GLUING OF DERIVED EQUIVALENCES OF DG CATEGORIES

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Abstract. A diagram consisting of differential graded (dg for short) categories and dg functors is formutated as a colax functor X from a small category I to the 2-category $\Bbbk$-**dgCat** of dg categories, dg functors and dg natural transformations for a fixed commutative ring $\Bbbk$. The dg categories $X(i)$ with i objects of I can be glued together to have a single dg category $\mathrm{Gr}(X)$, called the Grothendieck construction of X. In this paper, we consider colax functors X and X' from I to $\Bbbk$-**dgCat** such that $X(i)$ and $X'(i)$ are derived equivalent for all objects i of I, and give a way to glue these derived equivalences together and a sufficient condition for this gluing to be a derived equivalence between their Grothendieck constructions $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$. This generalizes the main result of [8] to the dg case. Finally, we give some examples to illustrate our main theorem.

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1. Introduction

Throughout this paper we fix a commutative ring $\Bbbk$, and all linear categories and all linear functors are considered to be over $\Bbbk$. Let $\mathscr{A}$ be a linear category. Then we have canonical embeddings $\mathscr{A} \hookrightarrow \operatorname{Mod}\mathscr{A} \hookrightarrow \mathscr{D}(\operatorname{Mod}\mathscr{A})$, where $\operatorname{Mod}\mathscr{A}$ denotes the category of (right) $\mathscr{A}$-modules, and $\mathscr{D}(\operatorname{Mod}\mathscr{A})$ stands for the derived module category of $\mathscr{A}$ that turns out to be a triangulated category. Two linear categories $\mathscr{A}$ and $\mathscr{A}'$ are said to be *derived equivalent* if $\mathscr{D}(\operatorname{Mod}\mathscr{A})$ and $\mathscr{D}(\operatorname{Mod}\mathscr{A}')$ are equivalent as triangulated categories. If $\mathscr{A}$ and $\mathscr{A}'$ are *Morita equivalent*, i.e., if $\operatorname{Mod}\mathscr{A}$ and $\operatorname{Mod}\mathscr{A}'$ are equivalent as linear categories, then $\mathscr{A}$ and $\mathscr{A}'$ are derived equivalent, but the converse is not true in most cases. Thus, the derived equivalence classification is usually rougher than the Morita equivalence classification. Broué's abelian defect conjecture in [14] made this notion more important. In this connection, Rickard classified Brauer tree algebras up to derived equivalence in [34], and one of the authors gave the derived equivalence classification for representation-finite selfinjective algebras in [3]. An essential tool for the classifications above was given by Rickard's Morita type theorem for derived categories of rings in [33], which was generalized later by Keller in [24] to differential graded (dg for short) categories with an alternative proof. Both theorems give very useful criteria to check for rings or dg categories to be derived equivalent in terms of tilting complexes or tilting subcategories, which will be also used in this paper as a fundamental tool.

Recall that a dg category is a graded category whose morphism spaces are endowed with differentials satisfying suitable compatibility with the grading, and note that a dg category with a single object is nothing but a dg algebra. Dg categories are used to enhance triangulated categories by Bondal–Kapranov in [13], which was motivated by the study of exceptional collections of coherent sheaves on projective varieties. Also, they are efficiently used in [26] by Keller to compute derived invariants such as K-theory, Hochschild (co-)homology and cyclic homology associated with a ring or a variety.

Now, we come back to derived equivalences of linear categories. If $\mathscr{A}$ and $\mathscr{A}'$ are derived equivalent linear categories, then they share invariants under derived equivalences, such as the center, the Grothendieck group, and those listed above. If we have the classification of a class $\mathscr{S}$ of linear categories under derived equivalences, then the computation of an invariant under derived equivalences in question for a complete set of representatives gives the invariant for all linear categories in the class $\mathscr{S}$. To obtain such a classification we need a tool that produces a lot of derived equivalent pairs $\mathscr{A}$ and $\mathscr{A}'$. In [8], for this purpose we have given a way to glue together derived equivalences between linear categories $\mathscr{A}_i$ and $\mathscr{A}_i'$ with $i \in I_0$ for an index small set I_0 to have a derived equivalence between a gluing $\mathscr{A}$ of $\mathscr{A}_i$ and a gluing $\mathscr{A}'$ of $\mathscr{A}_i'$, where the gluing of $\mathscr{A}_i$ was given as the Grothendieck construction $\operatorname{Gr}(X)$ of a colax functor X from a small category I whose object set is I_0 to the 2-category $\Bbbk$-**Cat** of linear categories with $X(i) := \mathscr{A}_i$ for all $i \in I_0$. This also shows us how to produce from $\{\mathscr{A}_i \mid i \in I_0\}$ and derived equivalences between $\mathscr{A}_i$ and $\mathscr{A}_i'$ with $i \in I_0$ a glued linear category $\mathscr{A}'$ that is derived equivalent to $\mathscr{A}$. The main result in [8] can be formulated as follows after defining a 2-category $\overleftarrow{\operatorname{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$ of colax functors $X\colon I \to \Bbbk$-**Cat** and a tilting colax functor for X:

Theorem. *Let $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Assume that X is $\Bbbk$-flat and that there exists a tilting colax functor $\mathscr{T}$ for X such that $\mathscr{T}$ and X' are equivalent in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$ are derived equivalent.*

In the above, X is said to be $\Bbbk$-flat if the $\Bbbk$-modules $X(i)(x, y)$ are flat for all $i \in I_0$ and for all objects x, y of $X(i)$.

As a special case when I is a group G (regarded as a groupoid with a single object), $\mathrm{Gr}(X) = \mathscr{A}/G$ is the orbit category (also called the skew group category and denoted by $\mathscr{A} * G$) of linear category $\mathscr{A}$ with a G-action, and hence it tells us when a derived equivalence between linear categories $\mathscr{A}$ and $\mathscr{A}'$ with G-actions have derived equivalent orbit categories $\mathscr{A}/G$ and $\mathscr{A}'/G$.

In this paper we will investigate the same problem for dg categories by considering the 2-category $\Bbbk$-**dgCat** of dg $\Bbbk$-categories. The main result can be stated as follows:

Theorem (Theorem 10.5 in the text)**.** *Let $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. Assume that X is $\Bbbk$-flat and that there exists a tilting colax functor $\mathscr{T}$ for X such that there exists a zigzag chain of* quasi-equivalences *between $\mathscr{T}$ and X' in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$ are derived equivalent.*

Also as a special case when $I = G$ is a group, $\mathrm{Gr}(X) = \mathscr{A}/G$ is again the orbit dg category of a dg category $\mathscr{A}$ with a G-action, and hence it gives us a sufficient condition for a derived equivalence between dg categories $\mathscr{A}$ and $\mathscr{A}'$ with G-actions to have derived equivalent dg orbit categories $\mathscr{A}/G$ and $\mathscr{A}'/G$. We will apply this to the complete Ginzburg dg algebras of quivers with potentials having a G-action. Recall that a quiver with potentials was introduced by Derksen, Weyman and Zelevinsky in [16] to study the theory of cluster algebras. From a quiver with potentials (Q, W), the Jacobian algebra $J(Q, W)$ and the completed Ginzburg dg algebra $\widehat{\Gamma}(Q, W)$ are defined, which are related as $H^0(\widehat{\Gamma}(Q, W)) = J(Q, W)$. Therefore, $\widehat{\Gamma}(Q, W)$ is regarded as an extension of Jacobian algebra to a dg algebra.

The orbit category (the skew group algebra) $J(Q, W)/G$ was computed up to Morita equivalence as the form $J(Q_G, W_G)$ for some quiver with potentials (Q_G, W_G) by Paquette–Schiffler in [29] in the case that G is a finite subgroup of the automorphism group of $J(Q_G, W_G)$ acting freely on vertices. On the other hand, the orbit dg category (the skew group dg algebra) $\widehat{\Gamma}(Q, W)/G$ was computed up to Morita equivalence as the form $\widehat{\Gamma}(Q_G, W_G)$ for some quiver with potentials (Q_G, W_G) by Le Meur in [30] in the case that G is a finite group (see also Amiot–Plamondon [1] for the case that $G = \mathbb{Z}/2\mathbb{Z}$, Giovannini and Pasquali [19] for the cyclic case, and Giovannini, Pasquali and Plamondon [20] for the finite abelian case). We remark that for both $J(Q, W)$ and $\widehat{\Gamma}(Q, W)$, the quiver Q_G can be computed by using a result by Demonet in [15] on the computation of the skew group algebra of the path algebra of a quiver with an action of a finite group, and in the arbitrary group case, Q_G can be computed from a non-admissible presentation given in [9] by making it as an admissible presentation.

By Keller–Yang [27], if (Q', W') is obtained as a mutation of (Q, W), then the dg algebras $\widehat{\Gamma}(Q, W)$ and $\widehat{\Gamma}(Q', W')$ are derived equivalent. Using our main theorem above, we can show that this derived equivalence sometimes induces a derived equivalence between $\widehat{\Gamma}(Q_G, W_G)$ and $\widehat{\Gamma}(Q'_G, W'_G)$, where even if (Q_G, W_G) and (Q'_G, W'_G) do not need to be obtained by a mutation from each other. For this phenomenon, an example will be given at the end of the paper.

The paper is organized as follows. In Section 2, we shall fix notations and prepare some basic facts for our proofs. In Section 3, we collect basic facts about enriched categories that will be needed later. In section 4, we will introduce the notion of I-coverings that is a generalization of that of G-coverings for a group G introduced in [5], which was obtained by generalizing the notion of Galois coverings introduced by Gabriel in [17]. This will be used in the proof of our main theorem. In Section 5, we define a 2-functor $\mathrm{Gr}\colon \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat}) \to \Bbbk\text{-}\mathbf{dgCat}$ whose correspondence on objects is a dg version of (the opposite version of) the original Grothendieck construction. In Section 6, we will show that the Grothendieck construction is a strict left adjoint to the diagonal 2-functor, and that I-coverings are essentially given by the unit of the adjunction. In Section 7, we will give the definition of dg module colax functors. In Section 8, we will review the quasi-equivalences and derived equivalences for dg categories. In Section 9, we define necessary terminologies such as 2-quasi-isomorphisms for 2-morphisms, quasi-equivalences for 1-morphisms, and the derived 1-morphism $\mathbf{L}\overline{(F,\psi)}\colon \mathscr{D}(\mathrm{dgMod}\, X) \to \mathscr{D}(\mathrm{dgMod}\, X')$ of a 1-morphism $(F,\psi)\colon X \to X'$ between colax functors, and show the fact that the derived 1-morphism of a quasi-equivalence between colax functors X, X' turns out to be an equivalence between derived dg module colax functors of X, X'. Also, we give definitions of tilting subfunctors and of derived equivalences. We will prove our main theorem in Section 10. Two examples are given in Section 11 which illustrate our main theorem.

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2. Preliminaries

In this section we recall the definition of the 2-category of colax functors from a small category I to a 2-category from [7] (see also Tamaki [35]).

We summarize necessary facts on 2-categories that will be used later.

Definition 2.1. A 2-category $\mathbf{C}$ is a sequence of the following data:

- A class $\mathbf{C}_0$ of objects,
- A family of categories $(\mathbf{C}(x,y))_{x,y\in\mathbf{C}_0}$,
- A family of functors $\circ := (\circ_{x,y,z} : \mathbf{C}(y,z) \circ \mathbf{C}(x,y) \to \mathbf{C}(x,z))_{x,y,z\in\mathbf{C}_0}$,
- A family of functors $(\mu_x : \mathbb{1}_x \to \mathbf{C}(x,x))_{x\in\mathbf{C}_0}$.

