CRYSTAL INVARIANT THEORY I: GEOMETRIC RSK

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ABSTRACT. Berenstein and Kazhdan's theory of geometric crystals gives rise to two commuting families of geometric crystal operators acting on the space of complex $m \times n$ matrices. These are birational actions, which we view as a crystal-theoretic analogue of the usual action of $\mathrm{SL}_m \times \mathrm{SL}_n$ on $m \times n$ matrices. We prove that the field of invariants of each family of geometric crystal operators is generated by a set of algebraically independent polynomials, which are generalizations of the elementary symmetric polynomials in m (or n) variables. We also give a set of algebraically independent generators for the intersection of these fields, and we explain how these fields are situated inside the larger fields of geometric R-matrix invariants, which were studied by Lam and the third-named author. The key tool in our proof is the geometric RSK correspondence of Noumi and Yamada, which we show to be an isomorphism of geometric crystals.

CONTENTS

1. INTRODUCTION

The original problem of classical invariant theory was to understand the invariants of the action of SL_m on the space of $m \times n$ matrices. The first fundamental theorem of invariant theory says that the ring of invariants is trivial if $n < m$, and generated by the $m \times m$ minors of the matrix if $n \geq m$. The second fundamental theorem describes the relations among these generators. We refer the reader to [Wey46] for a classical exposition of this subject and to [GW00, PV94] for more modern ones.

In this paper, we study the analogous problem for a "crystallized" version of the SL_m-action. Specifically, we consider the invariants of the GL_m-geometric crystal operators, which were introduced by Berenstein and Kazhdan [BK00] as "de-tropicalizations" of Kashiwara's crystal operators [Kas90, Kas91]. In contrast to the classical SL_m-action, these operators act birationally on $m \times n$ matrices. We prove that the ring of polynomial invariants is generated by a set of algebraically independent polynomials, which arise as the entries of a certain $n \times n$ matrix.

1.1. Main result.

Consider the space $\mathrm{Mat}_{m\times n}(\mathbb{C}^*)$ of $m \times n$ matrices over $\mathbb{C}^*$. We write $\mathbf{x} = (x_i^j)_{i\in\{1,\ldots,m\}}^{j\in\{1,\ldots,n\}}$ for an element of $\mathrm{Mat}_{m\times n}(\mathbb{C}^*)$, and we denote the ith row and jth column of $\mathbf{x}$ by $\mathbf{x}_i = (x_i^1, \ldots, x_i^n)$ and $\mathbf{x}^j = (x_1^j, \ldots, x_m^j)$, respectively.[1] For $i = 1, \ldots, m-1$, the geometric crystal operator e_i is a rational $\mathbb{C}^*$-action

2010 *Mathematics Subject Classification.* 05E10, 14E05, 13A50, 14R20, 05E05.
Key words and phrases. geometric crystal, RSK, invariant theory, tropicalization.

[1] This is the reverse of the usual convention for covariant and contravariant indices in tensor calculus.

on $\mathrm{Mat}_{m\times n}(\mathbb{C}^*)$, and we denote the action of $c\in\mathbb{C}^*$ on $\mathbf{x}$ by $e_i^c(\mathbf{x})$. The action is given by

$$e_i^c\colon \begin{pmatrix} & \vdots & \\ x_i^1 & \cdots & x_i^n \\ x_{i+1}^1 & \cdots & x_{i+1}^n \\ & \vdots & \end{pmatrix} \mapsto \begin{pmatrix} & \vdots & \\ \dfrac{\sigma^1(\mathbf{x}_i,\mathbf{x}_{i+1};c)}{\sigma^0(\mathbf{x}_i,\mathbf{x}_{i+1};c)}x_i^1 & \cdots & \dfrac{\sigma^n(\mathbf{x}_i,\mathbf{x}_{i+1};c)}{\sigma^{n-1}(\mathbf{x}_i,\mathbf{x}_{i+1};c)}x_i^n \\ \dfrac{\sigma^0(\mathbf{x}_i,\mathbf{x}_{i+1};c)}{\sigma^1(\mathbf{x}_i,\mathbf{x}_{i+1};c)}x_{i+1}^1 & \cdots & \dfrac{\sigma^{n-1}(\mathbf{x}_i,\mathbf{x}_{i+1};c)}{\sigma^n(\mathbf{x}_i,\mathbf{x}_{i+1};c)}x_{i+1}^n \\ & \vdots & \end{pmatrix},$$

where

$$\sigma^j\big((x^1,\ldots,x^n),(y^1,\ldots,y^n);c\big)=\sum_{r=1}^{n} c^{\mathbb{1}_{r\le j}}y^1\cdots y^{r-1}x^{r+1}\cdots x^n.$$

One can view this as analogous to the action of the copy of SL_2 inside GL_m corresponding to the ith simple root, whose generators add a multiple of row i to row $i+1$ (or vice versa), or scale rows i and $i+1$ by t,t^{-1}.

For example, when $m=3$ and $n=2$, the geometric crystal operator e_1 acts by

$$e_1^c\colon \begin{pmatrix} x_1^1 & x_1^2 \\ x_2^1 & x_2^2 \\ x_3^1 & x_3^2 \end{pmatrix} \mapsto \begin{pmatrix} c_1x_1^1 & c_2x_1^2 \\ c_1^{-1}x_2^1 & c_2^{-1}x_2^2 \\ x_3^1 & x_3^2 \end{pmatrix}, \quad \text{where} \quad c_1=\frac{cx_1^2+x_2^1}{x_1^2+x_2^1}, \quad c_2=\frac{cx_1^2+cx_2^1}{cx_1^2+x_2^1}. \tag{1.1}$$

When $m=2$ and $n=3$, e_1 acts by

$$e_1^c\colon \begin{pmatrix} x_1^1 & x_1^2 & x_1^3 \\ x_2^1 & x_2^2 & x_2^3 \end{pmatrix} \mapsto \begin{pmatrix} c_1x_1^1 & c_2x_1^2 & c_3x_1^3 \\ c_1^{-1}x_2^1 & c_2^{-1}x_2^2 & c_3^{-1}x_2^3 \end{pmatrix}, \tag{1.2}$$

where

$$c_1=\frac{cx_1^2x_1^3+x_2^1x_1^3+x_2^1x_2^2}{x_1^2x_1^3+x_2^1x_1^3+x_2^1x_2^2}, \qquad c_2=\frac{cx_1^2x_1^3+cx_2^1x_1^3+x_2^1x_2^2}{cx_1^2x_1^3+x_2^1x_1^3+x_2^1x_2^2}, \qquad c_3=\frac{cx_1^2x_1^3+cx_2^1x_1^3+cx_2^1x_2^2}{cx_1^2x_1^3+cx_2^1x_1^3+x_2^1x_2^2}.$$

Given $x_1,\ldots,x_n\in\mathbb{C}^*$, let $W(x_1,\ldots,x_n)$ be the $n\times n$ matrix with entries $x_1,\ldots,x_n$ along the main diagonal, 1's directly beneath the main diagonal, and 0's elsewhere. Associate to $\mathbf{x}\in\mathrm{Mat}_{m\times n}(\mathbb{C}^*)$ the $n\times n$ matrix

$$M(\mathbf{x})=W(\mathbf{x}_1)\cdots W(\mathbf{x}_m).$$

Theorem 1.1. *The non-trivial entries of $M(\mathbf{x})$ (i.e., those that are not 0 or 1) are algebraically independent generators of the subfield $\mathrm{Inv}_e\subset\mathbb{C}(x_i^j)$ of invariants of the GL_m-geometric crystal operators $e_1,\ldots,e_{m-1}$.*

For example, when $m=2$ and $n=3$, we have

$$M(\mathbf{x})=\begin{pmatrix} x_1^1 & 0 & 0 \\ 1 & x_1^2 & 0 \\ 0 & 1 & x_1^3 \end{pmatrix}\begin{pmatrix} x_2^1 & 0 & 0 \\ 1 & x_2^2 & 0 \\ 0 & 1 & x_2^3 \end{pmatrix}=\begin{pmatrix} x_1^1x_2^1 & 0 & 0 \\ x_1^2+x_2^1 & x_1^2x_2^2 & 0 \\ 1 & x_1^3+x_2^2 & x_1^3x_2^3 \end{pmatrix}. \tag{1.3}$$

Using (1.2), the reader may easily verify that each of the five non-trivial entries of this matrix is invariant under e_1. Theorem 1.1 asserts that every rational invariant of e_1 has a unique expression as a rational function in these five polynomials.

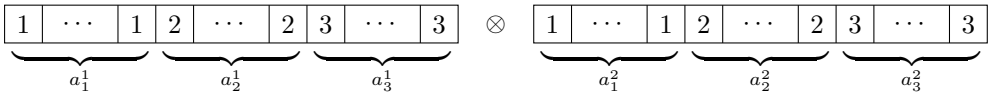
In fact, Theorem 1.1, together with a result of Lam and the third-named author, implies the stronger result that the non-trivial entries of $M(\mathbf{x})$ generate the ring of polynomial invariants of the e_i (see the last paragraph of §6.2).

1.2. **Crystal operators.** Let $\mathrm{Sym}^L\mathbb{C}^m$ denote the L-fold symmetric power of the vector representation of $\mathrm{GL}_m(\mathbb{C})$. This representation has a basis consisting of the degree L monomials $v_1^{a_1}\cdots v_m^{a_m}$, where $\{v_1,\ldots,v_m\}$ is a fixed basis of $\mathbb{C}^m$. The set of all exponent vectors $\mathbf{a}=(a_1,\ldots,a_m)\in(\mathbb{Z}_{\ge 0})^m$ labels a basis of the representation $\mathrm{Sym}\,\mathbb{C}^m=\bigoplus_{L\ge 0}\mathrm{Sym}^L\mathbb{C}^m$. One can also represent the monomial $v_1^{a_1}\cdots v_m^{a_m}$ as a semistandard Young tableau consisting of a single row, in which the number i appears a_i times.

Now consider the n-fold tensor product $(\mathrm{Sym}\,\mathbb{C}^m)^{\otimes n}$. A basis of this representation is labeled by $m\times n$ matrices of nonnegative integers $\mathbf{a}=(a_i^j)$, where the jth column $\mathbf{a}^j=(a_1^j,\ldots,a_m^j)$ represents a basis vector

of the jth tensor factor. The set of such matrices is endowed with GL_m-crystal operators $\widetilde{e}_1, \dots, \widetilde{e}_{m-1}$, where $\widetilde{e}_i$ modifies the entries in rows i and $i+1$ (see, *e.g.*, [Kas02, BS17]).

Example 1.2. Let $\mathbf{a} = \begin{pmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \\ a_3^1 & a_3^2 \end{pmatrix}$ represent a basis element of the GL_3-representation $\operatorname{Sym} \mathbb{C}^3 \otimes \operatorname{Sym} \mathbb{C}^3$. In tableau notation, $\mathbf{a}$ corresponds to



The GL_3-crystal operator $\widetilde{e}_1$ acts on $\mathbf{a}$ by the following piecewise-linear formula:

$$\widetilde{e}_1 \colon \begin{pmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \\ a_3^1 & a_3^2 \end{pmatrix} \mapsto \begin{pmatrix} a_1^1 + \widetilde{c}_1 & a_1^2 + \widetilde{c}_2 \\ a_2^1 - \widetilde{c}_1 & a_2^2 - \widetilde{c}_2 \\ a_3^1 & a_3^2 \end{pmatrix},$$

where

$$\widetilde{c}_1 = \min(1 + a_1^2, a_2^1) - \min(a_1^2, a_2^1), \qquad \widetilde{c}_2 = \min(1 + a_1^2, 1 + a_2^1) - \min(1 + a_1^2, a_2^1).$$

The reader will observe that the piecewise-linear formula in Example 1.2 is obtained from (1.1) by tropicalizing—replacing the operations $(+, \cdot, \div)$ with the operations $(\min, +, -)$—and setting $c = 1$ (we also rename the variables to emphasize that they now represent integers rather than nonzero complex numbers). This is no coincidence, as the geometric crystal operators were defined by Berenstein and Kazhdan (in the context of an arbitrary reductive algebraic group) so that they tropicalize to piecewise-linear formulas for the Kashiwara operators acting on the crystal bases of finite-dimensional representations [BK00, BK07] (see §2.5 for more details).

1.3. **Commuting crystal actions, RSK, and gRSK.** One way to prove the first and second fundamental theorems of invariant theory is to consider the GL_n- and GL_m-actions on $m \times n$ matrices simultaneously. These two actions commute, so one can ask how to decompose the polynomial ring $\mathbb{C}[x_i^j]$ into $\mathrm{GL}_n \times \mathrm{GL}_m$-irreducible representations. Using this decomposition (and some knowledge of the representation theory of GL_m), one can read off the first fundamental theorem by looking at the isotypic components in which the GL_m-irreducible is trivial as an SL_m-module. The second fundamental theorem can also be deduced from representation theory; for details of this approach to invariant theory, see [Ful97, §9.2].

Our proof of Theorem 1.1 proceeds in a similar spirit. The decomposition of $\mathbb{C}[x_i^j]$ into $\mathrm{GL}_n \times \mathrm{GL}_m$-irreducible representations has a beautiful analogue at the level of crystal bases. Just as there are GL_m-crystal operators acting on adjacent rows of a matrix of nonnegative integers, there are GL_n-crystal operators acting on adjacent columns. Although it is not obvious from the combinatorial definition, it turns out that these two sets of crystal operators commute [Las03, vL06] (this can be viewed as a crystal version of $(\mathrm{GL}_n, \mathrm{GL}_m)$-Howe duality [How89]). The connected components of the resulting $\mathrm{GL}_n \times \mathrm{GL}_m$-crystal, which correspond to the irreducible components of $\mathbb{C}[x_i^j]$ under the usual $\mathrm{GL}_n \times \mathrm{GL}_m$-action, are determined by the Robinson–Schensted–Knuth (RSK) correspondence (see, *e.g.,* [BS17, Ch. 9.1]). RSK is a well-known bijection between $m \times n$ matrices of nonnegative integers and pairs (P, Q) of semistandard Young tableaux of the same shape, where the entries of P lie in $\{1, \dots, n\}$, and the entries of Q lie in $\{1, \dots, m\}$ [Sch61, Knu70].

A geometric lifting of RSK was introduced by Kirillov [Kir01] and extensively developed by Noumi and Yamada [NY04]. This map was originally called the "tropical RSK correspondence," but following the convention of more recent work on this map, we will call it the *geometric RSK correspondence* and abbreviate it gRSK. Geometric RSK is a birational automorphism of the variety of $m \times n$ matrices over $\mathbb{C}^*$. For example, when $m = 2$ and $n = 3$, gRSK has the form

$$\mathrm{gRSK} \colon \begin{pmatrix} x_1^1 & x_1^2 & x_1^3 \\ x_2^1 & x_2^2 & x_2^3 \end{pmatrix} \mapsto \begin{pmatrix} z_{2,2} & z_{2,3} = z'_{2,2} & z'_{1,1} \\ z_{1,1} & z_{1,2} & z_{1,3} = z'_{1,2} \end{pmatrix},$$

where $z_{a,b}$ and $z'_{a,b}$ are rational functions in the x_i^j with positive integer coefficients. The rational functions $z_{a,b}$ and $z'_{a,b}$ tropicalize to piecewise-linear formulas for the semistandard tableaux P and Q (more precisely,

for the entries of the corresponding Gelfand–Tsetlin patterns) associated to a matrix $\mathbf{a}$ by RSK. In particular, the rational functions $z_{k,n} = z'_{k,m}$ tropicalize to formulas for the entries of the common shape of P and Q. We refer to the arrays consisting of the $z_{a,b}$ and $z'_{a,b}$ as the P-pattern and Q-pattern, respectively.

In addition to the GL_m-geometric crystal operators e_i acting on adjacent rows of $\mathbf{x} \in \mathrm{Mat}_{m\times n}(\mathbb{C}^*)$, there are GL_n-geometric crystal operators acting on adjacent columns of $\mathbf{x}$, which we denote by $\overline{e}_j$. These two families of actions are known to commute with each other [LP13a], so the variety $\mathrm{Mat}_{m\times n}(\mathbb{C}^*)$ has the structure of a $\mathrm{GL}_n \times \mathrm{GL}_m$-geometric crystal, which we denote by X^{Mat}. Berenstein and Kazhdan's theory also provides a geometric lifting of the crystal operators on Gelfand–Tsetlin patterns. By viewing an $m \times n$ matrix as a P-pattern and a Q-pattern glued together along a common shape, we obtain a different $\mathrm{GL}_n \times \mathrm{GL}_m$-geometric crystal structure on $\mathrm{Mat}_{m\times n}(\mathbb{C}^*)$, in which the GL_n- (resp., GL_m-) operators act on the P- (resp., Q-) pattern. We denote this geometric crystal by X^{GT}.

We prove in Theorem 4.10 that

$$\mathrm{gRSK}\colon X^{\mathrm{Mat}} \to X^{\mathrm{GT}}$$

is an isomorphism of $\mathrm{GL}_n \times \mathrm{GL}_m$-geometric crystals. Since the GL_m-geometric crystal operators on X^{GT} act only on the Q-pattern, this result implies that the entries $z_{i,j}$ of the P-pattern are invariant under the GL_m-geometric crystal operators. It is straightforward to show that the rational functions $z_{i,j}$ generate the same subfield of $\mathbb{C}(x_i^j)$ as the entries of the matrix $M(\mathbf{x})$ in Theorem 1.1. The key to proving that there are no other invariants is Theorem 5.6, which says that the action of the GL_m-geometric crystal operators on the Q-pattern has a dense orbit for each fixed shape. This result, in turn, relies on work of Kanakubo–Nakashima [KN19] and Kashiwara–Nakashima–Okado [KNO10].

By interchanging the roles of m and n in Theorem 1.1, one sees that the subfield $\mathrm{Inv}_{\overline{e}} \subset \mathbb{C}(x_i^j)$ fixed by the GL_n-geometric crystal operators is generated by the entries of the $m \times m$ matrix $M(\mathbf{x}^t)$, where $\mathbf{x}^t$ is the transpose of $\mathbf{x}$. We prove in Corollary 5.4 that the subfield fixed by both the GL_n- and GL_m-geometric crystal operators, $\mathrm{Inv}_e \cap \mathrm{Inv}_{\overline{e}}$, is generated by $\min(m, n)$ polynomials which are generalizations of Schur polynomials of rectangular shape. These polynomials are products of the rational functions $z_{k,n} = z'_{k,m}$ that describe the common shape of the P- and Q-patterns.

1.4. **Birational Weyl group actions.** Berenstein and Kazhdan's primary motivation in developing the theory of geometric crystals was to construct birational actions of Weyl groups. In the case of the GL_m-geometric crystal on $\mathrm{Mat}_{m\times n}(\mathbb{C}^*)$ described in §1.1, the resulting action of the Weyl group S_m is generated by the action of the *geometric* (or *birational*) *R-matrix* on adjacent rows. The geometric R-matrix was introduced by Yamada [Yam01] as a "de-tropicalization" of the *combinatorial R-matrix*, a map which comes from the theory of affine crystals [KKM+92, HKO+99, HKO+02b], and plays a central role in the study of solvable lattice models and the box-ball system [HHI+01, HKO+02a, HKT00, HKT01, KSY07, LS19, Tak05, TNS99, TS90, Yam01, Yam04].

The subfield $\mathrm{Inv}_R \subset \mathbb{C}(x_i^j)$ consisting of geometric R-matrix invariants has been previously studied [LP12, LP13b, LPS18, LP11, Lam12]. In particular, it is known that this field (and, in fact, its subring of polynomial invariants) is generated by a set of mn algebraically independent polynomials called *loop elementary symmetric functions*. When $n = 1$, these polynomials are simply the elementary symmetric polynomials in m variables; in general, there are n loop elementary symmetric functions of degree k for $k = 1, \ldots, m$, each of which can be viewed as a generalization of the kth elementary symmetric polynomial in m variables.

The Weyl group action on a geometric crystal is generated by the geometric crystal operators e_i^c for specific values of c, so Inv_e is a subfield of Inv_R. The generators of Inv_e described in Theorem 1.1 are in fact a subset of the loop elementary symmetric polynomials. For example, when $m = 2$ and $n = 3$, five of the six loop elementary symmetric polynomials appear as entries of $M(\mathbf{x})$ in (1.3), and the remaining loop elementary symmetric polynomial is $x_1^1 + x_2^3$. We emphasize that our proof of Theorem 1.1 does not make any reference to geometric R-matrix invariants. On the other hand, the stronger result that the entries of $M(\mathbf{x})$ generate the polynomial subring of Inv_e is deduced from the result that the loop elementary symmetric functions generate the polynomial subring of Inv_R.

1.5. **Future work.** In the sequel to this paper [BFPS], we study the fields $\mathrm{Inv}_R \cap \mathrm{Inv}_{\overline{e}}, \mathrm{Inv}_{\overline{R}} \cap \mathrm{Inv}_e$, and $\mathrm{Inv}_R \cap \mathrm{Inv}_{\overline{R}}$, where $\mathrm{Inv}_{\overline{R}}$ is the field of invariants of the S_n-action generated by applying geometric R-matrices to adjacent columns of $\mathbf{x}$. In particular, we give conjectural algebraically independent generating sets for each field. The field $\mathrm{Inv}_R \cap \mathrm{Inv}_{\overline{e}}$ contains two important functions: the *central charge* [BK07] and the *geometric*

energy function [KKM+92, HKO+99, HKO+02b, LP13b]. One of our motivations for this project was to find formulas for these functions which simultaneously exhibit their R- and $\overline{e}$-invariance, and we succeed in expressing both functions as polynomials in the (conjectural) generators of $\mathrm{Inv}_R \cap \mathrm{Inv}_{\overline{e}}$. We also obtain a new derivation of a piecewise-linear formula for cocharge due to Kirillov and Berenstein [KB95].

In a different direction, we propose the following general problem.

Problem 1.3. *Given geometric crystals $X_1, \ldots, X_n$ associated to a fixed reductive group, describe the invariants of the geometric crystal operators (or the birational Weyl group action) as a subfield of the fraction field $\mathbb{C}(X_1 \times \cdots \times X_n)$.*

The existence of commuting type A crystal structures on $m \times n$ matrices is a crystal interpretation of Howe duality [How89] for the dual reductive pair $(\mathrm{GL}_n, \mathrm{GL}_m)$, which states that

$$\mathrm{Sym}(\mathbb{C}^n \boxtimes \mathbb{C}^m) \cong \bigoplus_{\lambda:\, \ell(\lambda) \leq \min(m,n)} V_{\mathrm{GL}_n}(\lambda) \boxtimes V_{\mathrm{GL}_m}(\lambda)$$

as $\mathrm{GL}_n \times \mathrm{GL}_m$-representations. (Here $V_{\mathrm{GL}_k}(\lambda)$ is the irreducible highest weight representation of GL_k with highest weight λ, and $\ell(\lambda)$ is the number of parts in λ.) It may be fruitful to investigate products of geometric crystals that correspond to other examples of Howe duality or skew Howe duality.

