}_{m\in\mathbb{Z}}$$

with multiplicities is invariant under the multiplication by (-1). More precisely

$$\frac{1}{-\frac{1}{4}\pi + m\pi} = (-1) \times \frac{1}{-\frac{3}{4}\pi + (1-m)\pi}$$

holds for all $m \in \mathbb{Z}$. Hence $\dim \mathfrak{m}_\alpha \cap \mathfrak{p} = \dim \mathfrak{m}_\alpha \cap \mathfrak{h}$ and (c) follows. This proves the proposition. □

Remark 3.5. As proved above, if the orbit $P(G, H \times K) * \hat{w}$ is austere then $\langle e_i \pm e_j, w\rangle \in \frac{\pi}{4}$ holds for all $i < j$. Thus if the condition (ii) holds then $\langle e_i, w\rangle \in \frac{\pi}{8} + \frac{\pi}{4}\mathbb{Z}$ holds for *all* i.

Example 2. An example of the triple (G, H, K) satisfying the condition $\Delta_1 \cap \Delta_{-1} = \{\pm e_i\}_i \cup \{\pm(e_i \pm e_j)\}_{i<j}$ is given by $(SU(r+s), S(U(r)\times U(s)), SO(r+s))$ where $r > s$. In this case the H-orbit through $w := \frac{\pi}{8}\sum_{i=1}^{s} e_i$ satisfies the condition (ii) in Proposition 3.4 and it is not austere. On the other hand the $P(G, H \times K)$-orbit through $\hat{w}$ is austere (see [9, Section 8] for details).

Acknowledgements

This article is based on my talk at The 23rd International Differential Geometry Workshop on Submanifolds in Homogeneous Spaces & Related Topics - The 19th RIRCM-OCAMI Joint Differential Geometry Workshop on July 2-3 in 2021. The author would like to thank the organizers of this workshop for giving him the opportunity to give the talk and for their excellent organization of this workshop.

References

[1] O. Goertsches, G. Thorbergsson, *On the geometry of the orbits of Hermann actions*, Geom. Dedicata 129 (2007), 101–118.

[2] R. Harvey and H. B. Lawson, Jr., *Calibrated geometries*, Acta Math., **148** (1982), 47-157.

[3] R. Hermann, *Variational completeness for compact symmetric spaces.* Proc. Amer. Math. Soc. 11 (1960), 544-546.

[4] O. Ikawa, *The geometry of symmetric triad and orbit spaces of Hermann actions*, J. Math. Soc. Japan **63** (2011), no. 1, 79-136.

[5] O. Ikawa, *A note on symmetric triad and Hermann action Differential geometry of submanifolds and its related topics*, 220–229, World Sci. Publ., Hackensack, NJ, 2014.

[6] M. Morimoto, *On weakly reflective PF submanifolds in Hilbert spaces*, Tokyo J. Math. **44** (2021), no. 1, pp. 103-124.

[7] M. Morimoto, *Austere and arid properties for PF submanifolds in Hilbert spaces*, Differential Geom. Appl., Vol. 69 (2020) 101613, 24pp.

[8] M. Morimoto, *On weakly reflective submanifolds in compact isotropy irreducible Riemannian homogeneous spaces*, arXiv:2003.04674, to appear in Tokyo J. Math.

[9] M. Morimoto, *Curvatures and austere property of orbits of path group actions induced by Hermann actions*, arXiv:2105.12533.

[10] R. S. Palais, C.-L. Terng, *Critical Point Theory and Submanifold Geometry*, Lecture Notes in Math., vol 1353, Springer-Verlag, Berlin and New York, 1988.

[11] C.-L. Terng, *Proper Fredholm submanifolds of Hilbert space.* J. Differential Geom. **29** (1989), no. 1, 9-47.

[12] C.-L. Terng, *Polar actions on Hilbert space.* J. Geom. Anal. **5** (1995), no. 1, 129-150.

[13] C.-L. Terng, G. Thorbergsson, *Submanifold geometry in symmetric spaces.* J. Differential Geom. **42** (1995), no. 3, 665-718.