These data are required to satisfy the following axioms:

- (Associativity) The following diagram is commutative for all $x, y, z \in \mathbf{C}_0$

$$\begin{array}{ccc} \mathbf{C}(z,w)\circ\mathbf{C}(y,z)\circ\mathbf{C}(x,y) & \xrightarrow{\circ\times\mathbf{1}} & \mathbf{C}(y,w)\circ\mathbf{C}(x,y) \\ {\scriptstyle \mathbf{1}\times\circ}\downarrow & & \downarrow{\scriptstyle\circ} \\ \mathbf{C}(z,w)\circ\mathbf{C}(x,z) & \xrightarrow{\quad\circ\quad} & \mathbf{C}(x,w). \end{array}$$

- (Unitality) The following diagram is commutative for all $x, y \in \mathbf{C}_0$

$$\begin{array}{ccccc} \mathbb{1}\times\mathbf{C}(x,y) & & & & \mathbf{C}(x,y)\times\mathbb{1} \\ & \searrow^{\mathrm{prj}_1} & & {}^{\mathrm{prj}_2}\swarrow & \\ {\scriptstyle \mu_y\times\mathbf{C}(x,y)}\downarrow & & \mathbf{C}(x,y) & & \downarrow{\scriptstyle \mathbf{C}(x,y)\times\mu_x} \\ & \nearrow^{\circ} & & {}^{\circ}\nwarrow & \\ \mathbf{C}(y,y)\times\mathbf{C}(x,y) & & & & \mathbf{C}(x,y)\times\mathbf{C}(x,x). \end{array}$$

Remark 2.2. Elements of $\mathbf{C}_0$ are called objects of $\mathbf{C}$, objects (resp. morphisms, compositions) of $\mathbf{C}(x, y)$ are called are called 1-morphisms (resp. 2-morphisms, vertical compositions) of $\mathbf{C}$ with $x, y \in \mathbf{C}_0$. We sometimes abbreviate $x \in \mathbf{C}$ for $x \in \mathbf{C}_0$ if there seems to be no risk of confusion, and do the same even when $\mathbf{C}$ is a usual category.

Definition 2.3. Let I be a small category and $\mathbf{C}$ a 2-category. A *colax functor* (or an *oplax* functor) from I to $\mathbf{C}$ is a triple $(X, X_i, X_{b,a})$ of data:

- a quiver morphism $X : I \to \mathbf{C}$, where I and $\mathbf{C}$ are regarded as quivers by forgetting additional data such as 2-morphisms or compositions;
- a family $(X_i)_{i\in I_0}$ of 2-morphisms $X_i : X(\mathbb{1}_i) \Rightarrow \mathbb{1}_{X(i)}$ in $\mathbf{C}$ indexed by $i \in I_0$; and
- a family $(X_{b,a})_{(b,a)}$ of 2-morphisms $X_{b,a} : X(ba) \Rightarrow X(b)X(a)$ in $\mathbf{C}$ indexed by $(b, a) \in \mathrm{com}(I) := \{(b, a) \in I_1 \times I_1 \mid ba \text{ is defined}\}$

satisfying the axioms:

(a) For each $a : i \to j$ in I the following are commutative:

$$\begin{array}{ccc} X(a\mathbb{1}_i) & \overset{X_{a,1_i}}{\Longrightarrow} & X(a)X(\mathbb{1}_i) \\ & \diagdown\!\!\diagdown & \Downarrow{\scriptstyle X(a)X_i} \\ & & X(a)\mathbb{1}_{X(i)} \end{array} \quad \text{and} \quad \begin{array}{ccc} X(\mathbb{1}_j a) & \overset{X_{1_j,a}}{\Longrightarrow} & X(\mathbb{1}_j)X(a) \\ & \diagdown\!\!\diagdown & \Downarrow{\scriptstyle X_jX(a)} \\ & & \mathbb{1}_{X(j)}X(a) \end{array} \; ; \text{ and}$$

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ in I the following is commutative:

$$\begin{array}{ccc} X(cba) & \overset{X_{c,ba}}{\Longrightarrow} & X(c)X(ba) \\ {\scriptstyle X_{cb,a}}\Downarrow & & \Downarrow{\scriptstyle X(c)\theta_{b,a}} \\ X(cb)X(a) & \underset{X_{c,b}X(a)}{\Longrightarrow} & X(c)X(b)X(a). \end{array}$$

Definition 2.4. Let $\mathbf{C}$ be a 2-category and $X = (X, X_i, X_{b,a})$, $X' = (X', X'_i, X'_{b,a})$ be colax functors from I to $\mathbf{C}$. A 1-*morphism* (called a *left transformation*) from X to X' is a pair (F, ψ) of data

- a family $F := (F(i))_{i \in I_0}$ of 1-morphisms $F(i)\colon X(i) \to X'(i)$ in $\mathbf{C}$; and
- a family $\psi := (\psi(a))_{a \in I_1}$ of 2-morphisms $\psi(a)\colon X'(a)F(i) \Rightarrow F(j)X(a)$

$$\begin{array}{ccc} X(i) & \xrightarrow{F(i)} & X'(i) \\ {\scriptstyle X(a)}\downarrow & \overset{\psi(a)}{\swArrow} & \downarrow{\scriptstyle X'(a)} \\ X(j) & \xrightarrow[F(j)]{} & X'(j) \end{array}$$

  in $\mathbf{C}$ indexed by $a\colon i \to j$ in I_1

satisfying the axioms

(a) For each $i \in I_0$ the following is commutative:

$$\begin{array}{ccc} X'(\mathbb{1}_i)F(i) & \xRightarrow{\psi(\mathbf{1}_i)} & F(i)X(\mathbb{1}_i) \\ {\scriptstyle X'_iF(i)}\Downarrow & & \Downarrow{\scriptstyle F(i)X_i} \\ \mathbb{1}_{X'(i)}F(i) & = & F(i)\mathbb{1}_{X(i)} \end{array} \quad ; \text{ and}$$

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k$ in I the following is commutative:

$$\begin{array}{ccccc} X'(ba)F(i) & \xRightarrow{X'_{b,a}F(i)} & X'(b)X'(a)F(i) & \xRightarrow{X'(b)\psi(a)} & X'(b)F(j)X(a) \\ {\scriptstyle \psi(ba)}\Downarrow & & & & \Downarrow{\scriptstyle \psi(b)X(a)} \\ F(k)X(ba) & & \xRightarrow[F(k)\,X_{b,a}]{} & & F(k)X(b)X(a). \end{array}$$

A 1-morphism $(F, \psi)\colon X \to X'$ is said to be I-*equivariant* if $\psi(a)$ is a 2-isomorphism in $\mathbf{C}$ for all $a \in I_1$.

Definition 2.5. Let $\mathbf{C}$ be a 2-category, $X = (X, X_i, X_{b,a})$, $X' = (X', X'_i, X'_{b,a})$ be colax functors from I to $\mathbf{C}$, and (F, ψ), (F', ψ') 1-morphisms from X to X'. A 2-*morphism* from (F, ψ) to (F', ψ') is a family $\zeta = (\zeta(i))_{i \in I_0}$ of 2-morphisms $\zeta(i)\colon F(i) \Rightarrow F'(i)$ in $\mathbf{C}$ indexed by $i \in I_0$ such that the following is commutative for all $a\colon i \to j$ in I:

$$\begin{array}{ccc} X'(a)F(i) & \xRightarrow{X'(a)\zeta(i)} & X'(a)F'(i) \\ {\scriptstyle \psi(a)}\Downarrow & & \Downarrow{\scriptstyle \psi'(a)} \\ F(j)X(a) & \xRightarrow{\zeta(j)X(a)} & F'(j)X(a). \end{array}$$

Definition 2.6. Let $\mathbf{C}$ be a 2-category, $X = (X, X_i, X_{b,a}), X' = (X', X'_i, X'_{b,a})$ and $X'' = (X'', X''_i, X''_{b,a})$ colax functors from I to $\mathbf{C}$, and let $(F, \psi)\colon X \to X'$, $(F', \psi')\colon X' \to X''$ be 1-morphisms. Then the composite $(F', \psi')(F, \psi)$ of (F, ψ) and (F', ψ') is a 1-morphism from X to X'' defined by

$$(F', \psi')(F, \psi) := (F'F, \psi' \circ \psi),$$

where $F'F := ((F'(i)F(i))_{i\in I_0}$ and for each $a\colon i \to j$ in I, $(\psi' \circ \psi)(a) := F'(j)\psi(a) \circ \psi'(a)F(i)$ is the pasting of the diagram

$$\begin{array}{ccccc} X(i) & \xrightarrow{F(i)} & X'(i) & \xrightarrow{F'(i)} & X''(i) \\ {\scriptstyle X(a)}\downarrow & \overset{\psi(a)}{\Swarrow} & \downarrow{\scriptstyle X'(a)} & \overset{\psi'(a)}{\Swarrow} & \downarrow{\scriptstyle X''(a)} \\ X(j) & \xrightarrow[F(j)]{} & X'(j) & \xrightarrow[F'(j)]{} & X''(j). \end{array}$$

The following is straightforward to verify.

Proposition 2.7. *Let* $\mathbf{C}$ *be a* 2*-category. Then colax functors* $I \to \mathbf{C}$, 1*-morphisms between them, and* 2*-morphisms between* 1*-morphisms* (*defined above*) *define a* 2*-category, which we denote by* $\overleftarrow{\mathrm{Colax}}(I, \mathbf{C})$.

Notation 2.8. Let $\mathbf{C}$ be a 2-category. Then we denote by $\mathbf{C}^{\mathrm{op}}$ (resp. $\mathbf{C}^{\mathrm{co}}$) the 2-category obtained from $\mathbf{C}$ by reversing the 1-morphisms (resp. the 2-morphisms), and we set $\mathbf{C}^{\mathrm{coop}} := (\mathbf{C}^{\mathrm{co}})^{\mathrm{op}} = (\mathbf{C}^{\mathrm{op}})^{\mathrm{co}}$.

3. Enriched categories

In this section we collect basic facts about enriched categories which will be needed later. Throughout this section, we fix a symmetric monoidal category $\mathbb{V}$ and work in $\mathbb{V}$. Before starting our discussion we recall the definition of symmetric monoidal categories.

Definition 3.1. (1) A *monoidal category* is a sequence of the data

- a category $\mathbb{V}$,
- an object 1 of $\mathbb{V}$,
- a functor $\otimes\colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}$,
- a family of natural isomorphisms $a_{A,B,C}\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ indexed by the triples A, B, C of objects of $\mathbb{V}$, called the associator,
- a family of natural isomorphisms $\ell_A\colon 1 \otimes A \to A$ indexed by the objects A of $\mathbb{V}$,
- a family of natural isomorphisms $r_A\colon A \otimes 1 \to A$ indexed by the objects A of $\mathbb{V}$

that satisfies the following axioms:

(a) For any $A, B, C, D \in \mathbb{V}_0$, the following is commutative:

$$\begin{array}{ccccc} & & A \otimes (B \otimes (C \otimes D)) & & \\ & \swarrow^{a} & & \searrow^{1\otimes a} & \\ (A \otimes B) \otimes (C \otimes D) & & & & A \otimes ((B \otimes C) \otimes D) \\ {\scriptstyle a}\downarrow & & & & \downarrow{\scriptstyle a} \\ ((A \otimes B) \otimes C) \otimes D & & \xleftarrow[a\otimes 1]{} & & (A \otimes (B \otimes C)) \otimes D \end{array};$$

(b) For any $A, B \in \mathbb{V}_0$, the following is commutative:

$$\begin{array}{ccc} A \otimes (B \otimes 1) & \xrightarrow{1\otimes r} & A \otimes B \\ {\scriptstyle a}\downarrow & \nearrow_{r} & \\ (A \otimes B) \otimes 1 & & \end{array} \quad ;\text{and}$$

(c) $\ell_1 = r_1 \colon 1 \otimes 1 \to 1$.