1.6. **Outline of paper.** In §2, we review fundamental definitions and results in the theory of geometric crystals, and we explain how combinatorial crystals are obtained from decorated geometric crystals by tropicalization. We illustrate the theory with the example of the basic geometric crystal, the geometric analogue of the crystal structure on one-row tableaux.

In §3, we construct a geometric crystal structure on the geometric analogue of a Gelfand–Tsetlin pattern. This is essentially a special case of a general construction due to Berenstein and Kazhdan [BK07], but we derive many of its properties in an elementary manner, using only the Lindström/Gessel–Viennot Lemma. In particular, in §3.3 we obtain explicit formulas for the geometric crystal operators and decoration, and show that they tropicalize to a piecewise-linear description of the usual crystal structure on Gelfand–Tsetlin patterns.

In §4, we present two definitions of the geometric RSK correspondence, one due to Noumi–Yamada [NY04], and one due to O'Connell–Seppäläinen–Zygouras [OSZ14]. Using these two definitions, we derive several important properties of gRSK, including the new result that gRSK is an isomorphism of geometric crystals (Theorem 4.10). Our presentation is self-contained, aside from the fact that we refer to [OSZ14] for the proof that the two definitions agree. We caution the reader that our version of gRSK differs from the usual one by a rational involution that corresponds to interchanging min and max (see §4.4). The usual version is not compatible with the geometric crystal structures.

As an application of the results in §§3-4, we show in §4.6 that for m-fold products of the basic GL_n-geometric crystal, Berenstein and Kazhdan's central charge [BK07] is equal to the decoration of the Q-pattern (plus an extra term when $m = n$). This implies, in particular, that the central charge is positive.

In §5, we prove Theorem 1.1, and deduce the result about generators of $\mathrm{Inv}_e \cap \mathrm{Inv}_{\overline{e}}$ as a corollary.

In §6, we show that the Weyl group action on products of the basic geometric crystal agrees with the geometric R-matrix. We then review results of Lam and the third-named author about the field (and ring) of R-invariants, and deduce that our generators of Inv_e generate the subring of polynomial invariants.

Remark 1.4. The definitions and results in §§2.3-2.5, 3.3, and 4.5-4.6 are not needed for the proof of our main result. We have included these sections in the hope that this paper will be a convenient reference for readers interested in learning the basics of type A geometric crystals and geometric RSK. In addition, some of the results in these sections (especially those concerning the central charge) will be used in [BFPS].

Notation. For integers a, b, we write $[a, b]$ for the interval $\{k \in \mathbb{Z} \mid a \leq k \leq b\}$. We often abbreviate $[1, b]$ to $[b]$.

Acknowledgements. We thank Thomas Lam, during discussions with whom some of the ideas in this paper were born. We thank Jake Levinson for his help with the proof of Lemma 5.7.

This work benefited from computations using SageMath [Dev21, SCc08]. B.B. was supported in part by NSF grant DMS-2101392. G.F. was supported in part by the Canada Research Chairs program. P.P. was supported in part by NSF grant DMS-1949896. T.S. was supported in part by Grant-in-Aid for JSPS Fellows

21F51028. This work was partly supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849).

2. Background on geometric and unipotent crystals

In §§2.1-2.2, we recall the definitions of geometric and unipotent crystals in the case of GL_m. These definitions are due to Berenstein and Kazhdan [BK00, BK07]. In §§2.3-2.5, we describe how geometric crystals give rise to the combinatorial crystals introduced by Kashiwara [Kas90, Kas91] via the tropicalization functor.

2.1. **Geometric crystals.** Let $T = (\mathbb{C}^*)^m$ be the maximal torus of GL_m. For $i \in [m-1]$, let $\alpha_i \colon T \to \mathbb{C}^*$ and $\alpha_i^\vee \colon \mathbb{C}^* \to T$ denote the character and co-character

$$\alpha_i(x_1, \ldots, x_m) = \frac{x_i}{x_{i+1}} \qquad \text{and} \qquad \alpha_i^\vee(c) = (1, \ldots, c, c^{-1}, \ldots, 1),$$

where c and c^{-1} are in positions i and $i+1$.

Definition 2.1. A *GL_m-geometric crystal* is an irreducible complex algebraic variety X, together with a rational map $\gamma \colon X \to T$, and for each $i \in [m-1]$, rational functions[2] $\varepsilon_i, \varphi_i \colon X \to \mathbb{C}$ and a rational $\mathbb{C}^*$-action $e_i : \mathbb{C}^* \times X \to X$. We write $e_i^c(x)$ for the action of $c \in \mathbb{C}^*$ on $x \in X$. These maps must satisfy the following identities (when they are defined):

(1) $\dfrac{\varphi_i(x)}{\varepsilon_i(x)} = \alpha_i\big(\gamma(x)\big)$;

(2) $\gamma\big(e_i^c(x)\big) = \alpha_i^\vee(c)\gamma(x), \qquad \varepsilon_i\big(e_i^c(x)\big) = c^{-1}\varepsilon_i(x), \qquad \varphi_i\big(e_i^c(x)\big) = c\varphi_i(x)$;

(3) (a) if $|i-j| > 1$, then $e_i^c e_j^{c'} = e_j^{c'} e_i^c$;

(b) if $|i-j| = 1$, then $e_i^c e_j^{cc'} e_i^{c'} = e_j^{c'} e_i^{cc'} e_j^c$.

The maps e_i^c are called *geometric crystal operators*.

An *isomorphism* of GL_m-geometric crystals is a birational isomorphism of underlying varieties that commutes with the geometric crystal structures.

Remark 2.2. Axioms (1) and (2) are analogues of the axioms of a combinatorial crystal (see §2.3). The purpose of axiom (3) is to guarantee that the maps $s_i(\mathbf{x}) = e_i^{1/\alpha_i(\gamma(\mathbf{x}))}$ generate a birational action of the Weyl group S_m on X. This action is discussed further in §6.

The geometric analogue of the GL_m-representation $\operatorname{Sym} \mathbb{C}^m$ is the *basic geometric crystal* X_m introduced in [KOTY03], which has underlying variety $X_m = (\mathbb{C}^*)^m$, and the following geometric crystal structure: for $\mathbf{x} = (x_1, \ldots, x_m) \in X_m$,

$$\gamma(\mathbf{x}) = \mathbf{x}, \qquad \varepsilon_i(\mathbf{x}) = x_{i+1}, \qquad \varphi_i(\mathbf{x}) = x_i, \qquad e_i^c(\mathbf{x}) = (x_1, \ldots, cx_i, c^{-1}x_{i+1}, \ldots, x_m). \tag{2.1}$$

Given two geometric crystals X and X', Berenstein and Kazhdan [BK00] define

$$\gamma(x, x') = \gamma(x)\gamma(x'), \quad \varepsilon_i(x, x') = \frac{\varepsilon_i(x)\varepsilon_i(x')}{\varepsilon_i(x) + \varphi_i(x')}, \quad \varphi_i(x, x') = \frac{\varphi_i(x)\varphi_i(x')}{\varepsilon_i(x) + \varphi_i(x')}, \tag{2.2}$$

and

$$e_i^c(x, x') = \big(e_i^{c^+}(x), e_i^{c/c^+}(x')\big), \qquad \text{where } c^+ = \frac{c\varphi_i(x') + \varepsilon_i(x)}{\varphi_i(x') + \varepsilon_i(x)}. \tag{2.3}$$

This definition is associative, and $X \times X'$ satisfies the first two axioms of Definition 2.1. In order to guarantee that the third axiom is satisfied, however, additional structure on X and X' is needed. This additional structure is introduced in §2.2, and plays a crucial role throughout the paper.

We end this section with explicit formulas for the geometric crystal structure on $(X_m)^n$. For $j \in [0, n]$, $c \in \mathbb{C}^*$, and $x = (x^1, \ldots, x^n), y = (y^1, \ldots, y^n) \in (\mathbb{C}^*)^n$, define

$$\sigma^j(x, y; c) = \sum_{r=1}^{n} c^{\mathbb{1}_{r \le j}} y^1 \cdots y^{r-1} x^{r+1} \cdots x^n, \tag{2.4}$$

where $\mathbb{1}_{r \le j}$ is the indicator function for $r \le j$. Also define $\sigma(x, y) = \sigma^j(x, y; 1)$ (this is independent of j).

[2] Berenstein and Kazhdan's definition [BK00, BK07] differs from ours by replacing ε_i, φ_i with $1/\varepsilon_i, 1/\varphi_i$. Our convention has the advantage that ε_i, φ_i tropicalize to the corresponding functions $\widetilde{\varepsilon}_i, \widetilde{\varphi}_i$ on combinatorial crystals, rather than their negatives.

Lemma 2.3. *Suppose* $\mathbf{x} = (\mathbf{x}^1, \dots, \mathbf{x}^n) \in (X_m)^n$, *where* $\mathbf{x}^j = (x_1^j, \dots, x_m^j)$. *For* $k \in [m]$, *let* $\mathbf{x}_k = (x_k^1, \dots, x_k^n)$ *be the vector of* k*th coordinates of the* $\mathbf{x}^j$, *and let* $\pi_k(\mathbf{x}) = \prod_{j=1}^n x_k^j$. *The* GL_m*-geometric crystal structure on* $(X_m)^n$ *is given by the following formulas:*

$$\gamma(\mathbf{x}) = (\pi_1(\mathbf{x}), \dots, \pi_m(\mathbf{x})), \qquad \varepsilon_i(\mathbf{x}) = \frac{\pi_{i+1}(\mathbf{x})}{\sigma(\mathbf{x}_i, \mathbf{x}_{i+1})}, \qquad \varphi_i(\mathbf{x}) = \frac{\pi_i(\mathbf{x})}{\sigma(\mathbf{x}_i, \mathbf{x}_{i+1})},$$

$$e_i^c(\mathbf{x}) = (\mathbf{x}'), \quad \text{where } (x')_k^j = \begin{cases} \dfrac{\sigma^j(\mathbf{x}_i, \mathbf{x}_{i+1}; c)}{\sigma^{j-1}(\mathbf{x}_i, \mathbf{x}_{i+1}; c)} x_i^j & \text{if } k = i, \\ \dfrac{\sigma^{j-1}(\mathbf{x}_i, \mathbf{x}_{i+1}; c)}{\sigma^j(\mathbf{x}_i, \mathbf{x}_{i+1}; c)} x_{i+1}^j & \text{if } k = i+1, \\ x_k^j & \text{otherwise.} \end{cases}$$

The proof appears in Appendix A.

2.2. **Unipotent crystals.** Let $B^-, U \subset \mathrm{GL}_m(\mathbb{C})$ be the subgroups of lower triangular matrices and upper uni-triangular matrices, respectively. The subgroup U is generated by the *(upper) Chevalley generators*

$$x_i(a) = I + aE_{i,i+1} \quad (i \in [m-1], a \in \mathbb{C}),$$

where $E_{i,j}$ is the matrix unit with 1 in position (i,j) and 0's elsewhere. Define a rational action of U on B^- by $u.b = b'$ if $ub = b'u'$ for some $u' \in U, b' \in B^-$. If ub does not have such a factorization, then the action of u on b is undefined. It is easy to see that the action of a Chevalley generator on $M \in B^-$ is given by

$$x_i(a).M = x_i(a) \cdot M \cdot x_i\left(\frac{-aM_{i+1,i+1}}{M_{i,i} + aM_{i+1,i}}\right). \tag{2.5}$$

Definition 2.4. A *U-variety* is an irreducible complex algebraic variety X, together with a rational U-action $U \times X \to X$. A *morphism* of U-varieties is a rational map which commutes with the U-actions.

We view B^- as a U-variety with respect to the rational action defined above.

Definition 2.5. A *GL_m-unipotent crystal* is a pair (X, ι), where X is a U-variety, and $\iota\colon X \to B^-$ is a morphism of U-varieties such that for each $i \in [m-1]$, the function $\iota(x)_{i+1,i}$ is not identically zero. (We will omit GL_m from the name when there is no danger of confusion.)

Example 2.6. The pair (B^-, Id) is a unipotent crystal. More generally, define

$$(B^-)^{\leq n} = \{A \in \mathrm{GL}_m \mid A_{ij} = 0 \text{ if } i < j \text{ or } i - j > n, \text{ and } A_{ij} = 1 \text{ if } i - j = n\}$$

for $n \geq 1$ (note that $(B^-)^{\leq n} = B^-$ if $n \geq m$). It is clear that the rational action of U on B^- restricts to a rational action on $(B^-)^{\leq n}$, so the pair $((B^-)^{\leq n}, \iota)$ is a unipotent crystal, where ι is the inclusion of $(B^-)^{\leq n}$ into B.

Given unipotent crystals (X, ι) and (X', ι'), endow the product $X \times X'$ with the rational U-action

$$u.(x, x') = (u.x, u'.x'),$$

where $u' \in U$ is determined by $u\iota(x) = b'u'$ for some $b' \in B^-$ (if $u\iota(x)$ does not have such a factorization, then $u.(x,x')$ is undefined). Also define $\iota * \iota'\colon X \times X' \to B^-$ by

$$(x, x') \mapsto \iota(x)\iota'(x').$$

Berenstein and Kazhdan proved that the pair $(X \times X', \iota * \iota')$ is a unipotent crystal, and that this product of unipotent crystals is associative [BK00, Thm. 3.3, Prop. 3.4].

Theorem 2.7 ([BK00, Thm. 3.8, Lem. 3.9])**.** *Let* (X, ι) *be a* GL_m*-unipotent crystal. Fix* $i \in [m-1]$. *For* $x \in X$, *set* $M = \iota(x)$, *and define*

$$\gamma(x) = (M_{1,1}, \dots, M_{m,m}), \qquad \varepsilon_i(x) = \frac{M_{i+1,i+1}}{M_{i+1,i}}, \qquad \varphi_i(x) = \frac{M_{i,i}}{M_{i+1,i}},$$

$$e_i^c(x) = x_i((c-1)\varphi_i(x)).x \qquad \textit{(here . is the rational action of } U \textit{ on } X\textit{)}.$$

(1) *These maps make* X *into a* GL_m*-geometric crystal. We say that this geometric crystal is* induced *from the unipotent crystal* (X, ι).

(2) *If* (X', ι') *is another* GL_m*-unipotent crystal, then the geometric crystal induced from the product* $(X \times X', \iota * \iota')$ *is the product of the geometric crystals induced from* (X, ι) *and* (X', ι').

Part (2) implies that we need not worry about the product of geometric crystals failing to be a geometric crystal, as long as we restrict our attention to geometric crystals induced from unipotent crystals. This is not a serious restriction in practice, as to the best of our knowledge, all examples of geometric crystals appearing in the literature are induced from unipotent crystals. In addition, most computations with geometric crystals are done at the level of unipotent crystals, largely because of the following property.

Corollary 2.8. *If* X *is the geometric crystal induced from a unipotent crystal* (X, ι)*, then*

$$\iota(e_i^c(x)) = x_i\left((c-1)\varphi_i(x)\right) \cdot \iota(x) \cdot x_i\left((c^{-1}-1)\varepsilon_i(x)\right).$$

Proof. Since ι commutes with the U-actions on X and B^-, $\iota(e_i^c(x))$ is equal to the action of $x_i((c-1)\varphi_i(x))$ on $\iota(x)$. Using (2.5) and the definitions of $\varphi_i(x)$ and $\varepsilon_i(x)$, one obtains the result. □

We now explain how the basic geometric crystal X_m arises from a unipotent crystal. Identify the variety $X_m = (\mathbb{C}^*)^m$ with $(B^-)^{\leq 1} \subset \mathrm{GL}_m$ via the map

$$(x_1, \ldots, x_m) \mapsto W(x_1, \ldots, x_m) = \begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & x_2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & x_3 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & x_{m-1} & 0 \\ 0 & 0 & 0 & \cdots & 1 & x_m \end{pmatrix}.$$

By Example 2.6(1), this identification makes X_m into a unipotent crystal, and one can easily verify that the geometric crystal structure on X_m defined in the previous section is induced from this unipotent crystal. For example, the following calculation shows that the geometric crystal operator $e_2^c \colon X_4 \to X_4$ is induced from the action of e_2^c on the unipotent crystal $(B^-)^{\leq 1}$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & (c-1)x_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 1 & x_2 & 0 & 0 \\ 0 & 1 & x_3 & 0 \\ 0 & 0 & 1 & x_4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & (c^{-1}-1)x_3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 1 & cx_2 & 0 & 0 \\ 0 & 1 & c^{-1}x_3 & 0 \\ 0 & 0 & 1 & x_4 \end{pmatrix}.$$

Given $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^n) \in (X_m)^n$, define

$$M(\mathbf{x}) = W(\mathbf{x}^1) \cdots W(\mathbf{x}^n).$$

Theorem 2.7(2) and Corollary 2.8 imply that $\gamma \colon (X_m)^n \to T$ and $\varepsilon_i, \varphi_i \colon (X_m)^n \to \mathbb{C}$ are computed by

$$\gamma(\mathbf{x}) = (M(\mathbf{x})_{1,1}, \ldots, M(\mathbf{x})_{m,m}), \qquad \varepsilon_i(\mathbf{x}) = \frac{M(\mathbf{x})_{i+1,i+1}}{M(\mathbf{x})_{i+1,i}}, \qquad \varphi_i(\mathbf{x}) = \frac{M(\mathbf{x})_{i,i}}{M(\mathbf{x})_{i+1,i}}, \tag{2.6}$$

and that the geometric crystal operators on $(X_m)^n$ satisfy

$$M(e_i^c(\mathbf{x})) = x_i\big((c-1)\varphi_i(\mathbf{x})\big) \cdot M(\mathbf{x}) \cdot x_i\big((c^{-1}-1)\varepsilon_i(\mathbf{x})\big). \tag{2.7}$$

2.3. **Combinatorial crystals.** Let $\Lambda = \mathbb{Z}^m$ and $\Lambda^\vee = \mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z})$ denote the weight and coweight lattices associated to GL_m. For $i \in [m-1]$, define the simple root $\widetilde{\alpha}_i \in \Lambda$ and simple coroot $\widetilde{\alpha}_i^\vee \in \Lambda^\vee$ by

$$\widetilde{\alpha}_i = v_i - v_{i+1}, \qquad \widetilde{\alpha}_i^\vee(a_1 \ldots, a_m) = a_i - a_{i+1},$$

where v_i is the ith standard basis vector of $\mathbb{Z}^n$ and a_i is the corresponding dual basis vector.

Definition 2.9. A GL_m*-combinatorial crystal* (or *abstract crystal*) consists of a set $\mathcal{B}$, a *weight map* $\widetilde{\gamma} \colon \mathcal{B} \to \Lambda$, and for each $i \in [m-1]$, functions $\widetilde{\varepsilon}_i, \widetilde{\varphi}_i \colon \mathcal{B} \to \mathbb{Z}$ and *crystal operators* $\widetilde{e}_i, \widetilde{f}_i \colon \mathcal{B} \to \mathcal{B} \sqcup \{\mathbf{0}\}$. Here the symbol $\mathbf{0}$ represents an element which is not in $\mathcal{B}$, and one says that $\widetilde{e}_i(b)$ is *undefined* if $\widetilde{e}_i(b) = \mathbf{0}$. These maps must satisfy the following properties:

(1) For all $b \in \mathcal{B}$, $\widetilde{\varphi}_i(b) = \widetilde{\alpha}_i^\vee(\widetilde{\gamma}(b)) + \widetilde{\varepsilon}_i(b)$;

(2) If $b, b' \in \mathcal{B}$, then $\widetilde{e}_i(b) = b'$ if and only if $\widetilde{f}_i(b') = b$. In this case, one has

$$\widetilde{\gamma}(b') = \widetilde{\gamma}(b) + \widetilde{\alpha}_i, \qquad \widetilde{\varepsilon}_i(b') = \widetilde{\varepsilon}_i(b) - 1, \qquad \widetilde{\varphi}_i(b') = \widetilde{\varphi}_i(b) + 1.$$

The combinatorial crystal $\mathcal{B}$ is said to be *free* if the crystal operators $\widetilde{e}_i$ and $\widetilde{f}_i$ are mutually inverse bijections from $\mathcal{B} \to \mathcal{B}$, and *regular* (or *seminormal*) if for all $b \in \mathcal{B}$,

$$\widetilde{\varepsilon}_i(b) = \max\{k > 0 \mid \widetilde{e}_i^k(b) \neq \mathbf{0}\} \qquad \text{and} \qquad \widetilde{\varphi}_i(b) = \max\{k > 0 \mid \widetilde{f}_i^k(b) \neq \mathbf{0}\}. \tag{2.8}$$

Example 2.10. Let $\mathcal{B} = (\mathbb{Z}_{\geq 0})^m$. For $\mathbf{a} = (a_1, \ldots, a_m) \in \mathcal{B}$, define

$$\widetilde{\gamma}(\mathbf{a}) = \mathbf{a}, \qquad \widetilde{\varepsilon}_i(\mathbf{a}) = a_{i+1}, \qquad \widetilde{\varphi}_i(\mathbf{a}) = a_i.$$

Define $\widetilde{e}_i(\mathbf{a}) = (a_1, \ldots, a_i + 1, a_{i+1} - 1, \ldots, a_m)$ if $a_{i+1} > 0$, and otherwise $\widetilde{e}_i(\mathbf{a}) = \mathbf{0}$. Define $\widetilde{f}_i(\mathbf{a}) = (a_1, \ldots, a_i - 1, a_{i+1} + 1, \ldots, a_m)$ if $a_i > 0$, and otherwise $\widetilde{f}_i(\mathbf{a}) = \mathbf{0}$. These maps make $\mathcal{B}$ into a regular GL_m-combinatorial crystal. Moreover, each of the subsets

$$\mathcal{B}^L = \{(a_1, \ldots, a_m) \in (\mathbb{Z}_{\geq 0})^m \mid a_1 + \cdots + a_m = L\}$$

is a finite regular combinatorial crystal.

Kashiwara [Kas90, Kas91] proved that every finite-dimensional GL_m-representation gives rise to a GL_m-combinatorial crystal whose underlying set $\mathcal{B}$ is in bijection with a basis of the representation. The crystal operators $\widetilde{e}_i$ and $\widetilde{f}_i$ are a "combinatorial approximation" of the action of the Chevalley generators of the Lie algebra $\mathfrak{gl}_m$ on this basis. The map $\widetilde{\gamma}$ encodes the weight of a basis element with respect to the action of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_m$. The functions $\widetilde{\varepsilon}_i, \widetilde{\varphi}_i$ are defined by (2.8) (so regularity is automatic). For example, the finite crystal $\mathcal{B}^L$ defined in Example 2.10 arises from the Lth symmetric power of the vector representation of GL_m.