According to [28], it is known that both of the following diagrams automatically turn out to be commutative for all objects A, B in a monoidal category $\mathbb{V}$:

$$\begin{array}{ccc} 1 \otimes (A \otimes B) & \xrightarrow{\ell} & A \otimes B \\ {\scriptstyle a}\downarrow & \nearrow_{\ell\otimes 1} & \\ (1 \otimes A) \otimes B & & \end{array} \quad \text{and} \quad \begin{array}{ccc} A \otimes (1 \otimes B) & \xrightarrow{1\otimes \ell} & A \otimes B \\ {\scriptstyle a}\downarrow & \nearrow_{r\otimes 1} & \\ (A \otimes 1) \otimes B & & \end{array} \quad .$$

(2) A *switching operation* on $\mathbb{V}$ is a family $t = (t_{A,B} \colon A \otimes B \to B \otimes A)_{(A,B)\in\mathbb{V}_0\times\mathbb{V}_0}$ such that the following is commutative:

$$\begin{array}{ccc} A \otimes B & \xrightarrow{t_{A,B}} & B \otimes A \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ C \otimes D & \xrightarrow[t_{C,D}]{} & D \otimes C \end{array}$$

for all morphisms $f \colon A \to C$ and $g \colon B \to D$ in $\mathbb{V}$.

(3) A monoidal category $\mathbb{V}$ with a switching operation t is called a *symmetric monoidal category* if the following hold:

(a) $t_{A,B} \circ t_{B,A} = 1$ for all $A, B \in \mathbb{V}_0$; and

(b) For any $A, B, C \in \mathbb{V}_0$, the following is commutative:

$$\begin{array}{ccccc} & & A \otimes (B \otimes C) & & \\ & \swarrow^{t_{A,B\otimes C}} & & \searrow^{a_{A,B,C}} & \\ (B \otimes C) \otimes A & & & & (A \otimes B) \otimes C \\ {\scriptstyle a_{B,C,A}}\uparrow & & & & \downarrow{\scriptstyle t_{A,B}\otimes 1} \\ B \otimes (C \otimes A) & & & & (B \otimes A) \otimes C \\ & \nwarrow_{1\otimes t_{A,C}} & & \swarrow_{a^{-1}_{B,A,C}} & \\ & & B \otimes (A \otimes C) & & \end{array} \quad .$$

Example 3.2. The following give examples of symmetric monoidal categories:

(1) $\mathbb{V} :=$ **Cat**, the (1-)category of the small caetegories and functors. Here, 1 is given by a category with only one object and one morphism, $\otimes$ is given by the direct product of small categories and a, ℓ, r, t are given as the canonical isomorphisms.

(2) $\mathbb{V} := \mathrm{Mod}\,\Bbbk$, the category of $\Bbbk$-modules. In this case, 1 is given by $\Bbbk$, $\otimes$ is given by the tensor product over $\Bbbk$, and a, ℓ, r, t are also given as the canonical isomorphisms.

(3) $\mathbb{V} := \mathsf{Ch}(\mathrm{Mod}\,\Bbbk)$, the category of the (unbounded) chain complexes (here we use cocomplexes) over $\Bbbk$ and the chain morphisms, i.e., the degree-preserving morphisms commuting with differentials. In this case, 1 is given by the complex $\Bbbk$ concentrated in degree 0, for $A, B \in \mathbb{V}_0$, $A \otimes B$ is given as the tensor chain complex over $\Bbbk$, and also a, ℓ, r, t are given as the canonical isomorphisms. Note that for each $A \in \mathbb{V}_0$, the "underlying set" $\mathsf{Ch}(\mathrm{Mod}\,\Bbbk)(\Bbbk, A)$ is the set of 0-cocycles $Z^0(A)$ of A.

Definition 3.3. A category $\mathscr{A}$ *enriched over* $\mathbb{V}$, or simply a $\mathbb{V}$-*category* consists of the following data:

- a class of objects $\mathscr{A}_0$;
- for two objects a, b in $\mathscr{A}$, an object $\mathscr{A}(a, b)$ in $\mathbb{V}$;
- for three objects a, b, c in $\mathscr{A}$, a morphism
$$\circ : \mathscr{A}(b, c) \otimes \mathscr{A}(a, b) \to \mathscr{A}(a, c)$$
in $\mathbb{V}$; and
- for an object a in $\mathscr{A}$, a morphism in $\mathbb{V}$
$$1_a : 1 \to \mathscr{A}(a, a)$$

satisfying the following conditions:

(1) For any objects a, b, c, d, the following diagram is commutative:

$$\begin{array}{ccc}
(\mathscr{A}(c,d) \otimes \mathscr{A}(b,c)) \otimes \mathscr{A}(a,b) & \xrightarrow{\quad a \quad} & \mathscr{A}(c,d) \otimes (\mathscr{A}(b,c) \otimes \mathscr{A}(a,b)) \\
\downarrow{\scriptstyle \circ\times 1} & & \downarrow{\scriptstyle 1\times\circ} \\
\mathscr{A}(b,d) \otimes \mathscr{A}(a,b) & & \mathscr{A}(c,d) \otimes \mathscr{A}(a,c) \\
& \searrow{\scriptstyle \circ} \quad \swarrow{\scriptstyle \circ} & \\
& \mathscr{A}(a,d) &
\end{array} ; \text{and}$$

(2) For any objects a, b, the following diagram is commutative:

$$\begin{array}{ccccc}
\mathscr{A}(b,b) \otimes \mathscr{A}(a,b)) & \xrightarrow{\ \circ\ } & \mathscr{A}(a,b) & \xleftarrow{\ \circ\ } & \mathscr{A}(a,b) \otimes \mathscr{A}(a,a) \\
\uparrow & \nearrow & & \nwarrow & \uparrow \\
1 \otimes \mathscr{A}(a,b) & & & & \mathscr{A}(a,b) \otimes 1
\end{array} .$$

Definition 3.4. Given $\mathbb{V}$-categories $\mathscr{A}, \mathscr{B}$, a $\mathbb{V}$-functor or an enriched functor $F : \mathscr{A} \to \mathscr{B}$ consists of the following data:

- for each $a \in \mathscr{A}_0$, an object $F(a)$ of $\mathscr{B}$;
- for any $a, b \in \mathscr{A}_0$, a morphism in $\mathbb{V}$,
$$F_{a,b} : \mathscr{A}(a, b) \to \mathscr{B}(F(a), F(b))$$

that satisfies the following axioms:

(1) For any $a, b, c \in \mathscr{A}_0$, the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{A}(b,c)\otimes\mathscr{A}(a,b) & \xrightarrow{\quad\circ\quad} & \mathscr{A}(a,c) \\ {\scriptstyle F_{b,c}\times F_{a,b}}\big\downarrow & & \big\downarrow{\scriptstyle F_{a,c}} \\ \mathscr{B}(F(b),F(c))\otimes\mathscr{B}(F(a),F(b)) & \xrightarrow{\quad\circ\quad} & \mathscr{B}(F(a),F(c)) \end{array} \quad ;\text{ and}$$

(2) For each $a \in \mathscr{A}_0$, the following diagram is commutative:

$$\begin{array}{ccc} 1 & \xrightarrow{1_a} & \mathscr{A}(a,a) \\ & {\scriptstyle 1_{F(a)}}\searrow & \big\downarrow{\scriptstyle F_{a,a}} \\ & & \mathscr{A}(F(a),F(a)) \end{array} \quad .$$

Definition 3.5. Let $F, G \colon \mathscr{A} \to \mathscr{B}$ be $\mathbb{V}$-functors between $\mathbb{V}$-categories. A $\mathbb{V}$-*natural transformation* α from F to G, denoted by $\alpha\colon F \Rightarrow G$, is a family $\alpha = (\alpha(a))_{a\in\mathscr{A}_0}$ of morphisms $\alpha(a)\colon 1 \to \mathscr{B}(F(a), G(a))$ in $\mathbb{V}$ making the following diagram commutative for all $a, b \in \mathscr{A}_0$:

$$\begin{array}{ccccc} & & \mathscr{A}(a,b) & & \\ & {\scriptstyle r^{-1}}\swarrow & & \searrow{\scriptstyle \ell^{-1}} & \\ \mathscr{A}(a,b)\otimes 1 & & & & 1\otimes\mathscr{A}(a,b) \\ {\scriptstyle G\otimes\alpha(a)}\big\downarrow & & & & \big\downarrow{\scriptstyle \alpha(a')\otimes F} \\ \mathscr{B}(G(a),G(b))\otimes\mathscr{B}(F(a),G(a)) & & & & \mathscr{B}(F(b),G(b))\otimes\mathscr{B}(F(a),F(b)) \\ & {\scriptstyle \circ}\searrow & & \swarrow{\scriptstyle \circ} & \\ & & \mathscr{B}(F(a),G(b)) & & \end{array} \quad . \tag{3.1}$$

The composition of $\mathbb{V}$-natural transformations is defined in an obvious way.

Remark 3.6. Consider the case that $\mathbb{V} = \mathsf{Ch}(\mathrm{Mod}\,\Bbbk)$, and let $F, G\colon \mathscr{A} \to \mathscr{B}$ be dg functors between dg categories. Then $\mathbb{V}$-natural transformations is called *dg natural transformations*. By definition, a dg natural transformation $\alpha\colon F \Rightarrow G$ is a family $\alpha = (\alpha(a))_{a\in\mathscr{A}_0}$ of morphisms $\alpha(a)\colon \Bbbk \to \mathscr{B}(F(a), G(a))$ in $\mathsf{Ch}(\mathrm{Mod}\,\Bbbk)$ making the diagram (3.1) commutative. We set $\alpha_a := \alpha(a)(1_\Bbbk)$, where $1_\Bbbk$ is the identitiy of $\Bbbk$, and make the identification $\alpha = (\alpha_a)_{a\in\mathscr{A}_0}$. As in Exmaple 3.2 (3), $\alpha_a \in Z^0(\mathscr{B}(F(a), G(a)))$ for all $a \in \mathscr{A}_0$, and the commutativity of (3.1) is equivalent to saying that the following is commutative in $\mathscr{B}$ for all morphisms $f\colon a \to b$ in $\mathscr{A}$:

$$\begin{array}{ccc} F(a) & \xrightarrow{F(f)} & F(b) \\ {\scriptstyle \alpha_a}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_b} \\ G(a) & \xrightarrow[G(f)]{} & G(b) \end{array} \quad .$$

Here we have to remark that both α_a and α_b are 0-cocycles in $\mathscr{B}(F(a), F(b))$ and in $\mathscr{B}(G(a), G(b))$, respectively. Thus, we can set $F(f) = (F(f)^n)_{n\in\mathbb{Z}}$, $G(f) = (G(f)^n)_{n\in\mathbb{Z}}$,

$\alpha_a = (\alpha_a)^0$, and $\alpha_b = (\alpha_b)^0$, and the commutativity of the diagram above is equivalent to the equality

$$\alpha_b F(f)^n = G(f)^n \alpha_a$$

for all $n \in \mathbb{Z}$. In particular, this is used in the case where $\mathscr{B} = \mathrm{dgMod}\,\Bbbk$, the dg category of dg $\Bbbk$-modules, later. In this case the 0-cocycles are the chain morphisms.