Given two GL_m-combinatorial crystals $\mathcal{B}$ and $\mathcal{B}'$, Kashiwara defined the *tensor product* $\mathcal{B} \otimes \mathcal{B}'$ to be the crystal on the set $\mathcal{B} \times \mathcal{B}'$, with

$$\widetilde{\gamma}(b, b') = \widetilde{\gamma}(b) + \widetilde{\gamma}(b'),$$

$$\widetilde{\varepsilon}_i(b, b') = \widetilde{\varepsilon}_i(b) + \widetilde{\varepsilon}_i(b') - \min(\widetilde{\varepsilon}_i(b), \widetilde{\varphi}_i(b')), \qquad \widetilde{\varphi}_i(b, b') = \widetilde{\varphi}_i(b) + \widetilde{\varphi}_i(b') - \min(\widetilde{\varepsilon}_i(b), \widetilde{\varphi}_i(b')),$$

$$\widetilde{e}_i(b, b') = \begin{cases} (\widetilde{e}_i(b), b') & \text{if } \widetilde{\varepsilon}_i(b) > \widetilde{\varphi}_i(b') \\ (b, \widetilde{e}_i(b')) & \text{if } \widetilde{\varepsilon}_i(b) \leq \widetilde{\varphi}_i(b') \end{cases}, \qquad \widetilde{f}_i(b, b') = \begin{cases} (\widetilde{f}_i(b), b') & \text{if } \widetilde{\varepsilon}_i(b) \geq \widetilde{\varphi}_i(b') \\ (b, \widetilde{f}_i(b')) & \text{if } \widetilde{\varepsilon}_i(b) < \widetilde{\varphi}_i(b') \end{cases}.$$

This definition has the property that if $\mathcal{B}, \mathcal{B}'$ arise from GL_m-representations V, V', then $\mathcal{B} \otimes \mathcal{B}'$ is the crystal arising from the tensor product $V \otimes V'$. We follow the tensor product convention of [BS17], which is the opposite of Kashiwara's convention [Kas90, Kas91].

2.4. **Tropicalization.** Let $x_1, \ldots, x_d$ be variables. For $I = (i_1, \ldots, i_d) \in (\mathbb{Z}_{\geq 0})^d$, let $x^I = x_1^{i_1} \cdots x_d^{i_d}$. A rational function $f \in \mathbb{C}(x_1, \ldots, x_d)$ is *positive* (or *subtraction-free*) if it is nonzero, and has an expression of the form

$$f(x_1, \ldots, x_d) = \frac{\sum_I a_I x^I}{\sum_I b_I x^I} \tag{2.9}$$

with $a_I, b_I \in \mathbb{R}_{\geq 0}$, and $a_I, b_I = 0$ for all but finitely many I. If f is positive, define the *tropicalization* of f to be the piecewise-linear function given by

$$\mathrm{Trop}(f)(x_1, \ldots, x_d) = \min_{I\colon a_I \neq 0} (i_1 x_1 + \cdots + i_d x_d) - \min_{I\colon b_I \neq 0} (i_1 x_1 + \cdots + i_d x_d).$$

We view $\mathrm{Trop}(f)$ as a map from $\mathbb{Z}^d \to \mathbb{Z}$. More generally, a rational map $f = (f_1, \ldots, f_k)\colon (\mathbb{C}^*)^d \to (\mathbb{C}^*)^k$ is positive if each f_i is positive, and in this case its tropicalization is defined by

$$\mathrm{Trop}(f) = \big(\mathrm{Trop}(f_1), \ldots, \mathrm{Trop}(f_k)\big)\colon \mathbb{Z}^d \to \mathbb{Z}^k.$$

Example 2.11. The rational function $f = \dfrac{3x^2 + xy^3 + 5}{2x^5y^3 + 6xz}$ has tropicalization

$$\mathrm{Trop}(f) = \min(2x, x + 3y, 0) - \min(5x + 3y, x + z).$$

It is straightforward to verify (see, *e.g.*, [BFZ96, Lem. 2.1.6]) that tropicalization is independent of the choice of positive expression (2.9), and that it satisfies

$$\mathrm{Trop}(f + g) = \min\big(\mathrm{Trop}(f), \mathrm{Trop}(g)\big), \qquad \mathrm{Trop}(fg^{\pm 1}) = \mathrm{Trop}(f) \pm \mathrm{Trop}(g). \tag{2.10}$$

In other words, tropicalization is a homomorphism from the semi-field of positive rational functions over $\mathbb{C}$ with operations $(+, \cdot, \div)$ to the semi-field of piecewise-linear functions over $\mathbb{Z}$ with operations $(\min, +, -)$.

We say that a positive rational function f is a *geometric lift* of a piecewise-linear function F if $\mathrm{Trop}(f) = F$; note that every piecewise-linear function has many geometric lifts. We will make frequent use of a particular geometric lift of the maximum function. Given positive rational functions $f_1, \dots, f_k$, define the *geometric maximum* of $f_1, \dots, f_k$ by

$$\mathrm{gMax}(f_1, \dots, f_k) = \frac{1}{f_1^{-1} + \cdots + f_k^{-1}}. \tag{2.11}$$

Using (2.10), we see that

$$\mathrm{Trop}\big(\mathrm{gMax}(f_1, \dots, f_k)\big) = -\min\big(-\mathrm{Trop}(f_1), \dots, -\mathrm{Trop}(f_k)\big) = \max\big(\mathrm{Trop}(f_1), \dots, \mathrm{Trop}(f_k)\big).$$

2.5. **Tropicalization of geometric crystals.** Let X be a GL_m-geometric crystal of dimension d. A *parametrization* of X is a birational isomorphism $\Phi\colon (\mathbb{C}^*)^d \to X$. Given a parametrization Φ of X, we call the pair (X, Φ) a *positive GL_m-geometric crystal* if for each $i \in [m-1]$, the rational functions $\gamma\Phi, \varepsilon_i\Phi, \varphi_i\Phi\colon (\mathbb{C}^*)^d \to \mathbb{C}$ are positive, and the rational map $\Phi^{-1}e_i\Phi\colon (\mathbb{C}^*)^{d+1} \to (\mathbb{C}^*)^d$ given by

$$(c, x_1, \dots, x_d) \mapsto \Phi^{-1}e_i^c\Phi(x_1, \dots, x_d)$$

is positive. Given a positive GL_m-geometric crystal (X, Φ) of dimension d, set $\mathcal{B} = \mathbb{Z}^d$, and define $\widetilde{\gamma}\colon \mathcal{B} \to \mathbb{Z}^m$, $\widetilde{\varepsilon}_i, \widetilde{\varphi}_i : \mathcal{B} \to \mathbb{Z}$, and $\widetilde{e}_i, \widetilde{f}_i\colon \mathcal{B} \to \mathcal{B}$ by

$$\widetilde{\gamma} = \mathrm{Trop}(\gamma\Phi), \qquad \widetilde{\varepsilon}_i = \mathrm{Trop}(\varepsilon_i\Phi), \qquad \widetilde{\varphi}_i = \mathrm{Trop}(\varphi_i\Phi), \tag{2.12}$$

$$\widetilde{e}_i^{\mathrm{free}} = \mathrm{Trop}(\Phi^{-1}e_i^c\Phi)|_{c=1}, \qquad \widetilde{f}_i^{\mathrm{free}} = \mathrm{Trop}(\Phi^{-1}e_i^c\Phi)|_{c=-1}.$$

By comparing axioms (1) and (2) in Definitions 2.1 and 2.9, one sees immediately that these maps make $\mathcal{B}$ into a free GL_m-combinatorial crystal. We call $\mathcal{B}$ the *tropicalization of X (with respect to Φ)*, and we denote it by $\mathrm{Trop}(X)$.

Remark 2.12. In general, a positive geometric crystal associated to a reductive group G tropicalizes to a free combinatorial crystal for the Langlands dual group $G^\vee$. This is due to the fact that the roles of α_i and $\alpha_i^\vee$ in Definition 2.1 are played by $\widetilde{\alpha}_i^\vee$ and $\widetilde{\alpha}_i$ in Definition 2.9. We are able to ignore this subtlety because GL_m is its own Langlands dual.

In order to recover the crystals of finite-dimensional representations, Berenstein and Kazhdan introduced in [BK07] an elegant mechanism for "cutting out" finite vertex sets from $\mathbb{Z}^d$, which we now review. A *decoration* on a GL_m-geometric crystal X is a rational function $F : X \to \mathbb{C}$ such that

$$F(e_i^c(x)) = F(x) + (c-1)\varphi_i(x) + (c^{-1}-1)\varepsilon_i(x). \tag{2.13}$$

A *positive decorated GL_m-geometric crystal* is a triple (X, Φ, F), where (X, Φ) is a positive geometric crystal, F is a decoration on X, and $F\Phi$ is positive.

Theorem 2.13 ([BK07, Prop. 6.6, 6.7])**.** *Let (X, Φ, F) be a positive decorated GL_m-geometric crystal of dimension d. Define*

$$\mathcal{B}_F = \{b \in \mathbb{Z}^d \mid \mathrm{Trop}(F\Phi)(b) \geq 0\}.$$

Let $\widetilde{\gamma}, \widetilde{\varepsilon}_i, \widetilde{\varphi}_i$ be the restrictions to $\mathcal{B}_F$ of the maps (2.12), and define $\widetilde{e}_i, \widetilde{f}_i\colon \mathcal{B}_F \to \mathcal{B}_F \sqcup \{\mathbf{0}\}$ by

$$\widetilde{e}_i(b) = \begin{cases} \widetilde{e}_i^{\mathrm{free}}(b) & \text{if } \widetilde{e}_i^{\mathrm{free}}(b) \in \mathcal{B}_F, \\ \mathbf{0} & \text{otherwise,} \end{cases} \qquad \widetilde{f}_i(b) = \begin{cases} \widetilde{f}_i^{\mathrm{free}}(b) & \text{if } \widetilde{f}_i^{\mathrm{free}}(b) \in \mathcal{B}_F, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

(1) *These maps make $\mathcal{B}_F$ into a regular GL_m-combinatorial crystal. We call this crystal the tropicalization of (X, F) (with respect to Φ), and denote it by $\mathrm{Trop}(X, F)$.*

(2) *Suppose (X', Φ', F') is another positive decorated GL_m-geometric crystal. Define $\Phi \times \Phi'\colon (\mathbb{C}^*)^d \times (\mathbb{C}^*)^{d'} \to X \times X'$ and $F + F'\colon X \times X' \to \mathbb{C}$ by*

$$(\Phi \times \Phi')(z, z') = \big(\Phi(z), \Phi(z')\big), \qquad (F + F')(x, x') = F(x) + F(x').$$

The triple $(X \times X', \Phi \times \Phi', F + F')$ is a positive decorated GL_m-geometric crystal, and $\mathrm{Trop}(X \times X', F + F') = \mathrm{Trop}(X, F) \otimes \mathrm{Trop}(X', F')$.

Example 2.14. The function $F\colon X_m \to \mathbb{C}$ given by $F(x_1,\dots,x_m) = x_1 + \dots + x_m$ is easily seen to be a decoration. The tropicalization of (X_m, F) (with respect to $\Phi = \mathrm{Id}$) is the regular combinatorial crystal $\mathcal{B}$ defined in Example 2.10. Furthermore, Theorem 2.13(2) implies that the function

$$F(\mathbf{x}^1,\dots,\mathbf{x}^n) = \sum_{i\in[m],j\in[n]} x_i^j$$

is a decoration on $(X_m)^n$, and $\big((X_m)^n, F\big)$ tropicalizes to the n-fold tensor product of $\mathcal{B}$.

3. Gelfand–Tsetlin geometric crystal

3.1. **Gelfand–Tsetlin patterns.** A *Gelfand–Tsetlin (GT) pattern of height n* is a triangular array of nonnegative integers $(a_{i,j}), 1 \le i \le j \le n$, satisfying the inequalities

$$a_{i,j+1} \ge a_{i,j} \ge a_{i+1,j+1} \tag{3.1}$$

whenever $j < n$. We will represent GT patterns as triangles whose entries weakly increase from right to left along every diagonal, as illustrated below in the case $n = 4$:

$$\begin{array}{ccccccccccc} & & & & a_{11} & & & & & & \\ & & & a_{12} & & & a_{22} & & & & \\ & & a_{13} & & & a_{23} & & & a_{33} & & \\ & a_{14} & & & a_{24} & & & a_{34} & & & a_{44} \end{array} .$$

GT patterns of height n are in bijection with semistandard tableaux consisting of entries at most n: the partition $(a_{1,j},\dots,a_{j,j})$ in the jth row corresponds to the shape formed by the entries less than or equal to j in the tableau. In particular, the bottom row $(a_{1,n},\dots,a_{n,n})$ is the shape of the tableau, so we define the *shape* of the GT pattern to be this n-tuple. Another description of the bijection is that the number of j's in the ith row of the tableau is $a_{i,j} - a_{i,j-1}$ (with $a_{i,i-1} = 0$).

Example 3.1. Here is a GT pattern of height 4 and the corresponding semistandard tableau:

$$\begin{array}{ccccccccccc} & & & & 3 & & & & & & \\ & & & 6 & & 1 & & & & & \\ & & 6 & & 4 & & 1 & & & & \\ & 8 & & 5 & & 3 & & 0 & & & \end{array} \quad \longleftrightarrow \quad \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 2 & 4 & 4 \\ \hline 2 & 3 & 3 & 3 & 4 \\ \cline{1-5} 3 & 4 & 4 \\ \cline{1-3} \end{array} .$$

More generally, define a *Gelfand–Tsetlin pattern of height n and width m* to be a trapezoidal array of nonnegative integers $(a_{i,j})_{1\le i\le m, i\le j\le n}$ which satisfy the inequalities (3.1) whenever both sides of an inequality are in the array. For example, a GT pattern of height 5 and width 3 looks like this:

$$\begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{12} & & a_{22} & & & \\ & & a_{13} & & a_{23} & & a_{33} & & \\ & a_{14} & & a_{24} & & a_{34} & & & \\ a_{15} & & a_{25} & & a_{35} & & & & \end{array} .$$

These patterns are in bijection with semistandard tableaux with entries at most n, and at most m rows; again, we define the shape of the GT pattern to be the bottom row $(a_{1,n},\dots,a_{\min(m,n),n})$, since this is the shape of the corresponding tableau. Note that if $m \ge n$, then a GT pattern of height n and width m is simply a GT pattern of height n.

At the geometric level, we consider the torus

$$\mathrm{GT}_n^{\le m} = \{\mathbf{z} = (z_{i,j}) \mid z_{i,j} \in \mathbb{C}^*, 1 \le i \le m, i \le j \le n\}. \tag{3.2}$$

We refer to points of $\mathrm{GT}_n^{\le m}$ as *patterns*, and we define the *shape* of $\mathbf{z} \in \mathrm{GT}_n^{\le m}$ to be the vector

$$\mathrm{sh}(\mathbf{z}) = (z_{1,n},\dots,z_{p,n}) \in (\mathbb{C}^*)^p, \tag{3.3}$$

where $p = \min(m,n)$.

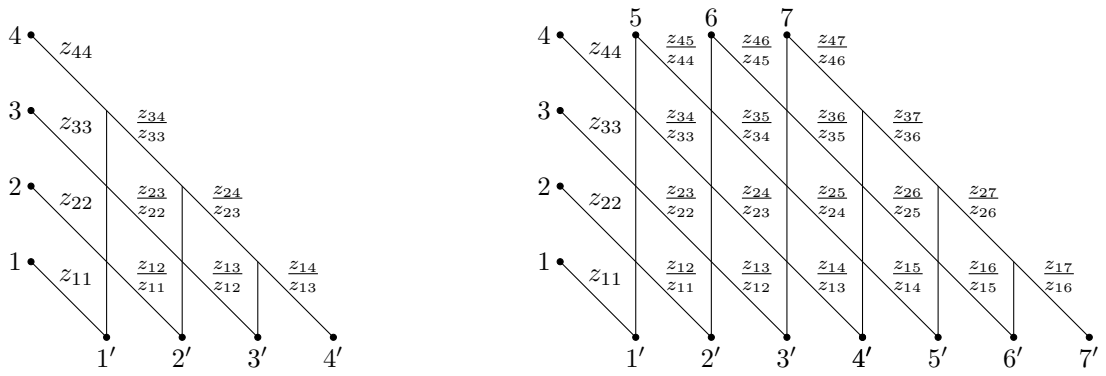


FIGURE 1. Examples of the planar network $\Gamma_n^{\leq m}$. The network on the left is Γ_4 (*i.e.*, $\Gamma_4^{\leq m}$ for any $m \geq 4$), and the network on the right is $\Gamma_7^{\leq 4}$. Edges are directed to the south and southeast, and all vertical edges have weight 1.

3.2. **Geometric crystal structure on** $\mathrm{GT}_n^{\leq m}$**.** In this section we work with GL_n-geometric crystals rather than GL_m-geometric crystals. Let B^- denote the subgroup of lower triangular matrices in $\mathrm{GL}_n(\mathbb{C})$, and consider the subsets of B^- defined by

$$(B^-)^{\leq m} = \{A \in \mathrm{GL}_n(\mathbb{C}) \mid A_{ij} = 0 \text{ if } i < j \text{ or } i - j > m, \text{ and } A_{ij} = 1 \text{ if } i - j = m\}$$

(note that $(B^-)^{\leq m} = B^-$ for $m \geq n$). By Example 2.6(a), $(B^-)^{\leq m}$ has the structure of a GL_n-unipotent crystal, and therefore of a GL_n-geometric crystal. The dimension of $(B^-)^{\leq m}$ is equal to the dimension of the torus $\mathrm{GT}_n^{\leq m}$. We will introduce a parametrization $\Phi_n^{\leq m} \colon \mathrm{GT}_n^{\leq m} \to (B^-)^{\leq m}$, and then obtain a geometric crystal structure on the torus $\mathrm{GT}_n^{\leq m}$ by "pulling back" the geometric crystal structure on $(B^-)^{\leq m}$. In §3.3, we give explicit positive formulas for this geometric crystal structure that make no reference to $(B^-)^{\leq m}$.

For $i \in [n]$, define an $n \times n$ matrix

$$W^i(z_i, \ldots, z_n) = \sum_{k=1}^{i-1} E_{kk} + \sum_{k=i}^{n} z_k E_{kk} + \sum_{k=i}^{n-1} E_{k+1,k}, \tag{3.4}$$

where $E_{i,j}$ is a matrix unit as in §2.2. Set $p = \min(m, n)$. Given $\mathbf{z} = (z_{i,j}) \in \mathrm{GT}_n^{\leq m}$, define

$$\Phi_n^{\leq m}(\mathbf{z}) = W^p\left(z_{p,p}, \frac{z_{p,p+1}}{z_{p,p}}, \ldots, \frac{z_{p,n}}{z_{p,n-1}}\right) \cdots W^1\left(z_{1,1}, \frac{z_{1,2}}{z_{1,1}}, \ldots, \frac{z_{1,n}}{z_{1,n-1}}\right).$$

Example 3.2. For $n = 4$ and $m = 2$, we have

$$\Phi_4^{\leq 2}(\mathbf{z}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z_{22} & 0 & 0 \\ 0 & 1 & \frac{z_{23}}{z_{22}} & 0 \\ 0 & 0 & 1 & \frac{z_{24}}{z_{23}} \end{pmatrix} \begin{pmatrix} z_{11} & 0 & 0 & 0 \\ 1 & \frac{z_{12}}{z_{11}} & 0 & 0 \\ 0 & 1 & \frac{z_{13}}{z_{12}} & 0 \\ 0 & 0 & 1 & \frac{z_{14}}{z_{13}} \end{pmatrix} = \begin{pmatrix} z_{11} & 0 & 0 & 0 \\ z_{22} & \frac{z_{12}z_{22}}{z_{11}} & 0 & 0 \\ 1 & \frac{z_{12}}{z_{11}} + \frac{z_{23}}{z_{22}} & \frac{z_{13}z_{23}}{z_{12}z_{22}} & 0 \\ 0 & 1 & \frac{z_{13}}{z_{12}} + \frac{z_{24}}{z_{23}} & \frac{z_{14}z_{24}}{z_{13}z_{23}} \end{pmatrix}.$$

We now introduce a very useful tool for working with the map $\Phi_n^{\leq m}$. Let $\Gamma_n^{\leq m}$ be a planar, edge-weighted, directed network, with

- vertex set
$$V = \{(x, y) \mid x \in [0, n], y \in [0, p], x + y \leq n\} \setminus \{(0, 0)\};$$
- a vertical edge of weight 1 from (x, y) to $(x, y - 1)$ whenever $x \geq 1$ and both vertices are in V;
- a diagonal edge of weight $z_{y,x+y}/z_{y,x+y-1}$ from (x, y) to $(x + 1, y - 1)$ whenever both vertices are in V (with $z_{y,y-1} = 1$).

This network has n source vertices $(0, 1), \ldots, (0, p), (1, p), \ldots, (n - m, p)$ and n sink vertices $(1, 0), \ldots, (n, 0)$, which we label $1, \ldots, n$ and $1', \ldots, n'$, respectively. Examples of this network are shown in Figure 1.

Given a planar, edge-weighted, directed network Γ with distinguished sources $1, \ldots, n$ and sinks $1', \ldots, n'$, we associate an $n \times n$ matrix $A(\Gamma)$ as follows: the weight of a path in the network is defined to be the sum of the weights of the edges in the path, and the entry $A(\Gamma)_{i,j}$ is defined to be the sum of the weights of

all (directed) paths from source i to sink j'. Using the fact that concatenation of networks corresponds to multiplication of the associated matrices, one sees that $\Phi_n^{\leq m}(\mathbf{z}) = A(\Gamma_n^{\leq m})$.

It is clear from inspection of the network $\Gamma_n^{\leq m}$ that $\Phi_n^{\leq m}(\mathbf{z}) \in (B^-)^{\leq m}$. To show that $\Phi_n^{\leq m}$ is a birational isomorphism, we will give an explicit formula for its (birational) inverse. For an $n \times n$ matrix A and subsets $I, J \subseteq [n]$ of the same size, let $\Delta_{I,J}(A)$ be the determinant of the submatrix of A consisting of the rows in I and the columns in J. When J consists of the first several columns, we will often omit it from the notation, so that $\Delta_I(A) = \Delta_{I,[1,|I|]}(A)$. We call the minor $\Delta_I(A)$ a *flag minor*. We use the convention that $\Delta_\emptyset(A) = \Delta_{\emptyset,\emptyset}(A) = 1$. Define a rational map $\Psi_n^{\leq m} \colon (B^-)^{\leq m} \to \mathrm{GT}_n^{\leq m}$ by $A \mapsto \mathbf{z} = (z_{i,j})$, where

$$z_{i,j} = \frac{\Delta_{[i,j]}(A)}{\Delta_{[i+1,j]}(A)} \tag{3.5}$$

for $1 \leq i \leq m, i \leq j \leq n$.

When $m \geq n$, the dependence on m in the above definitions disappears. In this case, we will sometimes omit the superscript "$\leq m$" and write $B^-, \mathrm{GT}_n, \Phi_n, \Psi_n, \Gamma_n$.

The following lemma appears in [NY04] but goes back at least to [BFZ96].

Lemma 3.3. *The map $\Phi_n^{\leq m}$ is a birational isomorphism from $\mathrm{GT}_n^{\leq m}$ to $(B^-)^{\leq m}$, with birational inverse $\Psi_n^{\leq m}$. In particular, Φ_n is a birational isomorphism from GT_n to B^-, with birational inverse Ψ_n.*

Proof. The Lindström/Gessel–Viennot Lemma [Lin73, GV85] says that the minor $\Delta_{I,J}(A(\Gamma))$ is equal to the the sum of the weights of non-intersecting collections of paths in Γ from the sources I to the sinks J' (the weight of a collection of paths is the product of all edges in the union of the paths, and non-intersecting means no two paths share a vertex). For i, j satisfying $1 \leq i \leq m$ and $i \leq j \leq n$, there is exactly one non-intersecting collection of paths in $\Gamma_n^{\leq m}$ from sources $[i, j]$ to sinks $[1, j - i + 1]'$, and this collection has weight $z_{i,j} z_{i+1,j} \cdots z_{j,j}$. Thus, the ratio of minors appearing in (3.5) is equal to $z_{i,j}$, so $\Psi_n^{\leq m} \circ \Phi_n^{\leq m}(\mathbf{z}) = \mathbf{z}$.

It remains to show that there is a (non-empty) Zariski open subset $V \subset (B^-)^{\leq m}$ such that $\Phi_n^{\leq m} \circ \Psi_n^{\leq m}(A) = A$ for all $A \in V$. Let V be the subset where the flag minors $\Delta_{[i,j]}$ are non-vanishing for all $1 \leq i \leq m$, $i \leq j \leq n$. If $A \in V$, then it follows from the above argument that $\Phi_n^{\leq m} \circ \Psi_n^{\leq m}(A)$ has the same flag minors as A. Now one argues that a point $A \in V$ is uniquely determined by its flag minors: the non-zero entries in the first column of A are equal to $\Delta_{[i,i]}(A)$, the non-zero entries in the second column of A are determined (from top to bottom) by $\Delta_{[i,i+1]}(A)$ and the entries in the first column, etc. □

Definition 3.4. Define $\overline{\gamma} \colon \mathrm{GT}_n^{\leq m} \to (\mathbb{C}^*)^n$ and $\overline{\varepsilon}_j, \overline{\varphi}_j \colon \mathrm{GT}_n^{\leq m} \to \mathbb{C}$ (for $j \in [n-1]$) by

$$\overline{\gamma}(\mathbf{z}) = (M_{1,1}, \ldots, M_{n,n}), \qquad \overline{\varepsilon}_j(\mathbf{z}) = \frac{M_{j+1,j+1}}{M_{j+1,j}}, \qquad \overline{\varphi}_j(\mathbf{z}) = \frac{M_{j,j}}{M_{j+1,j}},$$

where $M = \Phi_n^{\leq m}(\mathbf{z})$. Define $\overline{e}_j \colon \mathbb{C}^* \times \mathrm{GT}_n^{\leq m} \to \mathrm{GT}_n^{\leq m}$ by

$$\overline{e}_j^c(\mathbf{z}) = \Psi_n^{\leq m}\Big(x_j\left((c-1)\overline{\varphi}_j(\mathbf{z})\right) \cdot \Phi_n^{\leq m}(\mathbf{z}) \cdot x_j\left((c^{-1}-1)\overline{\varepsilon}_j(\mathbf{z})\right) \Big). \tag{3.6}$$

It follows from Theorem 2.7(1) and Corollary 2.8 that these maps make $\mathrm{GT}_n^{\leq m}$ into a GL_n-geometric crystal, which we call the *Gelfand–Tsetlin geometric crystal*.

Remark 3.5. The geometric crystal $\mathrm{GT}_n^{\leq 1}$ is isomorphic to the basic GL_n-geometric crystal defined in §2.1, via the map $(z_{1,1}, \ldots, z_{1,n}) \mapsto (z_{1,1}, z_{1,2}/z_{1,1}, \ldots, z_{1,n}/z_{1,n-1})$.

The following result shows that a fundamental property of the combinatorial crystal operators on semistandard tableaux carries over to the geometric setting. This result can be viewed as a special case of [BK07, Claim 2.9].

Lemma 3.6. *The geometric crystal operators on $\mathrm{GT}_n^{\leq m}$ preserve the shape of a pattern.*

Proof. By definition, the shape of $\overline{e}_j^c(\mathbf{z})$ is given by

$$\left(\frac{\Delta_{[1,n]}(M')}{\Delta_{[2,n]}(M')}, \ldots, \frac{\Delta_{[p,n]}(M')}{\Delta_{[p+1,n]}(M')} \right), \tag{3.7}$$

where M' is the matrix inside the large parentheses in (3.6), and $p = \min(m, n)$. Multiplying a matrix by an element of U (on either side) does not change the bottom-left justified minors, so we may replace M' with $\Phi_n^{\leq m}(\mathbf{z})$ in (3.7). By Lemma 3.3, the resulting vector is equal to the shape of $\mathbf{z}$. □

3.3. **Tropicalization of the Gelfand–Tsetlin geometric crystal.** In order to recover the combinatorial crystals associated to finite-dimensional GL_n-modules from $\mathrm{GT}_n^{\leq m}$, we first introduce a decoration.

Definition 3.7. For $\mathbf{z} = (z_{i,j})_{1 \leq i \leq m, i \leq j \leq n} \in \mathrm{GT}_n^{\leq m}$, define

$$F(\mathbf{z}) = \sum_{\substack{1 \leq i \leq m \\ i \leq j \leq n-1}} \frac{z_{i,j+1}}{z_{i,j}} + \sum_{\substack{1 \leq i \leq m-1 \\ i \leq j \leq n-1}} \frac{z_{i,j}}{z_{i+1,j+1}} + \mathbb{1}_{m<n} z_{n,n},$$

where $\mathbb{1}_{m<n}$ is the indicator function for $m < n$.

In the case $m \geq n$, this formula appears in [Lam16] (with a slightly different indexing convention).

Remark 3.8. An integer array $(a_{i,j})_{1 \leq i \leq m, i \leq j \leq n}$ satisfies $\mathrm{Trop}(F)(a_{i,j}) \geq 0$ if and only if the $a_{i,j}$ satisfy the interlacing inequalities (3.1), and (in the case $m < n$) the coordinate $a_{m,m}$ is nonnegative. The interlacing inequalities imply that $a_{m,m}$ is less than or equal to all other entries $a_{i,j}$, so for $m < n$, $\mathrm{Trop}(F)(a_{i,j}) \geq 0$ if and only if $(a_{i,j})$ is a Gelfand–Tsetlin pattern. For $m \geq n$, the inequality $\mathrm{Trop}(F)(a_{i,j}) \geq 0$ does not require the $a_{i,j}$ to be nonnegative; however, for a fixed partition shape $\lambda = (a_{1,n} \geq \ldots \geq a_{n,n} \geq 0)$, an array $(a_{i,j})$ of shape λ satisfies $\mathrm{Trop}(F)(a_{i,j}) \geq 0$ if and only if it is a GT pattern. (Adding a constant to all entries $a_{i,j}$ corresponds to tensoring with a power of the determinant representation of GL_n, so nothing fundamentally new is obtained by allowing the shape to have negative entries.)

Remark 3.8 explains that the tropicalization of F cuts out the underlying sets of finite-dimensional GL_n-crystals, but it is not clear that F satisfies (2.13). To prove this, we express F in terms of minors of the matrix $\Phi_n^{\leq m}(\mathbf{z})$, and appeal to a general result of Berenstein and Kazhdan.

Lemma 3.9. *The decoration on* $\mathrm{GT}_n^{\leq m}$ *is given by*

$$F(\mathbf{z}) = \sum_{k=1}^{\min(m-1,n-1)} \frac{\Delta_{\{k\} \cup [k+2,n],[1,n-k]}(M) + \Delta_{[k+1,n],[1,n-k-1] \cup \{n-k+1\}}(M)}{\Delta_{[k+1,n],[1,n-k]}(M)} + \mathbb{1}_{m<n} \left(\frac{\Delta_{\{m\} \cup [m+2,n],[1,n-m]}(M)}{\Delta_{[m+1,n],[1,n-m]}(M)} + \sum_{j=1}^{n-m} \frac{\Delta_{[m+1,m+j],[1,j-1] \cup \{j+1\}}(M)}{\Delta_{[m+1,m+j],[1,j]}(M)} \right),$$

where $M = \Phi_n^{\leq m}(\mathbf{z})$.

The proof appears in Appendix A. The expression for F appearing in Lemma 3.9 is a special case of Berenstein and Kazhdan's general formula for the decoration of a unipotent bicrystal of parabolic type. Indeed, in [BK07, Cor. 1.25], take P corresponding to nodes $\{m+1, \ldots, n-1\}$ in the A_{n-1} Dynkin diagram. This implies that F does in fact satisfy (2.13).

Next, one must show that the geometric crystal $(\mathrm{GT}_n^{\leq m}, \mathrm{Id})$ is positive, as this is not at all obvious from the definition (3.6) of the geometric crystal operators. We will give explicit positive formulas for the geometric crystal structure on GT_n (it is straightforward to generalize these formulas to the case of $\mathrm{GT}_n^{\leq m}$, but we focus on the case $m \geq n$ for clarity). In order to state the formulas compactly, we introduce the *diamond ratio*

$$\phi_{i,j} = \phi_{i,j}(\mathbf{z}) = \frac{z_{i-1,j} z_{i,j}}{z_{i-1,j-1} z_{i,j+1}}$$

for $2 \leq i \leq j \leq n-1$. (The entries of $\mathbf{z}$ that contribute to $\phi_{i,j}$ form the diamond $\begin{smallmatrix} & z_{i-1,j-1} & \\ z_{i-1,j} & & z_{i,j} \\ & z_{i,j+1} & \end{smallmatrix}$.)

We also use the rational function gMax defined in (2.11), with the convention that $\mathrm{gMax}(\emptyset) = 1$.

Lemma 3.10. *The* GL_n*-geometric crystal structure on* GT_n *is given by the explicit formulas*

$$\overline{\gamma}(\mathbf{z}) = \left(z_{1,1}, \frac{z_{1,2} z_{2,2}}{z_{1,1}}, \ldots, \frac{\prod_{i=1}^{n} z_{i,n}}{\prod_{i=1}^{n-1} z_{i,n-1}} \right),$$

$$\overline{\varepsilon}_j(\mathbf{z}) = \frac{z_{1,j+1}}{z_{1,j}} \operatorname{gMax}_{1\le k\le j}\left(\prod_{i=2}^{k} \phi_{i,j}^{-1}\right), \qquad \overline{\varphi}_j(\mathbf{z}) = \frac{z_{j,j}}{z_{j+1,j+1}} \operatorname{gMax}_{1\le k\le j}\left(\prod_{i=k+1}^{j} \phi_{i,j}\right),$$

$$\overline{e}_j^c(\mathbf{z}) = (\mathbf{z}'), \qquad \textit{where} \qquad z'_{i,r} = \begin{cases} z_{i,j}\dfrac{C_{i,j}}{C_{i+1,j}} & \textit{if } r = j, \\ z_{i,r} & \textit{otherwise,} \end{cases} \qquad \textit{and} \qquad C_{i,j} = \sum_{k=1}^{j} c^{\mathbb{1}_{k\ge i}} \prod_{\ell=2}^{k} \phi_{\ell,j}.$$

The proof appears in Appendix A.

Example 3.11. If $\mathbf{z} \in \mathrm{GT}_n$ with $n \ge 4$, then we have

$$\overline{\varepsilon}_3(\mathbf{z}) = \frac{z_{14}}{z_{13}} \frac{1}{1+\phi_{23}+\phi_{23}\phi_{33}}, \qquad \overline{\varphi}_3(\mathbf{z}) = \frac{z_{33}}{z_{44}} \frac{1}{\phi_{33}^{-1}\phi_{23}^{-1}+\phi_{33}^{-1}+1},$$

and $\overline{e}_3^c$ acts on the third row of $\mathbf{z}$ by

$$z_{13} \quad z_{23} \quad z_{33} \quad \mapsto \quad z_{13}\frac{c+c\phi_{23}+c\phi_{23}\phi_{33}}{1+c\phi_{23}+c\phi_{23}\phi_{33}} \quad z_{23}\frac{1+c\phi_{23}+c\phi_{23}\phi_{33}}{1+\phi_{23}+c\phi_{23}\phi_{33}} \quad z_{33}\frac{1+\phi_{23}+c\phi_{23}\phi_{33}}{1+\phi_{23}+\phi_{23}\phi_{33}},$$

and leaves the other entries unchanged.

Finally, we use the explicit formulas of Lemma 3.10 to show the the tropicalization of (GT_n, F) is a piecewise-linear description of the usual crystal structure on semistandard tableaux (see, *e.g.*, [Kas02, BS17]). For a partition λ with at most n parts, let $\mathcal{B}(\lambda)$ denote the GL_n-combinatorial crystal corresponding to the irreducible GL_n-representation with highest weight λ.

Proposition 3.12. *The tropicalization of* (GT_n, F) *is isomorphic to* $\bigsqcup_\lambda \mathcal{B}(\lambda)$*, where the union is taken over all partitions with at most* n *parts.*

Proof. In keeping with our usual convention, we write $\widetilde{\overline{\gamma}}, \widetilde{\overline{\varepsilon}}_j, \widetilde{\overline{\varphi}}_j, \widetilde{\overline{e}}_j, \widetilde{\overline{f}}_j$ for the maps of a GL_n-crystal.

Let $\mathbf{a} = (a_{i,j})$ be a Gelfand–Tsetlin pattern of height n, and let T be the corresponding semistandard Young tableau. Recall that $a_{i,j} - a_{i,j-1}$ is the number of j's in row i of T (with $a_{i,0} := 0$). The tropicalization of $\overline{\gamma}$ sends $\mathbf{a}$ to the vector $(\mu_1, \ldots, \mu_n)$, where

$$\mu_j = \sum_{i=1}^{j} a_{i,j} - \sum_{i=1}^{j-1} a_{i,j-1}.$$

Clearly μ_j is the number of j's in T, so $\mathrm{Trop}(\overline{\gamma}) = \widetilde{\overline{\gamma}}$.

Let r_i denote the ith row of T. The crystal structure on T can be computed by viewing T as the tensor product $r_n \otimes r_{n-1} \otimes \cdots \otimes r_1$, with each factor having the crystal structure on one-row tableaux described in Example 2.10. Fix $j \in [n-1]$, and define

$$A_{k,j} = \sum_{\ell=2}^{k} \left(\widetilde{\overline{\varphi}}_j(r_{\ell-1}) - \widetilde{\overline{\varepsilon}}_j(r_\ell)\right)$$

for $k \in [n]$. According to [KN94, Prop. 2.1.1], the functions $\widetilde{\overline{\varepsilon}}_j$ and $\widetilde{\overline{\varphi}}_j$ are given by the piecewise-linear formulas[3]

$$\widetilde{\overline{\varepsilon}}_j(T) = \widetilde{\overline{\varepsilon}}_j(r_1) - \min_{1\le k\le n}(A_{k,j}), \qquad \widetilde{\overline{\varphi}}_j(T) = \widetilde{\overline{\varphi}}_j(r_n) + \max_{k\in[n]}\left(\sum_{i=k+1}^{n}\left(\widetilde{\overline{\varphi}}_j(r_{i-1}) - \widetilde{\overline{\varepsilon}}_j(r_i)\right)\right), \tag{3.8}$$

and the crystal operators are given by

$$\widetilde{\overline{e}}_j(T) = r_n \otimes \cdots \otimes \widetilde{\overline{e}}_j(r_{i^*}) \otimes \cdots \otimes r_1, \qquad \widetilde{\overline{f}}_j(T) = r_n \otimes \cdots \otimes \widetilde{\overline{f}}_j(r_{i^{**}}) \otimes \cdots \otimes r_1, \tag{3.9}$$

where i^* (resp., i^{**}) is the smallest (resp., largest) value of i for which $A_{i,j} = \min_{k\in[n]}(A_{k,j})$.

[3]We have reversed both the tensor product convention and the labeling of the tensor factors used in [KN94], resulting in identical formulas.

It is straightforward to verify that the tropicalizations of the formulas in Lemma 3.10 agree with the formulas in (3.8) and (3.9). For the reader's convenience, we present the details for the functions $\widetilde{\varepsilon}_j$ and the crystal operators $\widetilde{e}_j$.

The first key observation is that

$$\mathrm{Trop}(\phi_{i,j}) = (a_{i-1,j} - a_{i-1,j-1}) - (a_{i,j+1} - a_{i,j}) = \widetilde{\varphi}_j(r_{i-1}) - \widetilde{\varepsilon}_j(r_i) \tag{3.10}$$

for $i \leq j$. This implies that

$$\mathrm{Trop}(\overline{\varepsilon}_j)(\mathbf{a}) = (a_{1,j+1} - a_{1,j}) + \max_{1 \leq k \leq j} \left(-\sum_{i=2}^{k} \mathrm{Trop}(\phi_{i,j}) \right) = \widetilde{\varepsilon}_j(r_1) - \min_{1 \leq k \leq j} (A_{k,j}). \tag{3.11}$$

If $k \geq j+1$, then $A_{k,j} = A_{j,j} + (a_{j,j} - a_{j+1,j+1}) \geq A_{j,j}$, so the last expression in (3.11) is equal to $\widetilde{\varepsilon}_j(T)$.

We will now show that

$$\widetilde{e}_j(T) = \mathrm{Trop}(\overline{e}_j^c)|_{c=1}(\mathbf{a})$$

if the latter is a Gelfand–Tsetlin pattern, and otherwise $\widetilde{e}_j(T)$ is undefined. By Lemma 3.10 and (3.10), $\mathrm{Trop}(\overline{e}_j^c)|_{c=1}(\mathbf{a}) = \mathbf{a}'$, where $a'_{i,r} = a_{i,r}$ for $r \neq j$, and

$$a'_{i,j} = a_{i,j} + \min_{1 \leq k \leq j} (\mathbb{1}_{k \geq i} + A_{k,j}) - \min_{1 \leq k \leq j} (\mathbb{1}_{k \geq i+1} + A_{k,j}) .$$

The second key observation is that the difference of the two minimums appearing on the right-hand side is equal to 1 if $i = i^*$, and 0 otherwise.

Suppose $\widetilde{e}_j(T)$ is defined. By (3.9), $\widetilde{e}_j(T)$ is obtained by replacing r_{i^*} with $\widetilde{e}_j(r_{i^*})$. In other words, $\widetilde{e}_j$ adds 1 to $a_{i^*,j}$, which is precisely what $\mathrm{Trop}(\overline{e}_j^c)|_{c=1}(\mathbf{a})$ does. Since $\widetilde{e}_j(T)$ is known to be a semistandard Young tableau, $\mathbf{a}'$ must be a Gelfand–Tsetlin pattern in this case (with a bit more work, one can see that $\mathbf{a}'$ is a Gelfand–Tsetlin pattern using only the piecewise-linear formulas).

Now suppose $\widetilde{e}_j(T)$ is undefined. This means that $\widetilde{\varepsilon}_j(T) = 0$. Since $A_{1,j} = 0$ and $\widetilde{\varepsilon}_j(r_1) \geq 0$, we must have $\widetilde{\varepsilon}_j(r_1) = 0$ and $A_{k,j} \geq 0$ for all k. The latter condition implies that $i^* = 1$, and the former condition implies that $a_{1,j+1} = a_{1,j}$. This means that $\mathbf{a}'$ violates the inequality $a'_{1,j+1} \geq a'_{1,j}$, so it is not a Gelfand–Tsetlin pattern. □

4. Geometric RSK

To motivate the definition of geometric RSK, we first give a brief review of the RSK correspondence, and explain how it can be viewed as a bijection from the set of all $m \times n$ nonnegative integer matrices to the set of $m \times n$ nonnegative integer matrices with weakly increasing rows and columns.

Let $\mathbf{a} = (a_i^j)_{i \in [m], j \in [n]}$ be an $m \times n$ matrix of nonnegative integers, and let $\mathbf{a}_i = (a_i^1, \ldots, a_i^n)$ denote the ith row of the matrix. We interpret $\mathbf{a}_i$ as the weakly increasing word in the alphabet $[n]$ consisting of a_i^1 1's, a_i^2 2's, etc., as in Example 2.10. The RSK correspondence sends the matrix $\mathbf{a}$ to a pair (P, Q) of semistandard tableaux of the same shape, such that P has entries in $[n]$, and Q has entries in $[m]$. Specifically, one takes P_0 to be the empty tableau, and then recursively defines P_i to be the tableau formed by row inserting the word $\mathbf{a}_i$ into P_{i-1}. The *insertion tableau* P is defined to be P_m, and the *recording tableau* Q is the tableau such that for each $i \in [m]$, the subtableau consisting of entries less than or equal to i has the same shape as P_i. (We refer the reader to [Sta99] for more details.)

To go from a pair $(P, Q) \in \mathrm{SSYT}_{\leq n}(\lambda) \times \mathrm{SSYT}_{\leq m}(\lambda)$ to an $m \times n$ matrix with weakly increasing rows and columns, one forms the GT patterns of width $\min(m, n)$ and heights n and m for P and Q, respectively (see §3.1). Then one glues the first GT pattern to the transpose of the second GT pattern along the diagonal specifying their common shape. (In this context, we represent GT patterns as left-justified arrays, with the ith row from the bottom containing the entries $a_{i,i}, \ldots, a_{i,n}$.) See Figure 2 for an example.

A fundamental property of RSK is that transposing the matrix $\mathbf{a}$ corresponds to interchanging the tableaux P and Q. In other words, Q is the tableau formed by inserting the words associated to the columns $\mathbf{a}^j = (a_1^j, a_2^j, \ldots, a_m^j)$, and P records this insertion procedure. If we view RSK as a map of $m \times n$ matrices, this property says that RSK commutes with transposition.

The *geometric RSK correspondence (gRSK)* is a birational isomorphism

$$\mathrm{gRSK}\colon \mathrm{Mat}_{m \times n}(\mathbb{C}^*) \to \mathrm{Mat}_{m \times n}(\mathbb{C}^*).$$

$$\mathbf{a} = \begin{pmatrix} 1 & 4 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow{\text{RSK}} (P, Q) = \left(\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 2 \\ \cline{1-3} \end{array} \, , \, \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} \end{array} \right)$$

$$(P, Q) \longleftrightarrow \begin{pmatrix} 3 & & , & 2 & 3 & \\ 4 & 6 & & 5 & 6 & 6 \end{pmatrix} \xrightarrow{\text{glue}} \begin{pmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 6 \end{pmatrix}$$

FIGURE 2. The first line shows the pair (P, Q) of semistandard tableaux associated to a matrix $\mathbf{a}$ by the RSK correspondence. The second line shows the Gelfand–Tsetlin patterns of width 2 corresponding to P and Q, and the matrix with weakly increasing rows and columns obtained by gluing these patterns together along their common shape $(6, 3)$.