Definition 3.7. The 2-category of small $\mathbb{V}$-categories, $\mathbb{V}$-functors, and $\mathbb{V}$-natural transformations is denoted by $\mathbb{V}$-**Cat**.

Example 3.8. The following are examples of $\mathbb{V}$-categories.

(1) In the case where $\mathbb{V} = \mathbf{Cat}$, the category of small categories, $\mathbb{V}$-categories are nothing but (strict) 2-categories. $\mathbb{V}$-functors are called 2-functors.
(2) In the case where $\mathbb{V} = \mathrm{Mod}\,\Bbbk$, the category of $\Bbbk$-modules, $\mathbb{V}$-categories are nothing but $\Bbbk$-linear categories.
(3) In the case where $\mathbb{V} = \mathsf{Ch}(\mathrm{Mod}\,\Bbbk)$, the category of chain complexes over $\Bbbk$, $\mathbb{V}$-categories are called dg (differential graded) categories over $\Bbbk$. In this case, $\mathbb{V}$-**Cat** is denoted by $\Bbbk$-**dgCat**. In most cases we only deal with small dg categories, therefore we sometimes omit the word "small" if there seems to be no confusion.

Remark 3.9. Since the last example above is our central subject, we here remind the explicit form of compositions in a dg category. Let $\mathscr{C}$ be a dg category, $x, y, z \in \mathscr{C}$, and $f = (f^i)_{i\in\mathbb{Z}} \in \mathscr{C}(x,y) = \bigoplus_{i\in\mathbb{Z}} \mathscr{C}^i(x,y)$, $g = (g^j)_{j\in\mathbb{Z}} \in \mathscr{C}(y,z) = \bigoplus_{j\in\mathbb{Z}} \mathscr{C}^j(y,z)$. Then we have the formula

$$g \circ f := \left(\sum_{i\in\mathbb{Z}} g^{n-i} \circ f^i\right)_{n\in\mathbb{Z}}. \tag{3.2}$$

On the other hand, in the opposite category $\mathscr{C}^{\mathrm{op}}$ of $\mathscr{C}$ having the composition $*$, we have $f \in \mathscr{C}^{\mathrm{op}}(y,x), g \in \mathscr{C}^{\mathrm{op}}(z,y)$, and

$$f * g = \left(\sum_{i\in\mathbb{Z}} (-1)^{(n-i)i}\, g^{n-i} \circ f^i\right)_{n\in\mathbb{Z}}. \tag{3.3}$$

Note that the representable functor $\mathscr{C}(\text{-}, z) = \mathscr{C}^{\mathrm{op}}(z, \text{-})$ is a functor $\mathscr{C}^{\mathrm{op}} \to \mathrm{dgMod}\,\Bbbk$, and hence $\mathscr{C}(f,z) \colon \mathscr{C}(y,z) \to \mathscr{C}(x,z)$ is defined as $\mathscr{C}^{\mathrm{op}}(z,f) \colon \mathscr{C}^{\mathrm{op}}(z,y) \to \mathscr{C}^{\mathrm{op}}(z,x)$ by

$$\mathscr{C}(f,z)(g) := \mathscr{C}^{\mathrm{op}}(z,f)(g) := f * g = \left(\sum_{i\in\mathbb{Z}} (-1)^{(n-i)i}\, g^{n-i} \circ f^i\right)_{n\in\mathbb{Z}}.$$

4. I-coverings

In this section we introduce the notion of I-coverings that is a generalization of that of G-coverings for a group G introduced in [5], which was obtained by generalizing the notion of Galois coverings introduced by Gabriel in [17]. This will be used in the proof of our main theorem.

In the following, we will consider I-coverings in $\Bbbk$-**dgCat**, i.e., in the case that $\mathbb{V} = \mathsf{Ch}(\mathrm{Mod}\,\Bbbk)$. The precise form in this case is described as follows.

Definition 4.1. We define a 2-functor $\Delta\colon \Bbbk\text{-}\mathbf{dgCat} \to \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$ as follows, which is called the *diagonal* 2-functor:

- Let $\mathscr{C} \in \Bbbk\text{-}\mathbf{dgCat}$. Then $\Delta(\mathscr{C})$ is defined to be a functor sending each morphism $a\colon i \to j$ in I to $\mathbb{1}_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$.
- Let $E\colon \mathscr{C} \to \mathscr{C}'$ be a 1-morphism in $\Bbbk\text{-}\mathbf{dgCat}$. Then $\Delta(E)\colon \Delta(\mathscr{C}) \to \Delta(\mathscr{C}')$ is a 1-morphism (F, ψ) in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$ defined by $F(i) := E$ and $\psi(a) := \mathbb{1}_E$ for all $i \in I_0$ and all $a \in I_1$:

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{E} & \mathscr{C}' \\ {\scriptstyle \mathbf{1}_{\mathscr{C}}}\downarrow & \overset{\mathbf{1}_E}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle \mathbf{1}_{\mathscr{C}'}} \\ \mathscr{C} & \xrightarrow[E]{} & \mathscr{C}'. \end{array}$$

- Let $E, E'\colon \mathscr{C} \to \mathscr{C}'$ be 1-morphisms (that is, dg functors) in $\Bbbk\text{-}\mathbf{dgCat}$, and $\alpha\colon E \Rightarrow E'$ a 2-morphism in $\Bbbk\text{-}\mathbf{dgCat}$. Then $\Delta(\alpha)\colon \Delta(E) \Rightarrow \Delta(E')$ is a 2-morphism in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$ defined by $\Delta(\alpha) := (\alpha)_{i \in I_0}$.

Remark 4.2. Let $\mathbf{C}$ be a 2-category, $X = (X, X_i, X_{b,a}) \in \overleftarrow{\mathrm{Colax}}(I, \mathbf{C})_0$, and $C \in \mathbf{C}_0$. Further let

- F be a family of 1-morphisms $F(i)\colon X(i) \to C$ in $\mathbf{C}$ indexed by $i \in I_0$; and
- ψ be a family of 2-morphisms $\psi(a)\colon F(i) \Rightarrow F(j)X(a)$ indexed by $a\colon i \to j$ in I:

$$\begin{array}{ccc} X(i) & \xrightarrow{F(i)} & C \\ {\scriptstyle X(a)}\downarrow & \overset{\psi(a)}{\swarrow\!\!\swarrow} & \| \\ X(j) & \xrightarrow[F(j)]{} & C \end{array}$$

Then (F, ψ) is in $\overleftarrow{\mathrm{Colax}}(I, \mathbf{C})(X, \Delta(C))$ if and only if the following hold.

(a) For each $i \in I_0$ the following is commutative:

$$\begin{array}{ccc} F(i) & \overset{\psi(\mathbf{1}_i)}{\Longrightarrow} & F(i)X(\mathbb{1}_i) \\ & \diagdown\!\!\diagdown & \Big\Downarrow{\scriptstyle F(i)X_i} \\ & & F(i)\mathbb{1}_{X(i)} \end{array} \quad ;\ \text{and}$$

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k$ in I the following is commutative:

$$\begin{array}{ccc} F(i) & \overset{\psi(a)}{\Longrightarrow} & F(j)X(a) \\ {\scriptstyle \psi(ba)}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \psi(b)X(a)} \\ F(k)X(ba) & \underset{F(k)\,X_{b,a}}{\Longrightarrow} & F(k)X(b)X(a). \end{array}$$

Definition 4.3. Let $\mathscr{C} \in \Bbbk\text{-}\mathbf{dgCat}$ and $(F, \psi)\colon X \to \Delta(\mathscr{C})$ be in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then

(1) (F, ψ) is called an *I-precovering* (of $\mathscr{C}$) if for any $i, j \in I_0$, $x \in X(i), y \in X(j)$, the morphism

$$(F,\psi)^{(1)}_{x,y} \colon \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y) \to \mathscr{C}(F(i)x, F(j)y)$$

of $\Bbbk$-complexes defined by the following is an isomorphism:

$$\begin{aligned}\bigoplus_{a\in I(i,j)} X(j)(X(a)x, y) &\xrightarrow{\bigoplus_{a\in I(i,j)} F(j)} \bigoplus_{a\in I(i,j)} \mathscr{C}(F(j)X(a)x, F(j)y)\\ &\xrightarrow{\bigoplus_{a\in I(i,j)} \mathscr{C}(\psi(a)_x, F(j)y)} \bigoplus_{a\in I(i,j)} \mathscr{C}(F(i)x, F(j)y)\\ &\xrightarrow{\text{summation}} \mathscr{C}(F(i)x, F(j)y),\end{aligned}$$

the precise form of which is given as follows:

$$\begin{aligned}(F,\psi)^{(1)}_{x,y}(((f^n_a)_{n\in\mathbb{Z}})_{a\in I(i,j)}) &= \sum_{a\in I(i,j)} \psi(a)_x * F(j)(f_a)\\ &= \left(\sum_{a\in I(i,j)} \sum_{r\in\mathbb{Z}} (-1)^{(n-r)r} F(j)(f_a)^{n-r} \circ \psi(a)^r_x\right)_{n\in\mathbb{Z}} \qquad (4.4)\\ &= \left(\sum_{a\in I(i,j)} F(j)(f_a)^n \circ \psi(a)_x\right)_{n\in\mathbb{Z}},\end{aligned}$$

where the second term is computed by using (3.3), and the last term uses the fact that $\psi(a)_x$ is concentrated in degree 0 (Remark 3.6).

(2) (F, ψ) is called an *I-covering* if it is an I-precovering and is *dense*, i.e., for each $c \in \mathscr{C}_0$ there exists an $i \in I_0$ and $x \in X(i)_0$ such that $F(i)(x)$ is isomorphic to c in $\mathscr{C}$.

5. Grothendieck constructions

In this section we define a 2-functor $\mathrm{Gr}\colon \overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat}) \to \mathbb{V}\text{-}\mathbf{Cat}$ whose correspondence on objects is a $\mathbb{V}$-enriched version of (the opposite version of) the original Grothendieck construction (cf. [35]). In particular, we deal with the case of $\Bbbk$-**dgCat** later.

Definition 5.1. We define a 2-functor $\mathrm{Gr}\colon \overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat}) \to \mathbb{V}\text{-}\mathbf{Cat}$, which is called the *Grothendieck construction*.

On objects. Let $X = (X(i), X_i, X_{b,a}) \in \overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat})_0$. Then $\mathrm{Gr}(X) \in \mathbb{V}\text{-}\mathbf{Cat}_0$ is defined as follows.