By analogy with classical RSK, we identify the output matrix with a pair of patterns $(P, Q) \in \mathrm{GT}_n^{\leq m} * \mathrm{GT}_m^{\leq n}$, where

$$\mathrm{GT}_n^{\leq m} * \mathrm{GT}_m^{\leq n} = \{(P, Q) \in \mathrm{GT}_n^{\leq m} \times \mathrm{GT}_m^{\leq n} \mid \mathrm{sh}(P) = \mathrm{sh}(Q)\}$$

(recall the definitions (3.2) and (3.3)). As in the combinatorial setting, our convention is that P sits in the bottom-left corner of the matrix, and the transpose of Q sits in the top-right corner.

We will give two equivalent definitions of gRSK. The first definition is modeled on the row insertion definition of RSK, and is due to Noumi and Yamada [NY04]; the second is similar in spirit to the growth diagram formulation of RSK, and is due to O'Connell, Seppäläinen, and Zygouras [OSZ14].

4.1. **Row insertion formulation of gRSK.** Suppose $\mathbf{x} = (x_a^b)_{a \in [m], b \in [n]} \in \mathrm{Mat}_{m \times n}(\mathbb{C}^*)$. Let $\mathbf{x}_a = (x_a^1, \ldots, x_a^n)$ be the ath row of $\mathbf{x}$, and let $p = \min(m, n)$. Recall from §2.2 the map $M \colon (X_n)^k \to \mathrm{GL}_n$ given by

$$M(\mathbf{x}_1, \ldots, \mathbf{x}_k) = W(\mathbf{x}_1) \cdots W(\mathbf{x}_k),$$

which makes $(X_n)^k$ into a GL_n-unipotent crystal. It is straightforward to show by induction on k that M is a surjection from $(X_n)^k$ onto $(B^-)^{\leq k}$.

Definition 4.1. Define $\mathrm{gRSK}(\mathbf{x}) = (P, Q)$, where $P = (z_{i,j}) \in \mathrm{GT}_n^{\leq m}$ and $Q = (z'_{j',i'}) \in \mathrm{GT}_m^{\leq n}$ are given by

$$z_{i,j} = \frac{\Delta_{[i,j]}\big(M(\mathbf{x}_1, \ldots, \mathbf{x}_m)\big)}{\Delta_{[i+1,j]}\big(M(\mathbf{x}_1, \ldots, \mathbf{x}_m)\big)}, \qquad z'_{j',i'} = \frac{\Delta_{[j',n]}\big(M(\mathbf{x}_1, \ldots, \mathbf{x}_{i'})\big)}{\Delta_{[j'+1,n]}\big(M(\mathbf{x}_1, \ldots, \mathbf{x}_{i'})\big)},$$

for $1 \leq i \leq m$, $i \leq j \leq n$ and $1 \leq j' \leq n$, $j' \leq i' \leq m$. We identify (P, Q) with a matrix in $\mathrm{Mat}_{m \times n}(\mathbb{C}^*)$ by gluing P and Q along their common shape, as described above. We refer to P and Q as the *P-pattern* and *Q-pattern* of the matrix $\mathbf{x}$.

It is clear that the shape $(z_{1,n}, \ldots, z_{p,n})$ of P is equal to the shape $(z'_{1,m}, \ldots, z'_{p,m})$ of Q, as required. Moreover, if we define

$$P_k = \Psi_n^{\leq k}\big(M(\mathbf{x}_1, \ldots, \mathbf{x}_k)\big) \in \mathrm{GT}_n^{\leq k},$$

then it follows from Lemma 3.3 that $P = P_m$, and the kth diagonal $(z'_{1,k}, z'_{2,k}, \ldots, z'_{\min(k,n),k})$ of Q is equal to the shape of the pattern P_k. We interpret this as saying that P is the result of "geometrically inserting" the rows of $\mathbf{x}$ (starting with the top row $\mathbf{x}_1$), and Q "records the growth" of P.

Example 4.2. We work out the geometric RSK correspondence in the case $m = 3, n = 2$. We start with the matrix $\mathbf{x} = \begin{pmatrix} x_1^1 & x_1^2 \\ x_2^1 & x_2^2 \\ x_3^1 & x_3^2 \end{pmatrix}$, and compute the matrices $M_1 = M(\mathbf{x}_1), M_2 = M(\mathbf{x}_1, \mathbf{x}_2), M_3 = M(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$:

$$M_1 = \begin{pmatrix} x_1^1 & 0 \\ 1 & x_1^2 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} x_1^1 x_2^1 & 0 \\ x_1^2 + x_2^1 & x_1^2 x_2^2 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} x_1^1 x_2^1 x_3^1 & 0 \\ x_1^2 x_2^2 + x_1^2 x_3^1 + x_2^1 x_3^1 & x_1^2 x_2^2 x_3^2 \end{pmatrix}.$$

Using these matrices, we compute $P \in \mathrm{GT}_2^{\leq 3} = \mathrm{GT}_2$ and $Q \in \mathrm{GT}_3^{\leq 2}$:

$$P = \begin{matrix} & z_{22} & \\ z_{11} & & z_{12} \end{matrix} = \begin{matrix} & \Delta_2(M_3) & \\ \Delta_1(M_3) & & \frac{\Delta_{12}(M_3)}{\Delta_2(M_3)} \end{matrix} = \begin{matrix} & x_1^2x_2^2 + x_1^2x_3^1 + x_2^1x_3^1 & \\ x_1^1x_2^1x_3^1 & & \frac{x_1^1x_2^1x_3^1x_1^2x_2^2x_3^2}{x_1^2x_2^2 + x_1^2x_3^1 + x_2^1x_3^1} \end{matrix}$$

$$Q = \begin{matrix} & z'_{22} & z'_{23} & \\ z'_{11} & & z'_{12} & & z'_{13} \end{matrix} = \begin{matrix} & \Delta_2(M_2) & \Delta_2(M_3) & \\ \frac{\Delta_{12}(M_1)}{\Delta_2(M_1)} & & \frac{\Delta_{12}(M_2)}{\Delta_2(M_2)} & & \frac{\Delta_{12}(M_3)}{\Delta_2(M_3)} \end{matrix} = \begin{matrix} & x_1^2 + x_2^1 & x_2^2x_2^2 + x_1^2x_3^1 + x_2^1x_3^1 & \\ x_1^1x_1^2 & & \frac{x_1^1x_2^1x_1^2x_2^2}{x_1^2 + x_2^1} & & \frac{x_1^1x_2^1x_3^1x_1^2x_2^2x_3^2}{x_1^2x_2^2 + x_1^2x_3^1 + x_2^1x_3^1} \end{matrix}.$$

Thus, we have

$$\begin{pmatrix} x_1^1 & x_1^2 \\ x_2^1 & x_2^2 \\ x_3^1 & x_3^2 \end{pmatrix} \xrightarrow{\text{gRSK}} \begin{pmatrix} x_1^2 + x_2^1 & x_1^1x_1^2 \\ x_1^2x_2^2 + x_1^2x_3^1 + x_2^1x_3^1 & \frac{x_1^1x_2^1x_1^2x_2^2}{x_1^2 + x_2^1} \\ x_1^1x_2^1x_3^1 & \frac{x_1^1x_2^1x_3^1x_1^2x_2^2x_3^2}{x_1^2x_2^2 + x_1^2x_3^1 + x_2^1x_3^1} \end{pmatrix}.$$

4.2. **Local move formulation of gRSK.** As above, denote an element of $\mathrm{Mat}_{m\times n}(\mathbb{C}^*)$ by $\mathbf{x} = (x_a^b)$, and set $p = \min(m, n)$. For each $a \in [m]$ and $b \in [n]$, we introduce two rational maps

$$\eta_a^b, T_a^b \colon \mathrm{Mat}_{m\times n}(\mathbb{C}^*) \to \mathrm{Mat}_{m\times n}(\mathbb{C}^*),$$

each of which changes only the entry x_a^b. These maps are defined by

$$\eta_a^b \colon x_a^b \mapsto x_a^b \,\mathrm{gMax}_a^b(\mathbf{x}), \qquad T_a^b \colon x_a^b \mapsto \frac{1}{x_a^b}\,\mathrm{gMax}_a^b(\mathbf{x})\,\mathrm{gMin}_a^b(\mathbf{x}), \tag{4.1}$$

where

$$\mathrm{gMax}_a^b(\mathbf{x}) = \begin{cases} \frac{x_a^{b-1}x_{a-1}^b}{x_a^{b-1} + x_{a-1}^b} & \text{if } a, b > 1, \\ x_1^{b-1} & \text{if } a = 1, b > 1, \\ x_{a-1}^1 & \text{if } a > 1, b = 1, \\ 1 & \text{if } a = b = 1, \end{cases} \qquad \mathrm{gMin}_a^b(\mathbf{x}) = \begin{cases} x_{a+1}^b + x_a^{b+1} & \text{if } a < m, b < n, \\ x_m^{b+1} & \text{if } a = m, b < n, \\ x_{a+1}^n & \text{if } a < m, b = n, \\ 1 & \text{if } a = m, b = n. \end{cases}$$

It is easy to see that T_a^b is an involution, and η_a^b has inverse given by $x_a^b \mapsto x_a^b \frac{1}{\mathrm{gMax}_a^b(\mathbf{x})}$.

Remark 4.3. The function gMax_a^b tropicalizes to $\max(x_a^{b-1}, x_{a-1}^b)$, and the function gMin_a^b tropicalizes to $\min(x_{a+1}^b, x_a^{b+1})$. (The various cases correspond to setting $x_0^b, x_a^0 = 0$ and $x_{m+1}^b = x_a^{n+1} = \infty$ in the tropicalized formulas. This rule does not work for gMin_m^n, but we will never use the operator T_m^n so this is not a problem.) Thus, if $(a, b) \neq (m, n)$, the map T_a^b tropicalizes to the *piecewise-linear toggle* at node (a, b) in the coordinate-wise partial order on $[m] \times [n]$ [EP21]. These toggles were first studied by Kirillov and Berenstein [KB95].

For $a \in [m]$ and $b \in [n]$, define τ_a^b to be the composition of the maps T_i^j at each position of the diagonal strictly to the northwest of (a, b), that is,

$$\tau_a^b = \begin{cases} T_1^{b-a+1} \circ \cdots \circ T_{a-2}^{b-2} \circ T_{a-1}^{b-1} & \text{if } a \leq b, \\ T_{a-b+1}^1 \circ \cdots \circ T_{a-2}^{b-2} \circ T_{a-1}^{b-1} & \text{if } a \geq b. \end{cases}$$

Define

$$\rho_a^b = \tau_a^b \circ \eta_a^b.$$

Note that τ_a^b is the identity map if $a = 1$ or $b = 1$, and $\rho_1^1 = \eta_1^1$ is also the identity map. Finally, define

$$\rho = (\rho_m^n \circ \cdots \circ \rho_m^1) \circ \cdots \circ (\rho_1^n \circ \cdots \circ \rho_1^1),$$

and define η and τ in the same way, but with ρ_a^b replaced by η_a^b and τ_a^b, respectively.

Lemma 4.4.

(1) *The map ρ (resp., η, τ) can be computed by composing the maps ρ_a^b (resp., η_a^b, τ_a^b) along any linear extension of the coordinate-wise partial order on $[m] \times [n]$.*
(2) *The map ρ factors as $\rho = \tau \circ \eta$.*

Proof. It is clear that when (a, b) and (a', b') are not adjacent, η_a^b commutes with both $\eta_{a'}^{b'}$ and $T_{a'}^{b'}$, and T_a^b commutes with $T_{a'}^{b'}$. This implies that ρ_a^b and $\rho_{a'}^{b'}$ commute (as do $\tau_a^b, \tau_{a'}^{b'}$ and $\eta_a^b, \eta_{a'}^{b'}$) when (a, b) and (a', b') are incomparable in $[m] \times [n]$, *i.e.*, when one position lies strictly to the southeast of the other. This proves part (1). Part (2) follows from the observation that η_a^b commutes with $\tau_{a'}^{b'}$ when $(a', b') \not\geq (a, b)$. □

Theorem 4.5 (O'Connell–Seppäläinen–Zygouras [OSZ14])**.** *The map ρ agrees with the geometric RSK correspondence of Definition 4.1.*

As mentioned in §1.6, our geometric RSK correspondence is different from the one studied in [NY04, OSZ14]. In §4.4, we discuss the precise connection between the two versions of gRSK, and explain why Theorem 4.5 follows from the analogous result proved in [OSZ14].

Example 4.6. We illustrate Theorem 4.5 in the case $m = 3, n = 2$. By Lemma 4.4(2), we have $\rho = \tau \circ \eta$, where

$$\eta = \eta_3^2 \circ \eta_3^1 \circ \eta_2^2 \circ \eta_2^1 \circ \eta_1^2 \circ \eta_1^1, \qquad \tau = \tau_3^2 \circ \tau_2^2 = T_2^1 \circ T_1^1.$$

We compute

$$\begin{pmatrix} x_1^1 & x_1^2 \\ x_2^1 & x_2^2 \\ x_3^1 & x_3^2 \end{pmatrix} \xrightarrow{\eta} \begin{pmatrix} x_1^1 & x_1^1 x_1^2 \\ x_1^1 x_2^1 & \dfrac{x_1^1 x_2^1 x_1^2 x_2^2}{x_1^2 + x_2^1} \\ x_1^1 x_2^1 x_3^1 & \dfrac{x_1^1 x_2^1 x_3^1 x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 + x_1^2 x_3^1 + x_2^1 x_3^1} \end{pmatrix} \xrightarrow{\tau} \begin{pmatrix} x_1^2 + x_2^1 & x_1^1 x_1^2 \\ x_1^2 x_2^2 + x_1^2 x_3^1 + x_2^1 x_3^1 & \dfrac{x_1^1 x_2^1 x_1^2 x_2^2}{x_1^2 + x_2^1} \\ x_1^1 x_2^1 x_3^1 & \dfrac{x_1^1 x_2^1 x_3^1 x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 + x_1^2 x_3^1 + x_2^1 x_3^1} \end{pmatrix},$$

obtaining the same result as in Example 4.2.

Corollary 4.7.

(1) *gRSK is a birational isomorphism.*
(2) *gRSK and its inverse are positive. This implies that gRSK restricts to a homeomorphism from $\mathrm{Mat}_{m \times n}(\mathbb{R}_{>0})$ to itself.*
(3) *gRSK satisfies the symmetry property*

$$\mathrm{gRSK}(\mathbf{x}) = (P, Q) \iff \mathrm{gRSK}(\mathbf{x}^t) = (Q, P),$$

where $\mathbf{x}^t$ is the transpose of $\mathbf{x}$. Thus, the patterns $P = (z_{i,j})$ and $Q = (z'_{j',i'})$ are given by the alternative formulas

$$z_{i,j} = \frac{\Delta_{[i,m]}(M(\mathbf{x}^1, \ldots, \mathbf{x}^j))}{\Delta_{[i+1,m]}(M(\mathbf{x}^1, \ldots, \mathbf{x}^j))}, \qquad z'_{j',i'} = \frac{\Delta_{[j',i']}(M(\mathbf{x}^1, \ldots, \mathbf{x}^n))}{\Delta_{[j'+1,i']}(M(\mathbf{x}^1, \ldots, \mathbf{x}^n))}, \tag{4.2}$$

for $1 \leq i \leq m$, $i \leq j \leq n$ and $1 \leq j' \leq n$, $j' \leq i' \leq m$, where $\mathbf{x}^j = (x_1^j, \ldots, x_m^j)$ is the jth column of $\mathbf{x}$.

Proof. Part (1) follows from the fact that the η_a^b and T_a^b are invertible rational maps, and part (2) follows from the fact that these maps and their inverses are positive. Part (3) follows from Lemma 4.4(1), which implies that the map ρ can be computed column-by-column, rather than row-by-row. □

Example 4.8. The reader may verify that in the case $m = 3, n = 2$, the P- and Q-patterns computed in Example 4.2 can be obtained from the matrices

$$M(\mathbf{x}^1) = \begin{pmatrix} x_1^1 & 0 & 0 \\ 1 & x_2^1 & 0 \\ 0 & 1 & x_3^1 \end{pmatrix}, \qquad M(\mathbf{x}^1, \mathbf{x}^2) = \begin{pmatrix} x_1^1 x_1^2 & 0 & 0 \\ x_2^1 + x_1^2 & x_2^1 x_2^2 & 0 \\ 1 & x_3^1 + x_2^2 & x_3^1 x_3^2 \end{pmatrix}$$

as the ratios of determinants specified by (4.2). In particular, the common shape $(z_{12}, z_{22}) = (z'_{13}, z'_{23})$ of P and Q is given by

$$\left(\frac{\Delta_{123}\big(M(\mathbf{x}^1,\mathbf{x}^2)\big)}{\Delta_{23}\big(M(\mathbf{x}^1,\mathbf{x}^2)\big)}, \frac{\Delta_{23}\big(M(\mathbf{x}^1,\mathbf{x}^2)\big)}{\Delta_{3}\big(M(\mathbf{x}^1,\mathbf{x}^2)\big)}\right) = \left(\frac{x^1_1x^1_2x^1_3x^2_1x^2_2x^2_3}{x^2_1x^2_2+x^2_1x^1_3+x^1_2x^1_3}, x^2_1x^2_2+x^2_1x^1_3+x^1_2x^1_3\right).$$

Remark 4.9. Results of Noumi and Yamada [NY04] imply that the geometric row insertion formulation of gRSK tropicalizes to RSK (see §4.4). There are also multiple proofs in the literature of the fact that the tropicalization of ρ is a piecewise-linear description of RSK. This was known to Pak [Pak02], who used a generalization of $\mathrm{Trop}(\rho)$ to give a simple proof of the hook-length formula for the number of standard Young tableaux of a given shape. An elementary proof that $\mathrm{Trop}(\rho) = \mathrm{RSK}$ can be found in [Hop14]. This result was recently reproved (and substantially generalized) in the context of quiver representations [GPT18].

4.3. **Geometric RSK is an isomorphism of geometric crystals.** We have described two GL_n- (resp., GL_m-) geometric crystal structures on the variety $X = \mathrm{Mat}_{m\times n}(\mathbb{C}^*)$. The first is the *basic geometric crystal structure*, which comes from identifying X with $(X_n)^m$ (resp., $(X_m)^n$); we denote this geometric crystal by X^{Mat}. The second is the *Gelfand–Tsetlin geometric crystal structure*, which comes from identifying X with $\mathrm{GT}_n^{\leq m} * \mathrm{GT}_m^{\leq n}$, and defining $\overline{e}_j^c(P,Q) = (\overline{e}_j^c(P), Q)$ (resp., $e_i^c(P,Q) = (P, e_i^c(Q))$). We denote this geometric crystal by X^{GT}.

Theorem 4.10. *The geometric RSK correspondence*

$$\mathrm{gRSK}\colon X^{\mathrm{Mat}} \to X^{\mathrm{GT}}$$

is an isomorphism of $\mathrm{GL}_n \times \mathrm{GL}_m$*-geometric crystals.*[4]

Proof. Since geometric RSK is a birational isomorphism of the underlying varieties and the GL_n- and GL_m-geometric crystal structures on X^{GT} clearly commute, it suffices to prove that gRSK is an isomorphism of GL_m-geometric crystals and GL_n-geometric crystals. By Corollary 4.7(3), it is enough to prove that gRSK is an isomorphism of GL_n-geometric crystals. To prove this, we must show that if $\mathrm{gRSK}(\mathbf{x}) = (P,Q)$, then

$$\overline{\gamma}(\mathbf{x}) = \overline{\gamma}(P), \qquad \overline{\varepsilon}_j(\mathbf{x}) = \overline{\varepsilon}_j(P), \qquad \overline{\varphi}_j(\mathbf{x}) = \overline{\varphi}_j(P), \qquad \mathrm{gRSK}(\overline{e}_j^c(\mathbf{x})) = (\overline{e}_j^c(P), Q). \tag{4.3}$$

By (2.6), $\overline{\gamma}(\mathbf{x})$ is the diagonal of the $n\times n$ matrix $M(\mathbf{x}_1,\ldots,\mathbf{x}_m)$, and $\overline{\varepsilon}_j(\mathbf{x}), \overline{\varphi}_j(\mathbf{x})$ are ratios of certain entries of this matrix. By Definition 3.4, $\overline{\gamma}(P)$, $\overline{\varepsilon}_j(P)$, and $\overline{\varphi}_j(P)$ are obtained in the exactly the same way from the matrix $\Phi_n^{\leq m}(P)$. The geometric RSK correspondence is defined in such a way that $P = \Psi_n^{\leq m}(M(\mathbf{x}_1,\ldots,\mathbf{x}_m))$, so $\Phi_n^{\leq m}(P) = M(\mathbf{x}_1,\ldots,\mathbf{x}_m)$ by Lemma 3.3. This proves the first three identities in (4.3).

Suppose $\mathrm{gRSK}(\overline{e}_j^c(\mathbf{x})) = (P', Q')$. By (2.7) and Definitions 3.4 and 4.1, we have

$$\begin{aligned} P' &= \Psi_n^{\leq m}\left(x_j\left((c-1)\overline{\varphi}_j(\mathbf{x})\right)\cdot M(\mathbf{x}_1,\ldots,\mathbf{x}_m)\cdot x_j\left((c^{-1}-1)\overline{\varepsilon}_j(\mathbf{x})\right)\right)\\ &= \Psi_n^{\leq m}\left(x_j\left((c-1)\overline{\varphi}_j(P)\right)\cdot \Phi_n^{\leq m}(P)\cdot x_j\left((c^{-1}-1)\overline{\varepsilon}_j(P)\right)\right)\\ &= \overline{e}_j^c(P). \end{aligned}$$

It remains to show that $Q' = Q$. Suppose $\overline{e}_j^c(\mathbf{x}_1,\ldots,\mathbf{x}_m) = (\mathbf{x}'_1,\ldots,\mathbf{x}'_m)$. The formula (2.7) implies that for each $k\in[m]$, there are rational functions a, a' in the entries of $\mathbf{x}$ and c such that

$$M(\mathbf{x}'_1,\ldots,\mathbf{x}'_k) = x_j(a)M(\mathbf{x}_1,\ldots\mathbf{x}_k)x_j(a').$$

As observed in the proof of Lemma 3.6, this means that $M(\mathbf{x}_1,\ldots\mathbf{x}_k)$ and $M(\mathbf{x}'_1,\ldots,\mathbf{x}'_k)$ have the same bottom-left justified minors. By definition, the entries of Q and Q' are ratios of bottom-left justified minors of these matrices, so we are done. □

Theorem 4.10 gives a new proof of the following result from [LP13a].