- $\mathrm{Gr}(X)_0 := \bigcup_{i\in I_0} \{i\} \times X(i)_0 = \{{}_ix := (i, x) \mid i \in I_0, x \in X(i)_0\}$.
- For each ${}_ix, {}_jy \in \mathrm{Gr}(X)_0$, we set

$$\mathrm{Gr}(X)({}_ix, {}_jy) := \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y).$$

- For any ${}_ix, {}_jy, {}_kz \in \mathrm{Gr}(X)_0$ and each $f = (f_a)_{a\in I(i,j)} \in \mathrm{Gr}(X)({}_ix, {}_jy)$, $g = (g_b)_{b\in I(j,k)} \in \mathrm{Gr}(X)({}_jy, {}_kz)$, we set

$$g \circ f := \left(\sum_{\substack{a\in I(i,j)\\ b\in I(j,k)\\ c=ba}} g_b \circ X(b)f_a \circ X_{b,a}x \right)_{c\in I(i,k)},$$

which is the composite of the following:

$$\begin{array}{ccc}
\mathrm{Gr}({}_jy, {}_kz) \times \mathrm{Gr}({}_ix, {}_jy) & \dashrightarrow & \mathrm{Gr}({}_ix, {}_kz) \\
\| & & \| \\
\bigoplus_{b\in I(j,k)} X(k)(X(b)y, z) \times \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y) & & \bigoplus_{c\in I(i,k)} X(k)(X(c)x, z) \\
\| & & \uparrow{\scriptstyle \text{summation}} \\
\bigoplus_{b,a}\{X(k)(X(b)y, z) \times X(j)(X(a)x, y)\} & & \bigoplus_{b,a} X(k)(X(ba)x, z) \\
{\scriptstyle \bigoplus_{b,a}(\mathbf{1}\times X(b))}\downarrow & & \uparrow{\scriptstyle \bigoplus_{b,a} X(k)(X_{b,a}x,\, z)} \\
\bigoplus_{b,a}\{X(k)(X(b)y, z) \times X(j)(X(b)X(a)x, X(b)y)\} & \longrightarrow & \bigoplus_{b,a} X(k)(X(b)X(a)x, z),
\end{array} \tag{5.5}$$

where elements are mapped as follows:

$$\begin{array}{ccc}
((g_b)_b, (f_a)_a) & \mapsto & (\sum_{c=ba} g_b \circ X(b)f_a \circ X_{b,a}x)_c \\
\downarrow & & \uparrow \\
(g_b, f_a)_{b,a} & & (g_b \circ X(b)f_a \circ X_{b,a}x)_{b,a} \\
\downarrow & & \uparrow \\
(g_b, X(b)f_a)_{b,a} & \longmapsto & (g_b \circ X(b)f_a)_{b,a}.
\end{array}$$

Note here that the composition with $X_{b,a}x$ is "contravariant", which is used in (5.7).

- For each ${}_ix \in \mathrm{Gr}(X)_0$ the identity $\mathbb{1}_{{}_ix}$ is given by

$$\mathbb{1}_{{}_ix} = (\delta_{a,\mathbf{1}_i} X_i\, x)_{a\in I(i,i)} \in \bigoplus_{a\in I(i,i)} X(i)(X(a)x, x),$$

where δ is the Kronecker delta[1].

On 1-morphisms. Let $X = (X, X_i, X_{b,a})$ and $X' = (X', X'_i, X'_{b,a})$ be objects of $\overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat})$, and let $(F, \psi)\colon X \to X'$ be a 1-morphism in $\overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat})$. Then a 1-morphism

$$\mathrm{Gr}(F, \psi)\colon\ \mathrm{Gr}(X) \to \mathrm{Gr}(X')$$

in $\mathbb{V}$-**Cat** is defined as follows.

- For each ${}_ix \in \mathrm{Gr}(X)_0$, $\mathrm{Gr}(F, \psi)({}_ix) := {}_i(F(i)x)$.

[1] This is used to mean that the a-th component is $\eta_i\, x$ if $a = \mathbb{1}_i$, or 0 otherwise.

- Let ${}_ix, {}_jy \in \mathrm{Gr}(X)_0$. Then we define

$$\mathrm{Gr}(F,\psi)\colon \mathrm{Gr}(X)({}_ix, {}_jy) \to \mathrm{Gr}(X')({}_i(F(i)x), {}_j(F(j)y))$$

as the composite

$$\begin{aligned}\bigoplus_{a\in I(i,j)} X(j)(X(a)x,y) &\xrightarrow{\bigoplus_{a\in I(i,j)} F(j)} \bigoplus_{a\in I(i,j)} X'(j)(F(j)X(a)x, F(j)y) \\ &\xrightarrow{\bigoplus_{a\in I(i,j)} X'(j)(\psi(a)_x, F(j)y)} \bigoplus_{a\in I(i,j)} X'(j)(X'(a)F(i)x, F(j)y).\end{aligned} \tag{5.6}$$

Namely, for each $f = (f_a)_{a\in I(i,j)} \in \mathrm{Gr}(X)({}_ix, {}_jy)$, we set

$$\mathrm{Gr}(F,\psi)(f) := (F(j)f_a \circ \psi(a)x)_{a\in I(i,j)}.$$

On 2-morphisms. Let $X = (X, X_i, X_{b,a})$ and $X' = (X', X'_i, X'_{b,a})$ be objects of $\overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat})$, $(F,\psi)\colon X \to X'$ a 1-morphism in $\overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat})$, and let $\zeta\colon (F,\psi) \Rightarrow (F',\psi')$ be a 2-morphism in $\overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat})$. Then a 2-morphism

$$\mathrm{Gr}(\zeta)\colon \mathrm{Gr}(F,\psi) \Rightarrow \mathrm{Gr}(F',\psi')$$

in $\mathbb{V}\text{-}\mathbf{Cat}$ is defined by

$$\mathrm{Gr}(\zeta)_{{}_ix} := \begin{cases} \zeta(i)_x \circ X'_i(F(i)x) & \text{if } a = \mathbb{1}_i \\ 0 & \text{if } a \neq \mathbb{1}_i \end{cases}$$

in $\mathrm{Gr}(X')$ for each ${}_ix \in \mathrm{Gr}(X)_0$.

In the following, we will consider the the case where $\mathbb{V} = \mathsf{Ch}(\mathrm{Mod}\,\mathbb{k})$, thus $\mathbb{V}\text{-}\mathbf{Cat} = \mathbb{k}\text{-}\mathbf{dgCat}$. In this case, the precise form of the Grothendieck construction

$$\mathrm{Gr}\colon \overleftarrow{\mathrm{Colax}}(I, \mathbb{k}\text{-}\mathbf{dgCat}) \to \mathbb{k}\text{-}\mathbf{dgCat}$$

is described as follows.

On objects. Let $X = (X(i), X_i, X_{b,a}) \in \overleftarrow{\mathrm{Colax}}(I, \mathbb{k}\text{-}\mathbf{dgCat})_0$. Then $\mathrm{Gr}(X) \in \mathbb{k}\text{-}\mathbf{dgCat}_0$ is defined as follows.

- $\mathrm{Gr}(X)_0 := \bigcup_{i\in I_0} \{i\} \times X(i)_0 = \{{}_ix := (i,x) \mid i \in I_0, x \in X(i)_0\}$.
- For each ${}_ix, {}_jy \in \mathrm{Gr}(X)_0$, we set

$$\mathrm{Gr}(X)({}_ix, {}_jy) := \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y) = \bigoplus_{a\in I(i,j)} \bigoplus_{n\in\mathbb{Z}} X(j)^n(X(a)x, y),$$

where note that $X(j)(X(a)x,y)$ is a dg $\mathbb{k}$-module.

- For any ${}_ix, {}_jy, {}_kz \in \mathrm{Gr}(X)_0$ and each $f = (f_a^p)_{a\in I(i,j), p\in\mathbb{Z}} \in \mathrm{Gr}(X)({}_ix, {}_jy)$, $g = (g_b^q)_{b\in I(j,k), q\in\mathbb{Z}} \in \mathrm{Gr}(X)({}_jy, {}_kz)$, it turns out that

$$\begin{aligned} g \circ f &= \left(\sum_{\substack{a\in I(i,j)\\ b\in I(j,k)\\ c=ba}} X_{b,a}x * (g_b \circ (X(b)f_a)) \right)_{c\in I(i,k), n\in\mathbb{Z}} \\ &= \left(\sum_{\substack{a\in I(i,j)\\ b\in I(j,k)\\ c=ba}} \sum_{p,r\in\mathbb{Z}} (-1)^{(n-r)r} g_b^{n-r-p} \circ (X(b)f_a)^p \circ (X_{b,a}x)^r \right)_{c\in I(i,k), n\in\mathbb{Z}} \end{aligned} \tag{5.7}$$

  because of the contravariant part in (5.5).
- For each ${}_ix \in \mathrm{Gr}(X)_0$ the identity $\mathbb{1}_{{}_ix}$ is given by

$$\mathbb{1}_{{}_ix} = (\delta_{a,\mathbf{1}_i} X_i\, x)_{a\in I(i,i)} \in \bigoplus_{a\in I(i,i)} X(i)(X(a)x, x) = \bigoplus_{a\in I(i,j)} \bigoplus_{p\in\mathbb{Z}} X(i)^p(X(a)x, x).$$

On 1-morphisms. Let $X = (X, X_i, X_{b,a})$ and $X' = (X', X'_i, X'_{b,a})$ be objects of $\overleftarrow{\mathrm{Colax}}(I, \mathbb{k}\text{-}\mathbf{dgCat})$, and let $(F,\psi)\colon X \to X'$ be a 1-morphism in $\overleftarrow{\mathrm{Colax}}(I, \mathbb{k}\text{-}\mathbf{dgCat})$. Then a 1-morphism

$$\mathrm{Gr}(F,\psi)\colon \mathrm{Gr}(X) \to \mathrm{Gr}(X')$$

in $\mathbb{k}$-**dgCat** is defined as follows.

- For each ${}_ix \in \mathrm{Gr}(X)_0$, $\mathrm{Gr}(F,\psi)({}_ix) := {}_i(F(i)x)$.
- Let ${}_ix, {}_jy \in \mathrm{Gr}(X)_0$. Then we define

$$\mathrm{Gr}(F,\psi)\colon \mathrm{Gr}(X)({}_ix, {}_jy) \to \mathrm{Gr}(X')({}_i(F(i)x), {}_j(F(j)y))$$

  as in (5.6). Namely, for each $f = ((f_a^n)_{n\in\mathbb{Z}})_{a\in I(i,j)} \in \mathrm{Gr}(X)({}_ix, {}_jy) = \bigoplus_{a\in I(i,j)} X(j)$ $(X(a)x, y)$, we have

$$\begin{aligned} ((f_a^n)_{n\in\mathbb{Z}})_{a\in I(i,j)} &\mapsto ((F(j)(f_a^n))_{n\in\mathbb{Z}})_{a\in I(i,j)} \\ &\mapsto \psi(a)_x * ((F(j))(f_a^n))_{n\in\mathbb{Z}})_{a\in I(i,j)} \\ &= ((F(j)(f_a)^n \circ \psi(a)_x)_{n\in\mathbb{Z}})_{a\in I(i,j)} \quad \text{(cf. (3.3))} \end{aligned}$$

  Thus we have

$$\mathrm{Gr}(F,\psi)(f) = ((F(j)(f_a)^n \circ \psi(a)_x)_{n\in\mathbb{Z}})_{a\in I(i,j)}. \tag{5.8}$$

On 2-morphisms. Let $X = (X, X_i, X_{b,a})$ and $X' = (X', X'_i, X'_{b,a})$ be objects of $\overleftarrow{\mathrm{Colax}}(I, \mathbb{k}\text{-}\mathbf{dgCat})$, $(F,\psi)\colon X \to X'$ a 1-morphism in $\overleftarrow{\mathrm{Colax}}(I, \mathbb{k}\text{-}\mathbf{dgCat})$, and let $\zeta\colon (F,\psi) \Rightarrow (F',\psi')$ be a 2-morphism in $\overleftarrow{\mathrm{Colax}}(I, \mathbb{k}\text{-}\mathbf{dgCat})$. Then a 2-morphism

$$\mathrm{Gr}(\zeta)\colon \mathrm{Gr}(F,\psi) \Rightarrow \mathrm{Gr}(F',\psi')$$

in $\Bbbk$-**dgCat** is defined by

$$\mathrm{Gr}(\zeta)_i x = \begin{cases} \zeta(i)_x \circ X'_i(F(i)x) = (\sum_{r\in\mathbb{Z}} \zeta(i)_x^{n-r} \circ X'_i(F(i)x)^r)_{n\in\mathbb{Z}} & \text{if } a = \mathbb{1}_i \\ 0 & \text{if } a \neq \mathbb{1}_i \end{cases}$$

in $\mathrm{Gr}(X')$ for each ${}_ix \in \mathrm{Gr}(X)_0$.