Corollary 4.11 ([LP13a])**.** *The basic* GL_n*- and* GL_m*-geometric crystal structures on* $\mathrm{Mat}_{m\times n}(\mathbb{C}^*)$ *commute.*

[4] A $\mathrm{GL}_n\times\mathrm{GL}_m$-geometric crystal is a variety equipped with GL_n- and GL_m-geometric crystal structures which commute with each other (*i.e.*, $\varepsilon_i\overline{e}_j^c = \varepsilon_i$, $\overline{\varepsilon}_j e_i^c = \overline{\varepsilon}_j$, $e_i^{c_1}\overline{e}_j^{c_2} = \overline{e}_j^{c_2}e_i^{c_1}$, etc.)

4.4. **Connection to Noumi and Yamada's geometric RSK.** Let $f\colon (\mathbb{C}^*)^d \to (\mathbb{C}^*)^k$ be a positive rational map. Define a piecewise-linear map $\mathrm{Trop}^\dagger(f)$ in the same way as $\mathrm{Trop}(f)$ was defined in §2.4, but using max instead of min. By definition, we have $\mathrm{Trop}(x+y) = \min(x,y)$ and $\mathrm{Trop}^\dagger(x+y) = \max(x,y)$. In addition, the identity $\max(x,y) = x+y-\min(x,y)$ implies that

$$\max(x,y) = \mathrm{Trop}\left(\frac{xy}{x+y}\right), \qquad \min(x,y) = \mathrm{Trop}^\dagger\left(\frac{xy}{x+y}\right).$$

Following [NY04, §1.3], define an involution $f \mapsto f^\dagger$ of positive rational functions by

$$f^\dagger(z_1,\dots,z_n) = \frac{1}{f\left(\frac{1}{z_1},\dots,\frac{1}{z_n}\right)}.$$

This involution interchanges the two tropicalizations; that is, $\mathrm{Trop}(f^\dagger) = \mathrm{Trop}^\dagger(f)$. This follows from the simple observation that

$$\left(\frac{1}{x}+\frac{1}{y}\right)^{-1} = \frac{xy}{x+y}.$$

The version of geometric RSK studied in [Kir01, NY04, OSZ14] tropicalizes to RSK using $\mathrm{Trop}^\dagger$. We will now show that our definition of gRSK is obtained from Noumi and Yamada's by the transformation $f \mapsto f^\dagger$. Since the Q-pattern in both definitions consists of the shapes of the P-pattern after inserting the first several rows, it suffices to consider the P-patterns. Let $y_{i,j}(\mathbf{x})$ be the entries of the P-pattern associated to $\mathbf{x}$ by the Noumi–Yamada version of gRSK. Noumi and Yamada define the P-pattern by a recursive "geometric row insertion" procedure that tropicalizes (using $\mathrm{Trop}^\dagger$) to combinatorial row insertion, and then they prove [NY04, Thm. 2.4] that the $y_{i,j}$ are given by the formula[5]

$$y_{i,j}(\mathbf{x}) = \frac{\Delta_{[1,i],[j-i+1,j]}(H(\mathbf{x}_1)\cdots H(\mathbf{x}_m))}{\Delta_{[1,i-1],[j-i+2,j]}(H(\mathbf{x}_1)\cdots H(\mathbf{x}_m))},$$

where $H(a^1,\dots,a^n)$ is the upper-triangular $n\times n$ matrix defined by

$$H(a^1,\dots,a^n) = \sum_{i\le j} a^i a^{i+1}\cdots a^j E_{ij}.$$

The following result shows that our definition of the P-pattern is related to Noumi and Yamada's definition by the transformation $f \mapsto f^\dagger$, and therefore our definition of geometric RSK tropicalizes to combinatorial RSK using Trop.

Lemma 4.12. *Let $\mathbf{x}_i^{-1} = (\frac{1}{x_i^1},\dots,\frac{1}{x_i^n})$. For $1\le i\le m$ and $i\le j\le n$, we have*

$$\frac{\Delta_{[1,i-1],[j-i+2,j]}(H(\mathbf{x}_1^{-1})\cdots H(\mathbf{x}_n^{-1}))}{\Delta_{[1,i],[j-i+1,j]}(H(\mathbf{x}_1^{-1})\cdots H(\mathbf{x}_n^{-1}))} = \frac{\Delta_{[i,j],[1,j-i+1]}(W(\mathbf{x}_1)\cdots W(\mathbf{x}_m))}{\Delta_{[i+1,j],[1,j-i]}(W(\mathbf{x}_1)\cdots W(\mathbf{x}_m))}. \tag{4.4}$$

Proof. For an invertible $n\times n$ matrix A, define $A^\dagger$ to be the matrix obtained from the transposed inverse $(A^{-1})^t$ by multiplying the ith row and column by $(-1)^{i-1}$. It is well-known that the minors of A and $A^\dagger$ are related by

$$\Delta_{I,J}(A^\dagger) = \frac{\Delta_{[n]\setminus I,[n]\setminus J}(A)}{\det(A)}. \tag{4.5}$$

Set $H = H(\mathbf{x}_1^{-1})\cdots H(\mathbf{x}_m^{-1})$ and $M = W(\mathbf{x}_1)\cdots W(\mathbf{x}_m)$. It is easy to see that $H(\mathbf{x}_i^{-1}) = W(\mathbf{x}_i)^\dagger$, so $H = M^\dagger$, and (4.4) asserts that

$$\frac{\Delta_{[1,i-1],[j-i+2,j]}(M^\dagger)}{\Delta_{[1,i],[j-i+1,j]}(M^\dagger)} = \frac{\Delta_{[i,j],[1,j-i+1]}(M)}{\Delta_{[i+1,j],[1,j-i]}(M)}.$$

Using (4.5), this is equivalent to the identity

$$\frac{\Delta_{[i,n],[1,j-i+1]\cup[j+1,n]}(M)}{\Delta_{[i+1,n],[1,j-i]\cup[j+1,n]}(M)} = \frac{\Delta_{[i,j],[1,j-i+1]}(M)}{\Delta_{[i+1,j],[1,j-i]}(M)}. \tag{4.6}$$

[5]Thm. 2.4 in [NY04] is stated in terms of coordinates p_i^j, which are related to our $y_{i,j}$ coordinates by $y_{i,j} = p_i^i p_{i+1}^i \cdots p_j^i$.

Since M is lower-triangular, the submatrices $M_{[i,j],[j+1,n]}$ and $M_{[i+1,j],[j+1,n]}$ consist entirely of zeroes, so we have

$$\frac{\Delta_{[i,n],[1,j-i+1]\cup[j+1,n]}(M)}{\Delta_{[i+1,n],[1,j-i]\cup[j+1,n]}(M)} = \frac{\Delta_{[i,j],[1,j-i+1]}(M)\Delta_{[j+1,n],[j+1,n]}(M)}{\Delta_{[i+1,j],[1,j-i]}(M)\Delta_{[j+1,n],[j+1,n]}(M)},$$

proving (4.6). □

Next, we consider the local move formulation. In [OSZ14], geometric RSK is written as a composition

$$\rho' = ({\rho'}_n^m \circ \cdots \circ {\rho'}_1^m) \circ \cdots \circ ({\rho'}_n^1 \circ \cdots \circ {\rho'}_1^1)$$

(the roles of m and n are reversed in that paper). It is clear from the discussion following [OSZ14, Eq. (3.4)] that ${\rho'}_b^a$ differs from the map ρ_a^b defined in §4.2 by interchanging the functions gMax and gMin in (4.1). The †-involution interchanges gMax and gMin, so we have ${\rho'}_b^a = (\rho_a^b)^\dagger$; since the †-involution commutes with composition of rational functions, we have $\rho = (\rho')^\dagger$. It is shown in [OSZ14, §4] that the map ρ' agrees with Noumi and Yamada's definition of geometric RSK in terms of geometric row insertion; together with the preceding discussion, this proves Theorem 4.5.

4.5. **Decorations and gRSK.** Recall that for $\mathbf{x} \in \mathrm{Mat}_{m\times n}(\mathbb{C}^*)$, the decoration $F(\mathbf{x})$ is defined to be the sum of the x_i^j (whether we view $\mathbf{x}$ as a point of $(X_n)^m$ or of $(X_m)^n$). We now show that $F(\mathbf{x})$ is (almost) the sum of the decorations of the P- and Q-patterns associated to $\mathbf{x}$ by gRSK. This result is essentially a special case of [OSZ14, Thm. 3.2], but since they use a different definition of gRSK, we provide a proof below.

Theorem 4.13 ([OSZ14])**.** *Suppose* $\mathbf{x} \in \mathrm{Mat}_{m\times n}(\mathbb{C}^*)$. *If* $\mathrm{gRSK}(\mathbf{x}) = (P, Q)$, *then*

$$F(\mathbf{x}) = F(P) + F(Q) + \delta_{m,n} z_{n,n},$$

where $\delta_{m,n}$ *is the Kronecker delta, and* $z_{n,n}$ *is the last part of the shape of* P *and* Q *(when* $m = n$*).*

Proof. Let $\mathbf{y} = (y_i^j)$ be the $m \times n$ matrix formed by gluing P and Q along their common shape. By our convention, P occupies the lower left corner of $\mathbf{y}$ and the transpose of Q occupies the upper right corner, with the common shape $(z_1, \ldots, z_p)$ given by the diagonal $(y_m^n, y_{m-1}^{n-1}, \ldots, y_{m-p+1}^{n-p+1})$. Define

$$G(\mathbf{x}) = x_1^1 + \sum_{1\le i\le m, 2\le j\le n} \frac{x_i^j}{x_i^{j-1}} + \sum_{2\le i\le m, 1\le j\le n} \frac{x_i^j}{x_{i-1}^j},$$

so that

$$G(\mathbf{y}) = F(P) + F(Q) + \delta_{m,n} z_{n,n}.$$

By Theorem 4.5 and Lemma 4.4, we have $\mathbf{y} = (\tau \circ \eta)(\mathbf{x})$, so it suffices to show that $G((\tau \circ \eta)(\mathbf{x})) = F(\mathbf{x})$. In fact, we will show that for any $m \times n$ matrix $\mathbf{x}$,

$$G(\eta(\mathbf{x})) = F(\mathbf{x}), \tag{4.7a}$$

$$G(\tau(\mathbf{x})) = G(\mathbf{x}). \tag{4.7b}$$

We first prove (4.7b). The map τ is a composition of the local maps T_a^b with $a < m$ and $b < n$; we will show that each of these maps is f-invariant. By definition, T_a^b replaces x_a^b with

$$\widehat{x}_a^b = \frac{1}{x_a^b} \mathrm{gMax}_a^b(\mathbf{x}) \, \mathrm{gMin}_a^b(\mathbf{x}).$$

If $(a, b) \neq (m, n)$, then the terms of $G(\mathbf{x})$ which depend on x_a^b can be written as

$$\frac{x_a^b}{\mathrm{gMax}_a^b(\mathbf{x})} + \frac{\mathrm{gMin}_a^b(\mathbf{x})}{x_a^b},$$

and it is clear that replacing x_a^b with $\widehat{x}_a^b$ does not change this sum. (Note that this argument would fail for $a = b = 1$ if $G(\mathbf{x})$ did not contain the term x_1^1.)

Now we prove (4.7a). For an order ideal S of the coordinate-wise partial order on $[m] \times [n]$, define

$$G_S(\mathbf{x}) = \sum_{(a,b)\in S} \frac{x_a^b}{\mathrm{gMax}_a^b(\mathbf{x})} + \sum_{(a,b)\notin S} x_a^b.$$

It is easy to see that $G_\emptyset(\mathbf{x}) = F(\mathbf{x})$ and $G_{[m]\times[n]}(\mathbf{x}) = G(\mathbf{x})$. Suppose S' is an order ideal obtained by adding a single element (a,b) to S. The map η_a^b replaces x_a^b with $x_a^b \operatorname{gMax}_a^b(\mathbf{x})$. The expressions $\operatorname{gMax}_i^j(\mathbf{x})$ for $(i,j) \in S$ do not depend on x_a^b, so we have

$$G_{S'}(\eta_a^b(\mathbf{x})) = G_S(\mathbf{x}).$$

Since η is the composition of the maps η_a^b along a linear extension of the lattice $[m] \times [n]$, we conclude that $G_{[m]\times[n]}(\eta(\mathbf{x})) = G_\emptyset(\mathbf{x})$. □

4.6. **Central charge.** Berenstein and Kazhdan defined a function called the *central charge* on products of decorated unipotent crystals[6] [BK07]. This function is invariant under the geometric crystal operators, so if it is positive, its tropicalization is constant on each connected component of the corresponding tensor product of combinatorial crystals, and thus defines a q-analogue of tensor product multiplicity. It is unknown in general whether the central charge is positive, and even in the cases where it is known to be positive (such as the product of "full" Gelfand–Tsetlin geometric crystals GT_n), this q-analogue does not seem to have received much attention.

For the product $(X_n)^m$ of the GL_n-unipotent crystal X_n, the central charge is given by

$$\Delta(\mathbf{x}) = F(\mathbf{x}) - F(P),$$

where $\mathrm{gRSK}(\mathbf{x}) = (P,Q)$. Theorem 4.13 and Corollary 4.7(2) yield the following.

Proposition 4.14. *The central charge of the GL_n-geometric crystal $(X_n)^m$ is given by*

$$\Delta(\mathbf{x}) = F(Q) + \delta_{m,n} z_{n,n}, \tag{4.8}$$

where $z_{n,n}$ is the last part of the shape of P and Q (when $m = n$). In particular, the central charge is positive.

The positivity of the central charge in this case was not previously known.

The components of the tensor product of one-row GL_n-crystals are labeled by the RSK recording tableau Q, and (4.8) gives a simple description of the q-weight assigned to Q by the tropicalization of the central charge. Interestingly, this q-weight is completely different from—and much simpler than—the charge statistic (or closely related energy function), which appears in the "usual" q-analogue of tensor product multiplicity for the product of one-row tableaux [LS78, NY97].

Example 4.15. Let $m = n = 2$, and consider the GL_2-representation $\mathrm{Sym}^{\mu_1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{\mu_2}(\mathbb{C}^2)$, with $\mu_1 \geq \mu_2$. The irreducible components of this representation are labeled by the semistandard Young tableaux of content (μ_1, μ_2), or equivalently, by the Gelfand–Tsetlin patterns

$$\begin{array}{ccccc} & & \mu_1 & & \\ \mu_1 + k & & & & \mu_2 - k \end{array}, \qquad 0 \leq k \leq \mu_2.$$

By Proposition 4.14, the tropicalization of the central charge evaluates to

$$\min(k, \mu_1 - \mu_2 + k, \mu_2 - k) = \min(k, \mu_2 - k)$$

on these Gelfand–Tsetlin patterns. Thus, the q-analogue of the character of $\mathrm{Sym}^{\mu_1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{\mu_2}(\mathbb{C}^2)$ coming from the tropicalization of the central charge is

$$s_{\mu_1,\mu_2} + q s_{\mu_1+1,\mu_2-1} + q^2 s_{\mu_1+2,\mu_2-2} + \ldots + q^2 s_{\mu_1+\mu_2-2,2} + q s_{\mu_1+\mu_2-1,1} + s_{\mu_1+\mu_2},$$

where s_λ denotes a Schur polynomial in two variables. By contrast, the charge-graded analogue of the character of $\mathrm{Sym}^{\mu_1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{\mu_2}(\mathbb{C}^2)$ is the modified Hall–Littlewood polynomial

$$Q'_{(\mu_1,\mu_2)}(x_1, x_2; q) = \sum_{k=0}^{\mu_2} q^k s_{\mu_1+k,\mu_2-k}.$$

[6] Technically, the central charge is only defined on products of unipotent crystals that come from *unipotent bicrystals*.

Example 4.16. For $m > 2$, the difference between the central charge-graded q-analogue and the modified Hall–Littlewood polynomial is more pronounced. For example, when $m = 3, n = 2$, and $(\mu_1, \mu_2, \mu_3) = (4, 3, 2)$, the central charge-graded q-analogue of the character of $\mathrm{Sym}^{\mu_1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{\mu_2}(\mathbb{C}^2) \otimes \mathrm{Sym}^{\mu_3}(\mathbb{C}^2)$ is

$$s_9 + (1+1)s_{81} + (1+1+q)s_{72} + (1+1+q)s_{63} + (1+1)s_{54},$$

whereas

$$Q'_{432}(x_1, x_2; q) = q^7 s_9 + (q^5 + q^6)s_{81} + (q^3 + q^4 + q^5)s_{72} + (q^2 + q^3 + q^4)s_{63} + (q^2 + q^3)s_{54}.$$

5. Invariants of the geometric crystal operators

For the remainder of the paper, we work in the field $\mathbb{C}(X)$ of rational functions in the mn indeterminates $X = \{x_i^j \mid i \in [m], j \in [n]\}$. The actions of the GL_m-geometric crystal operators e_i^c (resp., the GL_n-geometric crystal operators $\overline{e}_j^c$) on $(X_m)^n$ (resp., $(X_n)^m$) induce birational actions on $\mathbb{C}(X)$, which we denote by the same symbols. We write Inv_e (resp., $\mathrm{Inv}_{\overline{e}}$) for the subfield of $\mathbb{C}(X)$ consisting of fixed points of the e_i^c (resp., $\overline{e}_j^c$), and we refer to these fixed points as *e-invariants* (resp., *$\overline{e}$-invariants*). We also write $\mathrm{Inv}_{e\overline{e}}$ for the intersection $\mathrm{Inv}_e \cap \mathrm{Inv}_{\overline{e}}$, whose elements we call *$e\overline{e}$-invariants*.

Set $Z = \{z_{i,j}, z'_{j,i}\} \subset \mathbb{C}(X)$, where $z_{i,j}$ and $z'_{j,i}$ denote the rational functions in the x_i^j which give the entries of the P-pattern and Q-pattern, respectively (note that Z contains only one copy of the entries $z_{k,n} = z'_{k,m}$ of the common shape). Since geometric RSK is a birational isomorphism, it induces an automorphism $\mathrm{gRSK}\colon \mathbb{C}(X) \to \mathbb{C}(X)$ which sends the elements of X to the elements of Z. It follows that the elements of Z are algebraically independent generators of $\mathbb{C}(X)$, so we may view this field alternatively as the field $\mathbb{C}(Z)$ of rational functions in indeterminates Z.

In §5.1, we describe two algebraically independent generating sets for each of the fields Inv_e, $\mathrm{Inv}_{\overline{e}}$, and $\mathrm{Inv}_{e\overline{e}}$, one consisting of polynomials in the X variables, and one consisting of polynomials in the Z variables. We reduce the proof that these are indeed generating sets to Theorem 5.1, which is proved in §5.2.

5.1. Generating sets.

Since the GL_m-geometric crystal operators act only on the Q-pattern and geometric RSK commutes with the geometric crystal operators, one see immediately that the $z_{i,j}$ are e-invariants. The crucial step in proving Theorem 1.1 is the following result.

Theorem 5.1. *The subfield $\mathrm{Inv}_e \subset \mathbb{C}(Z)$ is equal to $\mathbb{C}(z_{i,j})$.*

We will prove Theorem 5.1 in §5.2.

Corollary 5.2. *$\mathrm{Inv}_{\overline{e}}$ is generated by the $z'_{j,i}$, and $\mathrm{Inv}_{e\overline{e}}$ is generated by the entries $z_{1,n}, \ldots, z_{\min(m,n),n}$ of the common shape.*

Proof. The first assertion follows immediately from Theorem 5.1 and the symmetry of gRSK (Corollary 4.7(3)). Since the set Z is algebraically independent, we have

$$\mathrm{Inv}_e \cap \mathrm{Inv}_{\overline{e}} = \mathbb{C}(z_{i,j}) \cap \mathbb{C}(z'_{j,i}) = \mathbb{C}(\{z_{i,j}\} \cap \{z'_{j,i}\}) = \mathbb{C}(z_{k,n}),$$

which proves the second assertion. □

Before giving our generating sets consisting of polynomials in the X variables, we explicitly describe the entries of the $n \times n$ matrix $M(\mathbf{x}) = W(\mathbf{x}_1) \cdots W(\mathbf{x}_m)$ defined in §2.2. For $k \in [m]$ and $r \in [n]$ such that $k + r \geq m + 1$, define

$$E_k^{(r)}(\mathbf{x}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} x_{i_1}^{r+1-i_1} x_{i_2}^{r+2-i_2} \cdots x_{i_k}^{r+k-i_k}.$$

The assumptions on k and r guarantee that the superscripts lie in $[n]$. Note that if we ignore superscripts, each $E_k^{(r)}(\mathbf{x})$ is just the kth elementary symmetric polynomial in m variables. We call $E_k^{(r)}$ a *P-type loop elementary symmetric function* (this terminology will be explained in §6).

It is straightforward to prove by induction on m that

$$M(\mathbf{x})_{ij} = \begin{cases} E^{(i)}_{m+j-i}(\mathbf{x}) & \text{if } i \geq j \text{ and } m + j - i > 0 \\ 1 & \text{if } m + j - i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

This shows that $M(\mathbf{x})$ lies in $(B^-)^{\leq m}$, and its non-trivial entries (*i.e.*, entries not equal to 0 or 1) are the P-type loop elementary symmetric functions. Interchanging the roles of m and n, we see that the $m \times m$ matrix $M(\mathbf{x}^t) = W(\mathbf{x}^1) \cdots W(\mathbf{x}^n)$ lies in $(B^-)^{\leq n}$, and has non-trivial entries

$$E_{k'}^{(r')}(\mathbf{x}^t) = \sum_{1 \leq j_1 < j_2 < \cdots < j_{k'} \leq n} x_{r'+1-j_1}^{j_1} x_{r'+2-j_2}^{j_2} \cdots x_{r'+k'-j_{k'}}^{j_{k'}}$$

for $k' \in [n], r' \in [m]$, and $k' + r' \geq n+1$. We call $E_{k'}^{(r')}(\mathbf{x}^t)$ a *Q-type barred loop elementary symmetric function* (these polynomials will be further examined in the sequel to this paper [BFPS]).

For $k = 1, \ldots, p = \min(m,n)$, define the *shape invariant*

$$S_k(\mathbf{x}) = \Delta_{[k,n],[1,n-k+1]}(M(\mathbf{x})) = \det\left(\left(E_{m-k+1+j-i}^{(k+i-1)}(\mathbf{x})\right)_{i,j \in [n-k+1]}\right),$$

where in the last expression we set $E_0^{(r)} = 1$, and $E_k^{(r)} = 0$ if $k < 0$ or $k > m$.

Remark 5.3. A reader familiar with symmetric functions will recognize that if we replace $E_k^{(r)}(\mathbf{x})$ with the elementary symmetric polynomial $e_k(x_1, \ldots, x_m)$, the last expression becomes the (dual) Jacobi–Trudi formula for the Schur function in m variables associated to the rectangular partition $\left((n-k+1)^{m-k+1}\right)$ (see, *e.g.*, [Sta99, Ch. 7]). In fact, we will see in §6.3 that $S_k(\mathbf{x})$ is a generating function for semistandard tableaux of this shape.

Corollary 5.4.

(1) *The P-type loop elementary symmetric functions are algebraically independent generators of the field* Inv_e *(this is the content of Theorem 1.1).*
(2) *The Q-type barred loop elementary symmetric functions are algebraically independent generators of the field* $\mathrm{Inv}_{\overline{e}}$.
(3) *The shape invariants are algebraically independent generators of the field* $\mathrm{Inv}_{e\overline{e}}$.

Proof. By definition, the P-pattern $(z_{i,j})$ associated to $\mathbf{x}$ is the image of $M(\mathbf{x})$ under $\Psi_n^{\leq m} \colon (B^-)^{\leq m} \to \mathrm{GT}_n^{\leq m}$. This means that each $z_{i,j}$ is a ratio of minors of the matrix $M(\mathbf{x})$, so $\mathbb{C}(z_{i,j}) \subseteq \mathbb{C}\big(E_k^{(r)}(\mathbf{x})\big)$. On the other hand, we have $M(\mathbf{x}) = \Phi_n^{\leq m}(z_{i,j})$ by Lemma 3.3, so each entry of $M(\mathbf{x})$ is a Laurent polynomial in the $z_{i,j}$, and we have the opposite inclusion $\mathbb{C}\big(E_k^{(r)}(\mathbf{x})\big) \subseteq \mathbb{C}(z_{i,j})$. Thus, $\mathrm{Inv}_e = \mathbb{C}\big(E_k^{(r)}(\mathbf{x})\big)$ by Theorem 5.1. The sets $\{E_k^{(r)}(\mathbf{x})\}$ and $\{z_{i,j}\}$ have the same cardinality (the dimension of $\mathrm{GT}_n^{\leq m}$), so the algebraic independence of the $z_{i,j}$ implies the algebraic independence of the $E_k^{(r)}(\mathbf{x})$. This proves part (1), and part (2) follows by symmetry.

The common shape of the P- and Q-patterns is, by definition, given by

$$(z_{1,n}, \ldots, z_{p-1,n}, z_{p,n}) = \left(\frac{S_1}{S_2}, \ldots, \frac{S_{p-1}}{S_p}, S_p\right).$$

This means the shape invariants generate the same field as the $z_{k,n}$, so part (3) follows from Corollary 5.2. □

Remark 5.5. The symmetry of gRSK implies that the shape invariant S_k is also equal to the $(m-k+1) \times (m-k+1)$ minor in the bottom-left corner of $M(\mathbf{x}^t)$, which can be viewed as a generalization of the Schur function in n variables associated to the partition $\left((m-k+1)^{n-k+1}\right)$.

5.2. **Connectedness and proof of Theorem 5.1.** A crystal is *connected* if one can get between any two elements by a sequence of crystal raising and lowering operators. An important property of crystals coming from representations is that a representation is irreducible if and only if its crystal is connected; thus, one can identify the irreducible components of a representation simply by looking at connected components of the crystal. As an example, the set of Gelfand–Tsetlin patterns with fixed shape λ form a connected crystal, corresponding to the Schur module indexed by λ.

We now address the analogous question for Gelfand–Tsetlin geometric crystals. The following result is based on combining the results of Kashiwara–Nakashima–Okado [KNO10] and Kanakubo–Nakashima [KN19].

Theorem 5.6. *Fix $\omega = (z_{1,n}, \ldots, z_{p,n}) \in (\mathbb{C}^*)^p$, where $p = \min(m,n)$, and let $\mathrm{GT}_n^{\leq m}(\omega)$ be the subvariety of $\mathrm{GT}_n^{\leq m}$ consisting of patterns of shape ω. The action of the geometric crystal operators $\overline{e}_1^c, \ldots, \overline{e}_{n-1}^c$ on $\mathrm{GT}_n^{\leq m}(\omega)$ has a dense orbit.*

Proof. We first show that it suffices to prove the result for $\mathrm{GT}_n^{\leq m}(1,\dots,1)$. Given $\omega \in (\mathbb{C}^*)^p$ and $\mathbf{z} \in \mathrm{GT}_n^{\leq m}$, let $\omega \cdot \mathbf{z}$ be the pattern obtained by replacing each $z_{i,j}$ with $\omega_i z_{i,j}$. We claim that

$$\overline{e}_j^c(\omega \cdot \mathbf{z}) = \omega \cdot \overline{e}_j^c(\mathbf{z}). \tag{5.1}$$

Multiplication by $(\omega_1^{-1},\dots,\omega_p^{-1})$ gives a bijection from $\mathrm{GT}_n^{\leq m}(\omega)$ to $\mathrm{GT}_n^{\leq m}(1,\dots,1)$, so (5.1) implies that the orbit structure under the geometric crystal operators is independent of ω.

To prove (5.1), let T_ω be the diagonal $n \times n$ matrix with entries $(\omega_1,\dots,\omega_p,1,\dots,1)$. It is easy to see that $\Phi(\omega \cdot \mathbf{z}) = T_\omega \Phi(\mathbf{z})$ and $\Psi(T_\omega M) = \omega \cdot \Psi(M)$ (here we write Φ, Ψ instead of $\Phi_n^{\leq m}, \Psi_n^{\leq m}$). Set $M = \Phi(\mathbf{z})$. Working through the various definitions, we compute

$$\begin{aligned}
\overline{e}_j^c(\omega \cdot \mathbf{z}) &= \Psi\left(x_j\left((c-1)\overline{\varphi}_j(T_\omega M)\right) \cdot T_\omega M \cdot x_j\left((c^{-1}-1)\overline{\varepsilon}_j(T_\omega M)\right)\right) \\
&= \Psi\left(x_j\left((c-1)\frac{\omega_j}{\omega_{j+1}}\overline{\varphi}_j(M)\right) \cdot T_\omega M \cdot x_j\left((c^{-1}-1)\overline{\varepsilon}_j(M)\right)\right) \\
&= \Psi\left(T_\omega x_j\left(c-1\right)\overline{\varphi}_j(M)\right) \cdot M \cdot x_j\left((c^{-1}-1)\overline{\varepsilon}_j(M)\right)\right) \\
&= \omega \cdot \overline{e}_j^c(\mathbf{z}).
\end{aligned}$$

Let $\mathbf{i} = (i_1,\dots,i_\ell) \in [n-1]^\ell$ be a reduced word for an element of the Weyl group S_n, and consider the map $\Phi_{\mathbf{i}} \colon (\mathbb{C}^*)^\ell \to B^- \subset \mathrm{GL}_n$ defined by

$$\Phi_{\mathbf{i}} \colon (z_1,\dots,z_\ell) \mapsto Y_{i_1}(z_1)\cdots Y_{i_\ell}(z_\ell), \quad \text{where} \quad Y_j(z) = I + (z^{-1}-1)E_{j,j} + (z-1)E_{j+1,j+1} + E_{j+1,j}.$$

One easily verifies that

$$Y_i(z_i)Y_{i+1}(z_{i+1})\cdots Y_{n-1}(z_{n-1}) = W^i\left(\frac{1}{z_i}, \frac{z_i}{z_{i+1}}, \dots, \frac{z_{n-2}}{z_{n-1}}, z_{n-1}\right),$$

where W^i is defined by (3.4). This implies that the restriction of the parametrization $\Phi_n^{\leq m}$ to $\mathrm{GT}_n^{\leq m}(1,\dots,1)$ is the same (up to a simple change of variables) as the parametrization $\Phi_{\mathbf{i}(m)}$, where

$$\mathbf{i}(m) = (p', p'+1, \dots, n-1, p'-1, p', p'+1, \dots, n-1, \dots, 1, 2, \dots, n-1), \qquad p' = \min(m, n-1).$$

(Note that $\mathbf{i}(m)$ is a reduced word for the longest element w_0 if $m \geq n-1$.)

One of the first results in the theory of geometric crystals is that for any reduced word $\mathbf{i}$, the tropicalization, with respect to $\Phi_{\mathbf{i}}$, of the geometric crystal induced from the unipotent crystal structure on B^- is the free combinatorial crystal $B_{\mathbf{i}} = B_{i_1} \otimes \cdots \otimes B_{i_\ell}$ [BK00, §4.4, Rem. 1]. Kanakubo and Nakashima proved that for any reduced word $\mathbf{i}$, the crystal $B_{\mathbf{i}}$ is connected [KN19, Thm. 8.7]. Kashiwara, Nakashima, and Okado proved that if the tropicalization of a geometric crystal is connected, then the geometric crystal has a dense orbit under the action of the geometric crystal operators [KNO10, Thm. 3.3]. □

The proof of Theorem 5.1 requires two additional general lemmas. For the first lemma, let X be an irreducible algebraic variety of dimension N over a field K, and let $G = \{G_i\}_{i \in I}$ be a collection of algebraic groups over K, each acting rationally on X. Define a *G-orbit* in K to be a subset obtained by starting with a point $x_0 \in X$, and repeatedly acting by elements of the groups G_i. Let $K(X)^G = \bigcap_{i \in I} K(X)^{G_i}$ denote the subfield of the fraction field of X consisting of invariants for all the G_i.

Lemma 5.7. *If every point of X is contained in a G-orbit closure of dimension at least $N-d$, then $K(X)^G$ has transcendence degree at most d over K.*

Proof. Suppose $f_1,\dots,f_k \in K(X)^G$ are algebraically independent over K. Let V be the open subset of X where all the f_j are defined. Consider the morphism

$$f = (f_1,\dots,f_k) \colon V \to \mathbb{A}^k.$$

Since the f_j are algebraically independent, the pullback f^* is injective, so f is dominant. A standard result in algebraic geometry (*e.g.*, [Har77, Ex. II.3.22]) says that the general fiber of a dominant morphism $f \colon X \to Y$, where X and Y are irreducible, has dimension $\dim X - \dim Y$. In our setting, this means that there is a non-empty open subset $V' \subseteq V$ such that for all $z \in f(V')$, the fiber $f^{-1}(z)$ has dimension $N-k$.

Choose $z \in f(V')$, and let $\overline{\mathcal{O}}$ be a G-orbit closure of dimension at least $N-d$ which intersects $f^{-1}(z)$. The irreducibility of V ensures that $\overline{\mathcal{O}} \cap U$ has the same dimension as $\overline{\mathcal{O}}$. The f_j are constant on each G-orbit, so we must have $\overline{\mathcal{O}} \cap V \subseteq f^{-1}(z)$, and thus $k \leq d$. □

The second lemma is a straightforward exercise in field theory.

Lemma 5.8. *If $x_1, \dots, x_\ell$ are independent transcendentals over a field K, then K has no proper algebraic extensions inside the field $K(x_1, \dots, x_\ell)$.*

Proof of Theorem 5.1. Let $|P| = \dim \mathrm{GT}_n^{\leq m}$ and $|Q| = \dim \mathrm{GT}_m^{\leq n}$ be the number of entries in the P- and Q-patterns, respectively. We observed at the beginning of §5.1 that Inv_e contains the $z_{i,j}$, so it has transcendence degree at least $|P|$ over $\mathbb{C}$. On the other hand, Theorem 5.6 implies that every point of $\mathrm{GT}_n^{\leq m} * \mathrm{GT}_m^{\leq n}$ is contained in a G-orbit closure of dimension $|Q| - \min(m,n) = mn - |P|$, where $G = \{G_i\}_{i \in [m-1]}$, with $G_i = \mathbb{C}^*$ acting by the geometric crystal operator e_i. Thus, Inv_e has transcendence degree at most $|P|$ by Lemma 5.7, so it is an algebraic extension of $\mathbb{C}(z_{i,j})$. By Lemma 5.8, $\mathbb{C}(z_{i,j})$ has no proper algebraic extensions inside $\mathbb{C}(Z)$, so we must have $\mathrm{Inv}_e = \mathbb{C}(z_{i,j})$. □

6. Invariants of the Weyl group action

6.1. **Weyl group action and geometric R-matrix.** Let X be a GL_m-geometric crystal. For $i \in [m-1]$, define a rational map $s_i \colon X \to X$ by

$$s_i(x) = e_i^{\frac{1}{\alpha_i(\gamma(x))}}(x) = e_i^{\frac{\varepsilon_i(x)}{\varphi_i(x)}}(x).$$

Berenstein and Kazhdan proved that the maps s_i generate a birational action of the Weyl group S_m on X [BK00, Prop. 2.3] (in fact, the purpose of axiom (3) in Definition 2.1 is to ensure that the s_i satisfy the braid relations). We will show that the birational Weyl group action on $(X_m)^n$ agrees with the action of the geometric R-matrix, a map coming from the representation theory of quantum groups of type $A_{n-1}^{(1)}$.

The *geometric R-matrix (of type $A_{n-1}^{(1)}$)* is the rational map

$$R \colon (\mathbb{C}^*)^n \times (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n \times (\mathbb{C}^*)^n$$
$$\big((x^1, \dots, x^n), (y^1, \dots, y^n)\big) \mapsto \big((\widehat{y}^1, \dots, \widehat{y}^n), (\widehat{x}^1, \dots, \widehat{x}^n)\big)$$

given by

$$\widehat{y}^j = y^j \frac{\kappa^{j+1}}{\kappa^j}, \quad \widehat{x}^j = x^j \frac{\kappa^j}{\kappa^{j+1}}, \qquad \kappa^r = \kappa^r(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{n-1} y^r \cdots y^{r+k-1} x^{r+k+1} \cdots x^{r+n-1},$$

with superscripts interpreted modulo n. For $i \in [m-1]$, define $R_i \colon (X_n)^m \to (X_n)^m$ by

$$R_i(\mathbf{x}_1, \dots, \mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_m) = (\mathbf{x}_1, \dots, \widehat{\mathbf{x}}_{i+1}, \widehat{\mathbf{x}}_i, \dots, \mathbf{x}_m),$$

where $(\widehat{\mathbf{x}}_{i+1}, \widehat{\mathbf{x}}_i) = R(\mathbf{x}_i, \mathbf{x}_{i+1})$.

Proposition 6.1. *If $(\mathbf{x}^1, \dots, \mathbf{x}^n) \in (X_m)^n$ are the columns of a matrix $\mathbf{x} \in \mathrm{Mat}_{m \times n}(\mathbb{C}^*)$, and $(\mathbf{x}_1, \dots, \mathbf{x}_m) \in (X_n)^m$ are the rows of $\mathbf{x}$, then we have*

$$s_i(\mathbf{x}^1, \dots, \mathbf{x}^n) = R_i(\mathbf{x}_1, \dots, \mathbf{x}_m).$$

Proof. By Lemma 2.3, s_i fixes row $\mathbf{x}_k$ for $k \neq i, i+1$, and acts on rows $\mathbf{x}_i, \mathbf{x}_{i+1}$ by

$$x_i^j \mapsto x_i^j \frac{\sigma^j\left(\mathbf{x}_i, \mathbf{x}_{i+1}; \frac{\pi_{i+1}}{\pi_i}\right)}{\sigma^{j-1}\left(\mathbf{x}_i, \mathbf{x}_{i+1}; \frac{\pi_{i+1}}{\pi_i}\right)}, \qquad x_{i+1}^j \mapsto x_{i+1}^j \frac{\sigma^{j-1}\left(\mathbf{x}_i, \mathbf{x}_{i+1}; \frac{\pi_{i+1}}{\pi_i}\right)}{\sigma^{j}\left(\mathbf{x}_i, \mathbf{x}_{i+1}; \frac{\pi_{i+1}}{\pi_i}\right)},$$

where $\pi_i = x_i^1 \cdots x_i^n$, and $\sigma^j(\mathbf{x}_i, \mathbf{x}_{i+1}; c)$ is defined by (2.4). For $j = 0, \ldots, n$, we compute

$$\begin{aligned}\sigma^j\left(\mathbf{x}_i, \mathbf{x}_{i+1}; \frac{\pi_{i+1}}{\pi_i}\right) &= \frac{\pi_{i+1}}{\pi_i} \sum_{k=1}^{j} x_{i+1}^1 \cdots x_{i+1}^{k-1} x_i^{k+1} \cdots x_i^n + \sum_{k=j+1}^{n} x_{i+1}^1 \cdots x_{i+1}^{k-1} x_i^{k+1} \cdots x_i^n \\ &= \frac{x_{i+1}^1 \cdots x_{i+1}^j}{x_i^1 \cdots x_i^j} \left(\sum_{k=1}^{j} x_{i+1}^{j+1} \cdots x_{i+1}^n x_{i+1}^1 \cdots x_{i+1}^{k-1} x_i^{k+1} \cdots x_i^j \right. \\ &\qquad \left. + \sum_{k=j+1}^{n} x_{i+1}^{j+1} \cdots x_{i+1}^{k-1} x_i^{k+1} \cdots x_i^n x_i^1 \cdots x_i^j \right) \\ &= \frac{x_{i+1}^1 \cdots x_{i+1}^j}{x_i^1 \cdots x_i^j} \kappa^{j+1}(\mathbf{x}_i, \mathbf{x}_{i+1}).\end{aligned}$$

The proposition follows. □

Remark 6.2. Yamada [Yam01] defined the geometric (or birational) R-matrix as a geometric lift of the *combinatorial R-matrix*, which is the unique crystal isomorphism that permutes adjacent factors in a tensor product of one-row affine crystals (see, *e.g.*, [HKO+99, HKO+02b]). Several different proofs of the fact that the maps R_i generate a birational S_m-action have appeared in the literature [Yam01, KNY02, Eti03, LP12, LP13a].

6.2. **Loop symmetric functions.** The geometric R-matrices $R_1, \ldots, R_{m-1}$ generate a group of automorphisms of the field $F = \mathbb{C}(x_i^j)$ isomorphic to S_m. The fixed field Inv_R of this group of automorphisms has been studied in several papers, starting with [LP12], under the name of loop symmetric functions. It is more convenient to describe loop symmetric functions in terms of variables $x_i^{(r)}$, where the superscript r is considered modulo n, and $x_i^j = x_i^{(j+i-1)}$. In other words, we make the identification

$$\mathbf{x}_i = (x_i^1, x_i^2, \ldots, x_i^n) = (x_i^{(i)}, x_i^{(i+1)}, \ldots x_i^{(i+n-1)}).$$

Definition 6.3. For $k \in [m]$ and $r \in [n]$, define the *loop elementary symmetric function*

$$E_k^{(r)} = E_k^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m} x_{i_1}^{(r)} x_{i_2}^{(r+1)} \cdots x_{i_k}^{(r+k-1)}.$$

It is convenient to allow arbitrary integers r in the superscript by setting $E_k^{(r)} = E_k^{(r \mod n)}$, and to define $E_0^{(r)} = 1$ and $E_k^{(r)} = 0$ if $k < 0$ or $k > m$.

Define $\mathrm{LSym} = \mathrm{LSym}_m(n)$ to be the ring generated (over $\mathbb{C}$) by the $E_k^{(r)}$. We call this the *ring of loop symmetric functions in m variables and n colors*.

When $k + r \ge m + 1$, $E_k^{(r)}$ is the P-type loop elementary symmetric function that appeared in §5.1 as an entry of the $n \times n$ matrix $M(\mathbf{x})$. We now describe an infinite "cyclic extension" of the matrix $M(\mathbf{x})$ that has all the loop elementary symmetric functions as entries.

Define an *n-periodic matrix* to be a $\mathbb{Z} \times \mathbb{Z}$ array $(A_{ij})_{i,j \in \mathbb{Z}}$ such that $A_{ij} = A_{i+n,j+n}$ for all i, j, and $A_{ij} = 0$ if $j - i$ is sufficiently large. The second hypothesis ensures that multiplication of these matrices is well-defined. Given an n-tuple of variables $(x^1, \ldots, x^n)$, let $\widetilde{W} = \widetilde{W}(x^1, \ldots, x^n)$ be the n-periodic matrix with $\widetilde{W}_{j,j-1} = 1$, $\widetilde{W}_{jj} = x^j$ (the superscript of x^j is interpreted mod n), and all other entries zero. This matrix is called a *whirl* in [LP12]. For example, when $n = 3$,

$$\widetilde{W}(x^1, x^2, x^3) = \begin{pmatrix} x^1 & 0 & 0 & 0 & \\ 1 & x^2 & 0 & 0 & \\ 0 & 1 & x^3 & 0 & \cdots \\ 0 & 0 & 1 & x^1 & \\ 0 & 0 & 0 & 1 & \\ & \vdots & & & \ddots \end{pmatrix}.$$

Note that we are depicting only the quadrant of the matrix with $i, j \ge 1$.

$$\left(\begin{array}{cc|cc|cc|c}
x_1^{(1)}x_2^{(2)}x_3^{(1)} = E_3^{(1)} & 0 & 0 & 0 & 0 & 0 & \cdots \\
x_1^{(2)}x_2^{(1)} + x_1^{(2)}x_3^{(1)} + x_2^{(2)}x_3^{(1)} = E_2^{(2)} & x_1^{(2)}x_2^{(1)}x_3^{(2)} = E_3^{(2)} & 0 & 0 & 0 & 0 & \\
\hline
x_1^{(1)} + x_2^{(1)} + x_3^{(1)} = E_1^{(1)} & x_1^{(1)}x_2^{(2)} + x_1^{(1)}x_3^{(2)} + x_2^{(1)}x_3^{(2)} = E_2^{(1)} & E_3^{(1)} & 0 & 0 & 0 & \\
1 & x_1^{(2)} + x_2^{(2)} + x_3^{(2)} = E_1^{(2)} & E_2^{(2)} & E_3^{(2)} & 0 & 0 & \\
\hline
0 & 1 & E_1^{(1)} & E_2^{(1)} & E_3^{(1)} & 0 & \\
0 & 0 & 1 & E_1^{(2)} & E_2^{(2)} & E_3^{(2)} & \\
\hline
0 & 0 & 0 & 1 & E_1^{(1)} & E_2^{(1)} & \\
0 & 0 & 0 & 0 & 1 & E_1^{(2)} & \\
\hline
\vdots & & & & & & \ddots
\end{array}\right)$$

FIGURE 3. The 2-periodic matrix $\widetilde{M}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, with 2×2 blocks indicated. We depict only the quadrant of the matrix with coordinates $i, j \geq 1$.

For $\mathbf{x}_1, \ldots, \mathbf{x}_m \in (\mathbb{C}^*)^n$, define $\widetilde{M}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \widetilde{W}(\mathbf{x}_1) \cdots \widetilde{W}(\mathbf{x}_m)$. It is straightforward to prove by induction that the entries of this matrix are the loop elementary symmetric functions:

$$\widetilde{M}(\mathbf{x}_1, \ldots, \mathbf{x}_m)_{ij} = E^{(i)}_{m+j-i}(\mathbf{x}_1, \ldots, \mathbf{x}_m). \tag{6.1}$$

An example of the matrix $\widetilde{M}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ appears in Figure 3. As one can see in this example, $\widetilde{M}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ consists of $n \times n$ blocks that repeat along (block) diagonals, and the top nonzero block is the matrix $M(\mathbf{x})$ that has appeared throughout the paper. The P-type loop elementary symmetric functions are exactly the loop elementary symmetric functions appearing in this block.