Example 5.2. Let A be a dg $\Bbbk$-algebra with the differential d_A regarded as a dg $\Bbbk$-category with a single object. Then $A \in \Bbbk\text{-}\mathbf{dgCat}_0$. Consider the functor $X := \Delta(A) \colon I \to \Bbbk\text{-}\mathbf{dgCat}$. Then it is straightforward to verify the following.

(1) If I is a free category defined by the quiver $1 \to 2$, then $\mathrm{Gr}(X)$ is isomorphic to the triangular dg algebra $\begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$.

(2) If I is a free category $\mathbb{P}Q$ defined by a quiver Q, then $\mathrm{Gr}(X)$ is isomorphic to the dg path-category AQ of Q over A defined as follows:
- $(AQ)_0 := Q_0$.
- For any $i, j \in Q_0$,

$$AQ(i,j) := \bigoplus_{\mu\in\mathbb{P}Q(i,j)} A\mu = \left\{ \sum_{\mu\in\mathbb{P}Q(i,j)} a_\mu \mu \;\middle|\; (a_\mu)_{\mu\in\mathbb{P}Q(i,j)} \in \bigoplus_{\mu\in\mathbb{P}Q(i,j)} A \right\}.$$

- For any $i, j, k \in Q_0$, the composition $AQ(j,k) \times AQ(i,j) \to AQ(i,k)$ is given by

$$\sum_{\nu\in\mathbb{P}Q(j,k)} b_\nu \nu \times \sum_{\mu\in\mathbb{P}Q(i,j)} a_\mu \mu \mapsto \sum_{\substack{\mu\in\mathbb{P}Q(i,j),\\ \nu\in\mathbb{P}Q(j,k)}} b_\nu a_\mu \nu\mu = \sum_{\lambda\in\mathbb{P}Q(i,k)} \left(\sum_{\lambda=\nu\mu} b_\nu a_\mu \right) \lambda.$$

- For any $i, j \in Q_0$ and any $n \in \mathbb{Z}$, we set $(AQ)^n(i,j) = \bigoplus_{\mu\in\mathbb{P}Q(i,j)} A^n\mu$.
- For any $i, j \in Q_0$ and any $n \in \mathbb{Z}$, the differential $d : (AQ)^n(i,j) \to (AQ)^{n+1}(i,j)$ is given by

$$d\left(\sum_{\mu\in\mathbb{P}Q(i,j)} a_\mu \mu \right) = \sum_{\mu\in\mathbb{P}Q(i,j)} d_A(a_\mu)\mu,$$

which automatically satisfies the graded Leibniz rule.

Indeed, we can define an isomorphism $\phi \colon AQ \to \mathrm{Gr}(X)$ as follows: We regard A as a category with a single objects $*$. Then for each $i \in Q_0$, we have $X(i)_0 = \{*\}$ and $X(i)_1 = A$. Then $\mathrm{Gr}(X)_0 = \bigsqcup_{i\in Q_0} X(i)_0 = \bigcup_{i\in Q_0}\{{}_i*\} = \{{}_i* \mid i \in Q_0\}$. Therefore, we define a bijection $\phi_0 \colon (AQ)_0 \to \mathrm{Gr}(X)_0$ by $i \mapsto {}_i*$. For any $i, j \in Q_0$, since we have $(AQ)(i,j) = \bigoplus_{\mu\in\mathbb{P}Q(i,j)} A\mu$, and

$$\mathrm{Gr}(X)({}_i*, {}_j*) := \bigoplus_{\mu\in I(i,j)} X(j)(X(\mu)*, *) = \bigoplus_{\mu\in I(i,j)} X(j)_1 = \bigoplus_{\mu\in\mathbb{P}Q(i,j)} A,$$

we define a bijection $\phi_1 \colon (AQ)(i,j) \to \mathrm{Gr}(X)({}_i*, {}_j*)$ by $\sum_{\mu\in\mathbb{P}Q} a_\mu\mu \mapsto (a_\mu)_{\mu\in\mathbb{P}Q}$. Then $\phi := (\phi_0, \phi_1) \colon AQ \to \mathrm{Gr}(X)$ turns out to be an isomorphism.

(3) If I is a poset S, then $\mathrm{Gr}(X)$ is isomorphic to the incidence dg category AS of S over A defined as follows:
- $(AS)_0 := S$ as a set.
- For any $i, j \in S$, $(AS)(i,j) := \begin{cases} A & \text{if } i \le j, \\ 0 & \text{otherwise.} \end{cases}$
- For any $i, j, k \in S$, the composition $AS(j,k) \times AS(i,j) \to AS(i,k)$ is given by the multiplication of A for the case that $i \le j \le k$, and as zero otherwise.
- For any $i, j \in Q_0$ and any $n \in \mathbb{Z}$, we set $(AS)^n(i,j) := \begin{cases} A^n & \text{if } i \le j, \\ 0 & \text{otherwise.} \end{cases}$
- For any $i, j \in Q_0$ and any $n \in \mathbb{Z}$, the differential $d : (AS)^n(i,j) \to (AS)^{n+1}(i,j)$ is given by $d_A \colon A^n \to A^{n+1}$ if $i \le j$, and as zero otherwise, which automatically satisfies the graded Leibniz rule.

Indeed, we can define an isomorphism $\phi \colon AS \to \mathrm{Gr}(X)$ as follows: We regard A as a category with a single objects $*$. Then for each $i \in S$, we have $X(i)_0 = \{*\}$ and $X(i)_1 = A$. Then $\mathrm{Gr}(X)_0 = \bigsqcup_{i \in I_0} X(i)_0 = \bigcup_{i \in I_0} \{{}_i*\} = \{{}_i* \mid i \in S\}$. Therefore, we define a bijection $\phi_0 \colon (AS)_0 \to \mathrm{Gr}(X)_0$ by $i \mapsto {}_i*$. For any $i, j \in S$, since we have $(AS)(i,j) = \begin{cases} A & \text{if } i \le j, \\ 0 & \text{otherwise} \end{cases}$, and

$$\mathrm{Gr}(X)({}_i*, {}_j*) := \bigoplus_{\mu \in S(i,j)} X(j)(X(\mu)*, *) = \bigoplus_{\mu \in S(i,j)} X(j)_1 = \bigoplus_{\mu \in S(i,j)} A = A, \text{if } i \le j,$$

we define a bijection $\phi_1 \colon (AS)(i,j) \to \mathrm{Gr}(X)({}_i*, {}_j*)$ by $\sum_{\mu \in S} a_\mu \mu \mapsto (a_\mu)_{\mu \in S}$. Then $\phi := (\phi_0, \phi_1) \colon AQ \to \mathrm{Gr}(X)$ turns out to be an isomorphism.

(4) If I is a monoid G, then $\mathrm{Gr}(X)$ is isomorphic to the monoid dg algebra[2] AG of G over A defined as follows:
- $AG := \bigoplus_{g \in G} Ag$.
- The multiplication $AG \times AG \to AG$ is defined by

$$\left(\sum_{g \in G} a_g g\right) \cdot \left(\sum_{h \in G} b_h h\right) := \sum_{g,h \in G} (a_g b_h) gh = \sum_{f \in G} \left(\sum_{gh = f} a_g b_h\right) f.$$

- For each $n \in \mathbb{Z}$, $(AG)^n := \bigoplus_{g \in G} A^n g$.
- The differential $d : (AG)^n \to (AG)^{n+1}$ is given by $d\left(\sum_{g \in G} a_g g\right) := \sum_{g \in G} d_A(a_g) g$, which automatically satisfies the graded Leibniz rule.

In (3) above, AS is defined to be the factor category of the dg path-category AQ modulo the ideal generated by the full commutativity relations in Q, where Q is the Hasse diagram of S regarded as a quiver by drawing an arrow $x \to y$ if $x \le y$ in Q. If S is a finite poset, then AS is identified with the usual incidence dg algebra.

[2]Since AG has the identity $1_A 1_G$, this is regarded as a category with a single object.

See [9] for further examples of the Grothendieck constructions of functors, further examples of the Grothendieck constructions of a functor $X\colon I \to \Bbbk\text{-}\mathbf{dgCat}$ will be done in the forthcoming paper.

Definition 5.3. Let $X \in \overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat})$. We define a left transformation $(P_X, \phi_X) := (P, \phi)\colon X \to \Delta(\mathrm{Gr}(X))$ (called the *canonical morphism*) as follows.

- For each $i \in I_0$, the functor $P(i)\colon X(i) \to \mathrm{Gr}(X)$ is defined by

$$\begin{cases} P(i)x := {}_i x \\ P(i)f := (\delta_{a,\mathbf{1}_i} f \circ (X_i\, x))_{a \in I(i,i)} \colon {}_i x \to {}_i y \text{ in } \mathrm{Gr}(X) \end{cases}$$

  for all $f\colon x \to y$ in $X(i)$.
- For each $a\colon i \to j$ in I, the natural transformation $\phi(a)\colon P(i) \Rightarrow P(j)X(a)$

$$\begin{array}{ccc} X(i) & \xrightarrow{P(i)} & \mathrm{Gr}(X) \\ {\scriptstyle X(a)}\downarrow & \overset{\phi(a)}{\swarrow} & \| \\ X(j) & \xrightarrow[P(j)]{} & \mathrm{Gr}(X) \end{array}$$

  is defined by $\phi(a)x := (\delta_{b,a} \mathbb{1}_{X(a)x})_{b \in I(i,j)}$ for all $x \in X(i)_0$.

Now let $X \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. The left transformation $(P_X, \phi_X) := (P, \phi)\colon X \to \Delta(\mathrm{Gr}(X))$ is as follows.

- For each $i \in I_0$, the dg functor $P(i)\colon X(i) \to \mathrm{Gr}(X)$ is defined by $P(i)x := {}_i x$ for all $x \in X(i)_0$, and by setting $P(i)f\colon {}_i x \to {}_i y$ as

$$\begin{aligned} P(i)f &:= (\delta_{a,\mathbf{1}_i} (X_i\, x) * f)_{a \in I(i,i)} \\ &= \left(\left(\delta_{a,\mathbf{1}_i} \sum_{r \in \mathbb{Z}} (-1)^{(n-r)r} f^{n-r} \circ (X_i\, x)^r \right)_{n \in \mathbb{Z}} \right)_{a \in I(i,i)} \end{aligned} \tag{5.9}$$

  for all $f\colon x \to y$ in $X(i)$. Note here that the map $\mathscr{C}(X_i x, y)\colon \mathscr{C}(x,y) \to \mathscr{C}(X(\mathbb{1}_i)x, y)$, $f \mapsto f \circ X_i x$ is given by the contravariant functor $\mathscr{C}(\text{-}, y)$ at $X_i x$.
- For each $a\colon i \to j$ in I, the dg natural transformation $\phi(a)\colon P(i) \Rightarrow P(j)X(a)$

$$\begin{array}{ccc} X(i) & \xrightarrow{P(i)} & \mathrm{Gr}(X) \\ {\scriptstyle X(a)}\downarrow & \overset{\phi(a)}{\swarrow} & \| \\ X(j) & \xrightarrow[P(j)]{} & \mathrm{Gr}(X) \end{array}$$

  is defined by $\phi(a)x := (\delta_{b,a} \mathbb{1}_{X(a)x})_{b \in I(i,j)}$ for all $x \in X(i)_0$.