Theorem 6.4 ([LP11, Thm. 4.1])**.** *The loop elementary symmetric functions are algebraically independent, and they generate the invariant field* Inv_R*. That is, we have*

$$\mathrm{Inv}_R = \mathrm{Frac}(\mathrm{LSym}) = \mathbb{C}(E_k^{(r)}).$$

Example 6.5.

(a) When $n = 1$, $E_i^{(1)}$ is the elementary symmetric polynomial in the m variables $x_1^{(1)}, \ldots, x_m^{(1)}$, and R_i simply swaps the variables $x_i^{(1)}$ and $x_{i+1}^{(1)}$. Theorem 6.4 reduces in this case to Newton's fundamental theorem of symmetric functions, which says that the ring of symmetric polynomials in m variables is the polynomial ring in the elementary symmetric polynomials $e_1, \ldots, e_m$.

(b) When $m = 1$, there are no maps R_i, and LSym is the polynomial ring in the n variables $x_1^{(r)} = E_1^{(r)}$, $r \in [n]$, so Theorem 6.4 is trivial in this case.

In fact, one has the stronger result that LSym is equal to the ring of polynomial R-invariants [Lam12, Thm. 4.4]. This result, together with Theorem 1.1, implies that the ring of polynomial e-invariants is generated by the P-type loop elementary symmetric functions.

6.3. **Loop Schur functions.** For our next definition, we need to recall some notions from tableau combinatorics. We identity a partition with its Young diagram, which we view, following the English convention, as a northwest-justified collection of unit cells in the plane. For partitions μ and λ, we write $\mu \subseteq \lambda$ if the Young diagram of μ is contained in that of λ, and if $\mu \subseteq \lambda$, we define the skew diagram λ/μ to be the cells in λ which are not in μ. For a cell $s = (i, j)$ in the ith row and jth column (using matrix coordinates), the *content* of s is $c(s) = i - j$ (this is the opposite of the usual definition). Let λ' denote the conjugate or transpose partition of λ. A *semistandard Young tableau of shape* λ/μ is a filling $T \colon \lambda/\mu \to \mathbb{Z}_{>0}$ of the cells of λ/μ with positive integers, such that rows are weakly increasing, and columns are strictly increasing. Let $\mathrm{SSYT}_{\leq m}(\lambda/\mu)$ be the set of semistandard Young tableaux of shape λ/μ with entries at most m.

For partitions $\mu \subseteq \lambda$ and a color $r \in [n]$, define the *loop skew Schur function*

$$s^{(r)}_{\lambda/\mu} = s^{(r)}_{\lambda/\mu}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \sum_{T \in \mathrm{SSYT}_{\leq m}(\lambda)} \mathbf{x}^T,$$

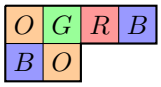
where $\mathbf{x}^T = \prod_{s\in\lambda/\mu} x_{T(s)}^{(c(s)+r)}$. If $\mu = \emptyset$, then $s^{(r)}_{\lambda/\emptyset} = s^{(r)}_\lambda$ is a *loop Schur function*. We refer to $c(s)+r$ (considered modulo n) as the *color* of the cell s.

The following analogue of the Jacobi–Trudi formula shows that the loop skew Schur functions lie in LSym.

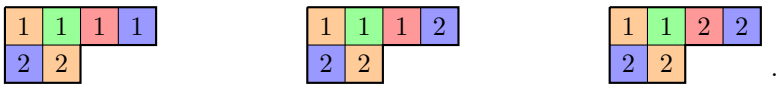
Proposition 6.6 ([LP12, Thm. 7.6]). *Suppose $\mu \subseteq \lambda$, and the conjugate partition λ' has length at most ℓ. Then*

$$s^{(r)}_{\lambda/\mu} = \det\left(E^{(r+\mu'_j-j+1)}_{\lambda'_i-\mu'_j+j-i}\right)_{i,j=1}^{\ell}.$$

Example 6.7. We compute the loop Schur function $s^{(1)}_{(4,2)}(\mathbf{x}_1,\mathbf{x}_2)$ in the case $n=4$. The colors $c(s)+1$ of the cells of $(4,2)$ are shown below, where Orange is 1, Blue is 2, Red is 3, and Green is 4:



There are three semistandard tableaux of shape λ with entries in $\{1,2\}$:



Thus, we have

$$s^{(1)}_{(4,2)}(\mathbf{x}_1,\mathbf{x}_2) = x_1^{(1)}x_1^{(2)}x_1^{(3)}x_1^{(4)}x_2^{(1)}x_2^{(2)} + x_1^{(1)}x_1^{(3)}x_1^{(4)}x_2^{(1)}(x_2^{(2)})^2 + x_1^{(1)}x_1^{(4)}x_2^{(1)}(x_2^{(2)})^2x_2^{(3)}.$$

The reader may verify that

$$s^{(1)}_{(4,2)}(\mathbf{x}_1,\mathbf{x}_2) = \det\begin{pmatrix} E_2^{(1)} & 0 & 0 & 0 \\ E_1^{(1)} & E_2^{(4)} & 0 & 0 \\ 0 & 1 & E_1^{(3)} & E_2^{(2)} \\ 0 & 0 & 1 & E_1^{(2)} \end{pmatrix}.$$

The Jacobi–Trudi formula implies that the minors of the n-periodic matrix $\widetilde{M}(\mathbf{x}_1,\ldots,\mathbf{x}_m)$ are precisely the loop skew Schur functions (one must reflect a submatrix of $\widetilde{M}(\mathbf{x}_1,\ldots,\mathbf{x}_m)$ over the anti-diagonal to get a matrix in "Jacobi–Trudi form," but this does not change the determinant). In particular, the entries of the gRSK P-pattern are ratios of loop Schur functions of rectangular shape. For $1 \le i \le m$ and $i \le j \le n$, let

$$\square(i,j) = s^{(j)}_{(\underbrace{j-i+1,\ldots,j-i+1}_{m-i+1 \text{ times}})}(\mathbf{x}_1,\ldots,\mathbf{x}_m)$$

be the loop Schur function associated with an $(m-i+1)\times(j-i+1)$ rectangle, where the unique northwest corner has color j. Set $\square(i+1,j) = 1$ if $i = j$ or $i = m$. Since $M(\mathbf{x})$ is the block of $\widetilde{M}(\mathbf{x}_1,\ldots,\mathbf{x}_m)$ consisting of rows and columns $1,\ldots,n$, the Jacobi–Trudi formula implies that

$$z_{i,j} = \frac{\Delta_{[i,j],[1,j-i+1]}(M(\mathbf{x}))}{\Delta_{[i+1,j],[1,j-i]}(M(\mathbf{x}))} = \frac{\square(i,j)}{\square(i+1,j)}. \tag{6.2}$$

The shape invariants are given by

$$S_k(\mathbf{x}) = \square(k,n) = s^{(n)}_{(\underbrace{n-k+1,\ldots,n-k+1}_{m-k+1 \text{ times}})}(\mathbf{x}_1,\ldots,\mathbf{x}_m),$$

as promised in Remark 5.3.

Appendix A. Proofs of technical lemmas

Proof of Lemma 2.3. The formula for $\gamma(\mathbf{x})$ is immediate from (2.1) and (2.2). For the other assertions, suppose $Y_1,\ldots,Y_n$ are GL_m-geometric crystals, and $y_j \in Y_j$. Set

$$A_k = \prod_{r=1}^{k-1}\varepsilon_i(y_r)\prod_{r=k+1}^{n}\varphi_i(y_r), \qquad A'_k = \prod_{r=1}^{k-1}\varepsilon_i(y_r)\prod_{r=k+1}^{n-1}\varphi_i(y_r)$$

for $k \in [n]$ and $k \in [n-1]$, respectively. Using (2.2) and the identity

$$\prod_{r=1}^{n-1} \varepsilon_i(y_r) + \sum_{k=1}^{n-1} A'_k \varphi_i(y_n) = \sum_{k=1}^{n} A_k, \tag{A.1}$$

one shows by induction on n that

$$\varepsilon_i(y_1, \dots, y_n) = \frac{\prod_{r=1}^{n} \varepsilon_i(y_r)}{\sum_{k=1}^{n} A_k}, \qquad \varphi_i(y_1, \dots, y_n) = \frac{\prod_{r=1}^{n} \varphi_i(y_r)}{\sum_{k=1}^{n} A_k}. \tag{A.2}$$

The formulas for $\varepsilon_i(\mathbf{x})$ and $\varphi_i(\mathbf{x})$ are obtained by taking $Y_j = X_m$ for all j, and using (2.1).

Now we prove by induction that

$$e_i^c(y_1, \dots, y_n) = (e_i^{c_1}(y_1), \dots, e_i^{c_n}(y_n)), \quad \text{where} \quad c_j = \frac{c \sum_{k=1}^{j} A_k + \sum_{k=j+1}^{n} A_k}{c \sum_{k=1}^{j-1} A_k + \sum_{k=j}^{n} A_k}, \tag{A.3}$$

from which the stated formula for $e_i^c(\mathbf{x})$ follows. For $n = 2$, this is just the definition (2.3). For $n > 2$, we have $e_i^c(y_1, \dots, y_n) = (e_i^{c^+}(y_1, \dots, y_{n-1}), e_i^{c/c^+}(y_n))$, where

$$c^+ = \frac{c\varphi_i(y_n) + \dfrac{\prod_{r=1}^{n-1} \varepsilon_i(y_r)}{\sum_{k=1}^{n-1} A'_k}}{\varphi_i(y_n) + \dfrac{\prod_{r=1}^{n-1} \varepsilon_i(y_r)}{\sum_{k=1}^{n-1} A'_k}} = \frac{c \sum_{k=1}^{n-1} A_k + A_n}{\sum_{k=1}^{n} A_k}$$

by (A.1) and (A.2). This shows that $c/c^+ = c_n$. For the other c_j, induction gives $e_i^{c^+}(y_1, \dots, y_{n-1}) = (e_i^{c_1^+}(y_1), \dots, e_i^{c_{n-1}^+}(y_{n-1}))$, where

$$c_j^+ = \frac{c^+ \sum_{k=1}^{j} A'_k + \sum_{k=j+1}^{n-1} A'_k}{c^+ \sum_{k=1}^{j-1} A'_k + \sum_{k=j}^{n-1} A'_k} = \frac{\dfrac{c \sum_{k=1}^{n-1} A_k + A_n}{\sum_{k=1}^{n} A_k} \sum_{k=1}^{j} A'_k + \sum_{k=j+1}^{n-1} A'_k}{\dfrac{c \sum_{k=1}^{n-1} A_k + A_n}{\sum_{k=1}^{n} A_k} \sum_{k=1}^{j-1} A'_k + \sum_{k=j}^{n-1} A'_k}. \tag{A.4}$$

The numerator of (A.4) can be rewritten as

$$\frac{c \sum_{k=1}^{n-1} A_k \sum_{k=1}^{j} A'_k + \sum_{k=1}^{n-1} A_k \sum_{k=j+1}^{n-1} A'_k + A_n \sum_{k=1}^{n-1} A'_k}{\sum_{k=1}^{n} A_k} = \frac{\left(c \sum_{k=1}^{j} A_k + \sum_{k=j+1}^{n} A_k \right) \sum_{k=1}^{n-1} A'_k}{\sum_{k=1}^{n} A_k},$$

where the second expression is obtained by using $A_k = A'_k \varphi_i(y_n)$ for $k \le n-1$. The denominator of (A.4) can be rewritten in the same way with j replaced by $j-1$; this proves (A.3). □

Proof of Lemma 3.9. We first rewrite the decoration as

$$F(\mathbf{z}) = \sum_{j=1}^{n-1} \sum_{i=1}^{\min(m,n-j)} \frac{z_{i,i+j}}{z_{i,i+j-1}} + \sum_{i=1}^{\min(m-1,n-1)} \sum_{j=1}^{n-i} \frac{z_{i,i+j-1}}{z_{i+1,i+j}} + \mathbb{1}_{m<n} z_{m,m}.$$

It suffices to show that

$$\sum_{i=1}^{\min(m,n-j)} \frac{z_{i,i+j}}{z_{i,i+j-1}} = \begin{cases} \dfrac{\Delta_{[m+1,m+j],[1,j-1]\cup\{j+1\}}(M)}{\Delta_{[m+1,m+j],[1,j]}(M)} & \text{if } j \le n-m, \\ \dfrac{\Delta_{[n-j+1,n],[1,j-1]\cup\{j+1\}}(M)}{\Delta_{[n-j+1,n],[1,j]}(M)} & \text{if } j \ge n-m+1, \end{cases} \tag{A.5}$$

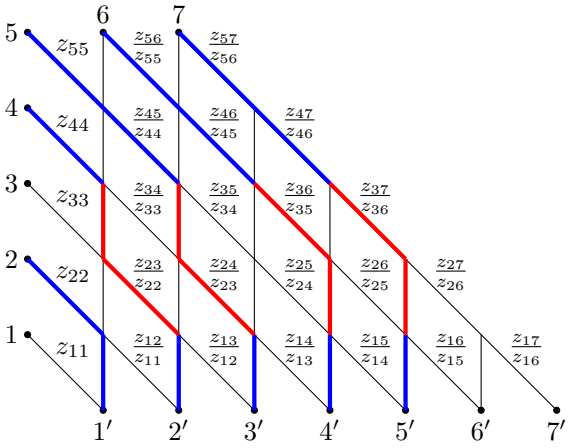


FIGURE 4. A non-intersecting collection of paths in the network $\Gamma_7^{\leq 5}$ which contributes to the minor $\Delta_{24567,12345}(M)$. The blue edges appear in all non-intersecting collections of paths from sources $\{2,4,5,6,7\}$ to sinks $\{1',2',3',4',5'\}$; the red edges appear in the collection corresponding to the choice $j=3$ in the proof of Lemma 3.9. The weight of this collection is $z_{2,4}\frac{z_{3,7}}{z_{3,5}}z_{4,7}z_{5,7}$.

for $j = 1, \ldots, n-1$,

$$\sum_{j=1}^{n-i} \frac{z_{i,i+j-1}}{z_{i+1,i+j}} = \frac{\Delta_{\{i\}\cup[i+2,n],[1,n-i]}(M)}{\Delta_{[i+1,n],[1,n-i]}(M)} \tag{A.6}$$

for $i = 1, \ldots, \min(m-1, n-1)$, and

$$z_{m,m} = \frac{\Delta_{\{m\}\cup[m+2,n],[1,n-m]}(M)}{\Delta_{[m+1,n],[1,n-m]}(M)} \tag{A.7}$$

if $m < n$.

Each of these equations is verified by applying the Lindström/Gessel–Viennot Lemma to the network $\Gamma_n^{\leq m}$ introduced in the proof of Lemma 3.3. We explain only (A.6); the arguments for the other two equations are a bit easier. The denominator of the right-hand side of (A.6) is given by

$$\Delta_{[i+1,n],[1,n-i]}(M) = \prod_{k=i+1}^{\min(m,n)} z_{k,n}, \tag{A.8}$$

since this is the weight of the unique non-intersecting collection of paths from $[i+1,n]$ to $[1,n-i]$.

Now consider non-intersecting collections of paths from $\{i\}\cup[i+2,n]$ to $[1,n-i]$. Since $i \leq m-1$, there is a unique path from i to $1'$. For $r = 2, \ldots, n-i$, the path from source $i+r-1$ to sink r' must start by taking diagonal steps down to level $i+1$ (the height of source $i+1$), and it must end with vertical steps from level $i-1$ down to the bottom of the network. This leaves two choices for each path: to get from level $i+1$ to level $i-1$, it can travel diagonally and then vertically (DV), or vertically and then diagonally (VD). If the path ending in sink r' chooses DV, then the path ending in sink $(r+1)'$ must also choose DV to avoid a collision. Thus, there must be some $j \in [1, n-i]$ such that the paths ending at $2', \ldots, j'$ choose VD, and the paths ending at $(j+1)', \ldots, (n-i)'$ choose DV (see Figure 4 for an illustration). We conclude that

$$\Delta_{\{i\}\cup[i+2,n],[1,n-i]}(M) = \prod_{k=i+2}^{\min(m,n)} z_{k,n} \times \sum_{j=1}^{n-i} z_{i,i+j-1}\frac{z_{i+1,n}}{z_{i+1,i+j}}.$$

Dividing by (A.8), we obtain (A.6). □

Proof of Lemma 3.10. Let $M = \Phi_n(\mathbf{z})$. By considering paths in the network Γ_n from source j (resp., $j+1$) to sink j', one sees that

$$M_{j,j} = \frac{z_{1,j} \cdots z_{j,j}}{z_{1,j-1} \cdots z_{j-1,j-1}}, \qquad M_{j+1,j} = z_{j+1,j+1} \sum_{k=1}^{j} \frac{z_{1,j} \cdots z_{k-1,j}}{z_{1,j-1} \cdots z_{k-1,j-1}} \frac{z_{k+1,j+1} \cdots z_{j,j+1}}{z_{k+1,j} \cdots z_{j,j}}.$$

The formula for $\overline{\gamma}(\mathbf{z})$ follows immediately. For $\overline{\varepsilon}_j(\mathbf{z})$, we compute

$$\begin{aligned}\overline{\varepsilon}_j(\mathbf{z}) = \frac{M_{j+1,j+1}}{M_{j+1,j}} &= \frac{z_{j+1,j+1} \displaystyle\prod_{i=1}^{j} \frac{z_{i,j+1}}{z_{i,j}}}{z_{j+1,j+1} \displaystyle\sum_{k=1}^{j} \prod_{i=1}^{k-1} \frac{z_{i,j}}{z_{i,j-1}} \prod_{i=k+1}^{j} \frac{z_{i,j+1}}{z_{i,j}}} \\ &= \frac{1}{\displaystyle\sum_{k=1}^{j} \frac{z_{1,j}}{z_{1,j+1}} \prod_{i=1}^{k-1} \frac{z_{i,j} z_{i+1,j}}{z_{i,j-1} z_{i+1,j+1}}} = \frac{z_{1,j+1}}{z_{1,j}} \operatorname{gMax}_{1 \le k \le j} \left(\prod_{i=2}^{k} \phi_{i,j}^{-1} \right).\end{aligned}$$

A similar computation gives the formula for $\overline{\varphi}_j(\mathbf{z})$.

It remains to consider the geometric crystal operators. Set

$$M' = x_j\left((c-1)\overline{\varphi}_j(\mathbf{z})\right) \cdot M \cdot x_j\left((c^{-1}-1)\overline{\varepsilon}_j(\mathbf{z})\right).$$

By definition, the entries of the GT pattern $\mathbf{z}' = \overline{e}_j^c(\mathbf{z})$ are given by

$$z'_{i,j} = \left(\frac{\Delta_{[i,j]}(M')}{\Delta_{[i+1,j]}(M')} \right)_{1 \le i \le j \le n}.$$

The matrix M' is obtained from M by adding a multiple of row $j+1$ to row j, and a multiple of column j to column $j+1$. The latter operation does not affect flag minors, and the former operation only affects minors of the form $\Delta_{[s,r]}$ if $r = j$. This implies that $z'_{i,r} = z_{i,r}$ if $r \neq j$, and

$$z'_{i,j} = \frac{\Delta_{[i,j]}(M) + (c-1)\overline{\varphi}_j(\mathbf{z})\Delta_{[i,j-1]\cup\{j+1\}}(M)}{\Delta_{[i+1,j]}(M) + (c-1)\overline{\varphi}_j(\mathbf{z})\Delta_{[i+1,j-1]\cup\{j+1\}}(M)}$$

(when $i = j$, the denominator is equal to 1). We will show that

$$\Delta_{[i,j]}(M) + (c-1)\overline{\varphi}_j(\mathbf{z})\Delta_{[i,j-1]\cup\{j+1\}}(M) = \prod_{k=i}^{j} z_{k,j} \frac{\displaystyle\sum_{k=1}^{j} c^{\mathbb{1}_{k \ge i}} \prod_{\ell=k+1}^{j} \phi_{\ell,j}^{-1}}{\displaystyle\sum_{k=1}^{j} \prod_{\ell=k+1}^{j} \phi_{\ell,j}^{-1}} = \prod_{k=i}^{j} z_{k,j} \;\times\; \frac{C_{i,j}}{C_{j+1,j}}$$

for $i = 1, \ldots, j$, from which it follows that $z'_{i,j} = z_{i,j} \dfrac{C_{i,j}}{C_{i+1,j}}$.

Applying the Lindström/Gessel–Viennot Lemma to Γ_n, we obtain

$$\Delta_{[i,j]}(M) = \prod_{k=i}^{j} z_{k,j}, \qquad \Delta_{[i,j-1]\cup\{j+1\}}(M) = z_{j+1,j+1} \prod_{k=i}^{j-1} z_{k,j-1} \sum_{k=i}^{j} \prod_{\ell=i}^{k-1} \frac{z_{\ell,j}}{z_{\ell,j-1}} \prod_{\ell=k+1}^{j} \frac{z_{\ell,j+1}}{z_{\ell,j}}.$$

Using these expressions and the formula $\overline{\varphi}_j(\mathbf{z}) = \frac{z_{j,j}}{z_{j+1,j+1}} \operatorname{gMax}_{1\leq k\leq j}\left(\prod_{\ell=k+1}^{j}\phi_{\ell,j}\right)$, we compute

$$\begin{aligned}
\Delta_{[i,j]}(M) + (c-1)\overline{\varphi}_j(\mathbf{z})\Delta_{[i,j-1]\cup\{j+1\}}(M) &= \prod_{k=i}^{j} z_{k,j}\left(1 + (c-1)\frac{\displaystyle\prod_{\ell=i}^{j-1}\frac{z_{\ell,j-1}}{z_{\ell,j}}\sum_{k=i}^{j}\prod_{\ell=i}^{k-1}\frac{z_{\ell,j}}{z_{\ell,j-1}}\prod_{\ell=k+1}^{j}\frac{z_{\ell,j+1}}{z_{\ell,j}}}{\displaystyle\sum_{k=1}^{j}\prod_{\ell=k+1}^{j}\phi_{\ell,j}^{-1}}\right)\\
&= \prod_{k=i}^{j} z_{k,j}\left(1 + (c-1)\frac{\displaystyle\sum_{k=i}^{j}\prod_{\ell=k}^{j-1}\frac{z_{\ell,j-1}}{z_{\ell,j}}\prod_{\ell=k+1}^{j}\frac{z_{\ell,j+1}}{z_{\ell,j}}}{\displaystyle\sum_{k=1}^{j}\prod_{\ell=k+1}^{j}\phi_{\ell,j}^{-1}}\right)\\
&= \prod_{k=i}^{j} z_{k,j}\left(1 + (c-1)\frac{\displaystyle\sum_{k=i}^{j}\prod_{\ell=k+1}^{j}\phi_{\ell,j}^{-1}}{\displaystyle\sum_{k=1}^{j}\prod_{\ell=k+1}^{j}\phi_{\ell,j}^{-1}}\right)\\
&= \prod_{k=i}^{j} z_{k,j} \times \frac{C_{i,j}}{C_{j+1,j}},
\end{aligned}$$

as claimed. □

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