Lemma 5.4. *The (P, ϕ) defined above is a 1-morphism in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$.*

Proof. This is straightforward by using Remark 4.2. □

Proposition 5.5. *Let $X \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})_0$. Then the canonical morphism $(P, \phi)\colon X \to \Delta(\mathrm{Gr}(X))$ is an I-covering. More precisely, the morphism*

$$(P,\phi)^{(1)}_{x,y}\colon \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y) \to \mathrm{Gr}(X)(P(i)x, P(j)y)$$

is the identity for all $i, j \in I_0$ and all $x \in X(i)_0$, $y \in X(j)_0$.

Proof. By the definitions of $\mathrm{Gr}(X)_0$ and of P it is obvious that (P, ϕ) is dense. Let $i, j \in I_0$ and $x \in X(i)$, $y \in X(j)$. We only have to show that

$$(P,\phi)^{(1)}_{x,y}\colon \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y) \to \mathrm{Gr}(X)(P(i)x, P(j)y)$$

is the identity. Let $f = (f_a)_{a\in I(i,j)} \in \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y)$. Then by noting the form of $f_a\colon X(a)x \to y$ in $X(j)$, we have the following equalities for each $n \in \mathbb{Z}$ by (4.4), (5.9) and (5.7):

$$
\begin{aligned}
&(P,\phi)^{(1)}_{x,y}(f)^n = \sum_{a\in I(i,j)} P(j)(f_a)^n \circ \phi(a)_x\\
&= \sum_{a\in I(i,j)} \left(\delta_{b,\mathbf{1}_j} \sum_{s\in\mathbb{Z}} (-1)^{(n-s)s} f_a^{n-s} \circ X_j(X(a)x)^s\right)_{b\in I(j,j)} \circ \phi(a)_x\\
&= \sum_{a\in I(i,j)} \left(\delta_{b,\mathbf{1}_j} \sum_{s\in\mathbb{Z}} (-1)^{(n-s)s} f_a^{n-s} \circ X_j(X(a)x)^s\right)_{b\in I(j,j)} \circ (\delta_{c,a}\mathbb{1}_{X(a)x})_{c\in I(i,j)}\\
&= \sum_{a\in I(i,j)} \left(\sum_{\substack{b\in I(j,j)\\ c\in I(i,j)\\ d=bc}} \delta_{b,\mathbf{1}_j} \sum_{r,s\in\mathbb{Z}} (-1)^{(n-r)r}(-1)^{(n-r-s)s} f_a^{n-r-s} \circ X_j(X(a)x)^s \circ X(b)(\delta_{c,a}\mathbb{1}_{X(a)x})^0 \circ (X_{b,c}x)^r\right)_{d\in I(i,j)}\\
&= \sum_{a\in I(i,j)} \left(\delta_{d,a} \sum_{r,s\in\mathbb{Z}} (-1)^{(n-r)r}(-1)^{(n-r-s)s} f_a^{n-r-s} \circ X_j(X(a)x)^s \circ X(\mathbb{1}_j)(\mathbb{1}_{X(a)x})^0 \circ (X_{\mathbf{1}_j,a}x)^r\right)_{d\in I(i,j)}\\
&= \sum_{a\in I(i,j)} \left(\delta_{d,a} \sum_{r,s\in\mathbb{Z}} (-1)^{(n-r)r}(-1)^{(n-r-s)s} f_a^{n-r-s} \circ X_j(X(a)x)^s \circ (X_{\mathbf{1}_j,a}x)^r\right)_{d\in I(i,j)}\\
&= \sum_{a\in I(i,j)} \left(\delta_{d,a} \sum_{\substack{r,s,t\in\mathbb{Z}\\ n=r+s+t}} (-1)^{rs+rt+st} f_a^t \circ X_j(X(a)x)^s \circ (X_{\mathbf{1}_j,a}x)^r\right)_{d\in I(i,j)} \qquad (m := r+s)\\
&= \sum_{a\in I(i,j)} \left(\delta_{d,a} \sum_{\substack{r,m,t\in\mathbb{Z}\\ n=m+t}} (-1)^{r(m-r)+mt} f_a^t \circ X_j(X(a)x)^{(m-r)} \circ (X_{\mathbf{1}_j,a}x)^r\right)_{d\in I(i,j)}\\
&= \sum_{a\in I(i,j)} \left(\delta_{d,a} \sum_{\substack{m,t\in\mathbb{Z}\\ n=m+t}} (-1)^{mt} f_a^t \circ \sum_{r\in\mathbb{Z}} (-1)^{(m-r)r} X_j(X(a)x)^{(m-r)} \circ (X_{\mathbf{1}_j,a}x)^r\right)_{d\in I(i,j)}
\end{aligned}
$$

$$= \sum_{a\in I(i,j)} \left(\delta_{d,a}((X_{\mathbf{1}_j,a}x * X_j(X(a)x))) * f_a)^n\right)_{d\in I(i,j)}$$
$$\overset{*}{=} \sum_{a\in I(i,j)} \left(\delta_{d,a}(\mathbb{1}_{X(a)x} * f_a)^n\right)_{d\in I(i,j)} = f^n.$$

In the above the equality $\overset{*}{=}$ holds. Indeed, let $(\text{-})^{\mathrm{op}}\colon X(j) \to X(j)^{\mathrm{op}}$ be the canonical contaravariant functor defined by $u^{\mathrm{op}} := u$ for all $u \in X(j)_0 \cup X(j)_1$, and $(h\circ g)^{\mathrm{op}} = g*h$ for all morphisms $g\colon u \to v, h\colon v \to w$ in $X(j)$. If we have an equality $h \circ g = \mathbb{1}_u$ in $X(j)$, then we have $g * h = (h \circ g)^{\mathrm{op}} = \mathbb{1}_u^{\mathrm{op}} = \mathbb{1}_u$. By applying this fact to the case that $g = X_{\mathbf{1}_j,a}x, h = X_j(X(a)x), u = X(a)x$, we have $X_{\mathbf{1}_j,a}x * X_j(X(a)x) = \mathbb{1}_{X(a)x}$. □

Lemma 5.6. *Let $X \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})_0$ and $H\colon \mathrm{Gr}(X) \to \mathscr{C}$ be in $\Bbbk$-$\mathbf{dgCat}$ and consider the composite 1-morphism $(F,\psi)\colon X \xrightarrow{(P,\phi)} \Delta(\mathrm{Gr}(X)) \xrightarrow{\Delta(H)} \Delta(\mathscr{C})$. Then (F,ψ) is an I-covering if and only if H is an equivalence.*

Proof. Obviously (F,ψ) is dense if and only if so is H. Further for each $i,j \in I_0$, $x \in X(i)$ and $y \in X(j)$, $(F,\psi)^{(1)}_{x,y}$ is an isomorphism if and only if so is $H_{{}_ix,{}_jy}$ because we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{a\in I(i,j)} X(j)(X(a)x,y) & \xrightarrow{(F,\psi)^{(1)}_{x,y}} & \mathscr{C}(F(i)x, F(j)y) \\ {\scriptstyle (P,\phi)^{(1)}_{x,y}} \Big\| & \nearrow_{H_{{}_ix,{}_jy}} & \\ \mathrm{Gr}(X)({}_ix, {}_jy) & & \end{array}$$

by Proposition 5.5. □

6. Adjoints

In this section we will show that the Grothendieck construction is a strict left adjoint to the diagonal 2-functor, and that I-coverings are essentially given by the unit of the adjunction.

Definition 6.1. Let $\mathscr{C} \in \mathbb{V}\text{-}\mathbf{Cat}$. We define a functor $Q_{\mathscr{C}}\colon \mathrm{Gr}(\Delta(\mathscr{C})) \to \mathscr{C}$ by

- $Q_{\mathscr{C}}({}_ix) := x$ for all ${}_ix \in \mathrm{Gr}(\Delta(\mathscr{C}))_0$; and
- $Q_{\mathscr{C}}((f_a)_{a\in I(i,j)}) := \sum_{a\in I(i,j)} f_a$ for all $(f_a)_{a\in I(i,j)} \in \mathrm{Gr}(\Delta(\mathscr{C}))({}_ix, {}_jy)$ and for all ${}_ix, {}_jy \in \mathrm{Gr}(\Delta(\mathscr{C}))_0$.

It is easy to verify that $Q_{\mathscr{C}}$ is a $\mathbb{V}$-functor.

Theorem 6.2. *The 2-functor $\mathrm{Gr}\colon \overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat}) \to \mathbb{V}\text{-}\mathbf{Cat}$ is a strict left 2-adjoint to the 2-functor $\Delta\colon \mathbb{V}\text{-}\mathbf{Cat} \to \overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat})$. The unit is given by the family of canonical morphisms $(P_X, \phi_X)\colon X \to \Delta(\mathrm{Gr}(X))$ indexed by $X \in \overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat})$, and the counit is given by the family of $Q_{\mathscr{C}}\colon \mathrm{Gr}(\Delta(\mathscr{C})) \to \mathscr{C}$ defined as above indexed by $\mathscr{C} \in \mathbb{V}\text{-}\mathbf{Cat}$.*

In particular, (P_X, ϕ_X) has a strict universality in the comma category $(X \downarrow \Delta)$, i.e., for each $(F,\psi)\colon X \to \Delta(\mathscr{C})$ in $\overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat})$ with $\mathscr{C} \in \mathbb{V}\text{-}\mathbf{Cat}$, there exists a *unique* $H\colon \mathrm{Gr}(X) \to \mathscr{C}$ *in* $\mathbb{V}$-**Cat** *such that the following is a commutative diagram in* $\overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat})$*:*

$$\begin{array}{ccc} X & \xrightarrow{(F,\psi)} & \Delta(\mathscr{C}). \\ {\scriptstyle (P_X,\phi_X)}\downarrow & \nearrow_{\Delta(H)} & \\ \Delta(\mathrm{Gr}(X)) & & \end{array}$$

Proof. For simplicity set $\eta := ((P_X, \phi_X))_{X \in \overleftarrow{\mathrm{Colax}}(I,\mathbb{V}\text{-}\mathbf{Cat})_0}$ and $\varepsilon := (Q_{\mathscr{C}})_{\mathscr{C} \in \mathbb{V}\text{-}\mathbf{Cat}_0}$.

Claim 1. $\Delta\varepsilon \cdot \eta\Delta = \mathbb{1}_\Delta$.

Indeed, let $\mathscr{C} \in \mathbb{V}$-**Cat**. It is enough to show that $\Delta(Q_{\mathscr{C}}) \cdot (P_{\Delta(\mathscr{C})}, \phi_{\Delta(\mathscr{C})}) = \mathbb{1}_{\Delta(\mathscr{C})}$. Now

$$\begin{aligned} \mathrm{LHS} &= \left((Q_{\mathscr{C}} P_{\Delta(\mathscr{C})}(i))_{i \in I_0}, (Q_{\mathscr{C}} \phi_{\Delta(\mathscr{C})}(a))_{a \in I_1}\right), \text{ and} \\ \mathrm{RHS} &= \left((\mathbb{1}_{\mathscr{C}})_{i \in I_0}, (\mathbb{1}_{\mathbf{1}_{\mathscr{C}}})_{a \in I_1}\right). \end{aligned}$$

First entry: Let $i \in I$. Then $Q_{\mathscr{C}} P_{\Delta(\mathscr{C})}(i) = \mathbb{1}_{\mathscr{C}}$ because for each $x, y \in \mathscr{C}_0$ and each $f \in \mathscr{C}(x, y)$ we have $(Q_{\mathscr{C}} P_{\Delta(\mathscr{C})}(i))(x) = Q_{\mathscr{C}}({}_i x) = x$; and $(Q_{\mathscr{C}} P_{\Delta(\mathscr{C})}(i))(f) = (\delta_{a,\mathbf{1}_i} f \cdot ((\eta_{\Delta(\mathscr{C})})_i\, x))_{a \in I_1} = \sum_{a \in I(i,i)} \delta_{a,\mathbf{1}_i} f = f$.

Second entry: Let $a\colon i \to j$ in I. Then $Q_{\mathscr{C}} \phi_{\Delta(\mathscr{C})}(a) = \mathbb{1}_{\mathbf{1}_{\mathscr{C}}}$ because for each $x \in \mathscr{C}_0$ we have $Q_{\mathscr{C}}\left(\phi_{\Delta(\mathscr{C})}(a) x\right) = Q_{\mathscr{C}}\left((\delta_{b,a} \mathbb{1}_{\Delta(\mathscr{C})(a)x})_{b \in I(i,j)}\right) = \sum_{b \in I(i,j)} \delta_{b,a} \mathbb{1}_x = \mathbb{1}_x = \mathbb{1}_{\mathbf{1}_{\mathscr{C}} x}$. This shows that LHS = RHS.

Claim 2. $\varepsilon\,\mathrm{Gr} \cdot \mathrm{Gr}\,\eta = \mathbb{1}_{\mathrm{Gr}}$.

Indeed, let $X \in \overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat})$. It is enough to show that $Q_{\mathrm{Gr}(X)} \cdot \mathrm{Gr}(P_X, \phi_X) = \mathbb{1}_{\mathrm{Gr}(X)}$.

On objects: Let ${}_i x \in \mathrm{Gr}(X)_0$. Then $Q_{\mathrm{Gr}(X)}\left(\mathrm{Gr}(P_X, \phi_X)(x)\right) = Q_{\mathrm{Gr}(X)}({}_i(P_X(i)x)) = {}_i x$.

On morphisms: Let $f = (f_a)_{a \in I(i,j)}\colon {}_i x \to {}_j y$ be in $\mathrm{Gr}(X)$. Then we have

$$\begin{aligned} Q_{\mathrm{Gr}(X)}\,\mathrm{Gr}(P_X, \phi_X)(f) &= Q_{\mathrm{Gr}(X)}((P_X(j)(f_a) \circ \phi_X(a)x)_{a \in I(i,j)}) \\ &= \sum_{a \in I(i,j)} P_X(j)(f_a) \circ \phi_X(a)x = (P_X, \phi_X)^{(1)}_{x,y}(f) = f. \end{aligned}$$

Thus the claim holds. The two claims above prove the assertion. □

The 2-functor $\mathrm{Gr}\colon \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat}) \to \Bbbk\text{-}\mathbf{dgCat}$ is a strict left 2-adjoint to the 2-functor $\Delta\colon \Bbbk\text{-}\mathbf{dgCat} \to \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. The unit is given by the family of canonical morphisms $(P_X, \phi_X)\colon X \to \Delta(\mathrm{Gr}(X))$ indexed by $X \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$, and the counit is given by the family of $Q_{\mathscr{C}}\colon \mathrm{Gr}(\Delta(\mathscr{C})) \to \mathscr{C}$ defined as above indexed by $\mathscr{C} \in \Bbbk$-**dgCat**.

In particular, (P_X, ϕ_X) has a strict universality in the comma category $(X \downarrow \Delta)$, i.e., for each $(F, \psi)\colon X \to \Delta(\mathscr{C})$ in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$ with $\mathscr{C} \in \Bbbk$-**dgCat**, there exists a unique $H\colon \mathrm{Gr}(X) \to \mathscr{C}$ in $\Bbbk$-**dgCat** such that the following is a commutative diagram in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$:

$$\begin{array}{ccc} X & \xrightarrow{(F,\psi)} & \Delta(\mathscr{C}). \\ {\scriptstyle (P_X,\phi_X)}\downarrow & \nearrow_{\Delta(H)} & \\ \Delta(\mathrm{Gr}(X)) & & \end{array}$$

Corollary 6.3. *Let $(F, \psi)\colon X \to \Delta(\mathscr{C})$ be in $\overleftarrow{\mathrm{Colax}}(I, \mathbb{V}\text{-}\mathbf{Cat})$. Then the following are equivalent.*

(1) *(F, ψ) is an I-covering;*
(2) *There exists an equivalence $H\colon \mathrm{Gr}(X) \to \mathscr{C}$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{(F,\psi)} & \Delta(\mathscr{C}) \\ {\scriptstyle (P_X,\phi_X)}\downarrow & \nearrow_{\Delta(H)} & \\ \Delta(\mathrm{Gr}(X)) & & \end{array}$$

is strictly commutative.

Proof. This immediately follows by Theorem 6.2 and Lemma 5.6. More precisely,

$$\begin{aligned} (F,\psi)^{(1)}_{x,y}(((f^n_a)_{n\in\mathbb{Z}})_{a\in I(i,j)}) &= \sum_{a\in I(i,j)} \psi(a)_x * F(j)(f_a) \\ &= \sum_{a\in I(i,j)} H\phi(a)_x * HP(j)(f_a) \\ &= H(\sum_{a\in I(i,j)} \phi(a)_x * P(j)(f_a)) \\ &= H(P,\phi)^{(1)}_{x,y}(f). \end{aligned} \tag{6.10}$$

□

7. The DG Module colax functor

Let $X\colon I \to \Bbbk\text{-}\mathbf{dgCat}$ be a colax functor. In this section we formulate the definition of the "dg module category $\mathrm{Mod}\, X$" of X as a colax functor $I \to \Bbbk\text{-}\mathbf{dgCat}$ by modifying the definition given in the previous paper [7]. Recall that the *dg module category* $\mathrm{dgMod}\,\mathscr{C}$ of a dg category $\mathscr{C} \in \Bbbk\text{-}\mathbf{dgCat}$ is defined to be the functor category $\mathrm{dgMod}\,\mathscr{C} := \Bbbk\text{-}\mathbf{dgCat}(\mathscr{C}^{\mathrm{op}}, \mathrm{dgMod}\,\Bbbk)$. Since $\Bbbk\text{-}\mathbf{dgCat}$ is a 2-category, this is extended to a representable 2-functor

$$\mathrm{dgMod}' := \Bbbk\text{-}\mathbf{dgCat}((\text{-})^{\mathrm{op}}, \mathrm{dgMod}\,\Bbbk)\colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{Ab}^{\mathrm{coop}}$$

(see Notation 2.8).

As is easily seen the composite $\mathrm{dgMod}' \circ X$ turns out to be a colax functor $I \to \Bbbk\text{-}\mathbf{Ab}^{\mathrm{coop}}$, i.e., a contravariant lax functor $I \to \Bbbk\text{-}\mathbf{Ab}$. When X is a group action, namely when I is a group G and $X\colon G \to \Bbbk\text{-}\mathbf{dgCat}$ is a functor, the usual dg module category $\mathrm{dgMod}\, X$ with a G-action of X was defined to be the composite functor

$\operatorname{dgMod} X := \operatorname{dgMod}' \circ X \circ i$, where $i\colon G \to G$ is the group anti-isomorphism defined by $x \mapsto x^{-1}$ for all $x \in G$. In this way we can change $\operatorname{dgMod}' \circ X$ to a covariant one. But in general we cannot assume the existence of such an isomorphism i. Instead in this paper we will use a covariant "pseudofunctor" $\operatorname{dgMod}\colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{Ab}$ defined below and will define $\operatorname{dgMod} X$ as the composite $\operatorname{dgMod} \circ X$, which can be seen as a "lax" extended version of the dg module category construction of a dg category with a G-action stated above. We start with a notion of colax functors from a 2-category to a 2-category. Compare our definitions of colax functors, left transformations (1-morphisms) and 2-morphisms in the setting of 2-categories given below with definitions of morphisms, transformations and modifications in the setting of bicategories (see Leinster [31] for instance).

Definition 7.1. Let $\mathbf{B}$ and $\mathbf{C}$ be 2-categories.

(1) A *colax functor* from $\mathbf{B}$ to $\mathbf{C}$ is a triple (X, η, θ) of data:

- a triple $X = (X_0, X_1, X_2)$ of maps $X_i\colon \mathbf{B}_i \to \mathbf{C}_i$ ($\mathbf{B}_i$ denotes the collection of i-morphisms of $\mathbf{B}$ for each $i = 0, 1, 2$) preserving domains and codomains of all 1-morphisms and 2-morphisms (i.e. $X_1(\mathbf{B}_1(i,j)) \subseteq \mathbf{C}_1(X_0 i, X_0 j)$ for all $i, j \in \mathbf{B}_0$ and $X_2(\mathbf{B}_2(a,b)) \subseteq \mathbf{C}_2(X_1 a, X_1 b)$ for all $a, b \in \mathbf{B}_1$ (we omit the subscripts of X below));
- a family $\eta := (\eta_i)_{i \in \mathbf{B}_0}$ of 2-morphisms $\eta_i\colon X(\mathbb{1}_i) \Rightarrow \mathbb{1}_{X(i)}$ in $\mathbf{C}$ indexed by $i \in \mathbf{B}_0$; and
- a family $\theta := (\theta_{b,a})_{(b,a)}$ of 2-morphisms $\theta_{b,a}\colon X(ba) \Rightarrow X(b)X(a)$ in $\mathbf{C}$ indexed by $(b,a) \in \operatorname{com}(\mathbf{B}) := \{(b,a) \in \mathbf{B}_1 \times \mathbf{B}_1 \mid ba \text{ is defined}\}$

satisfying the axioms:

(i) $(X_1, X_2)\colon \mathbf{B}(i,j) \to \mathbf{C}(X_0 i, X_0 j)$ is a functor for all $i, j \in \mathbf{B}_0$;

(ii) For each $a\colon i \to j$ in $\mathbf{B}_1$ the following are commutative:

$$\begin{array}{ccc} X(a\mathbb{1}_i) & \overset{\theta_{a,\mathbb{1}_i}}{\Longrightarrow} & X(a)X(\mathbb{1}_i) \\ & \diagdown\!\!\diagdown & \Big\Downarrow {\scriptstyle X(a)\eta_i} \\ & & X(a)\mathbb{1}_{X(i)} \end{array} \quad \text{and} \quad \begin{array}{ccc} X(\mathbb{1}_j a) & \overset{\theta_{\mathbb{1}_j,a}}{\Longrightarrow} & X(\mathbb{1}_j)X(a) \\ & \diagdown\!\!\diagdown & \Big\Downarrow {\scriptstyle \eta_j X(a)} \\ & & \mathbb{1}_{X(j)}X(a) \end{array} ;$$

(iii) For each $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ in $\mathbf{B}_1$ the following is commutative:

$$\begin{array}{ccc} X(cba) & \overset{\theta_{c,ba}}{\Longrightarrow} & X(c)X(ba) \\ {\scriptstyle \theta_{cb,a}} \Big\Downarrow & & \Big\Downarrow {\scriptstyle X(c)\theta_{b,a}} \\ X(cb)X(a) & \underset{\theta_{c,b}X(a)}{\Longrightarrow} & X(c)X(b)X(a) \end{array} \; ; \text{ and}$$

(iv) For each $a, a'\colon i \to j$ and $b, b'\colon j \to k$ in $\mathbf{B}_1$ and each $\alpha\colon a \to a'$, $\beta\colon b \to b'$ in $\mathbf{B}_2$ the following is commutative:

$$\begin{array}{ccc} X(ba) & \xRightarrow{\theta_{b,a}} & X(b)X(a) \\ {\scriptstyle X(\beta*\alpha)}\Big\Downarrow & & \Big\Downarrow{\scriptstyle X(\beta)*X(\alpha)} \\ X(b'a') & \xRightarrow{\theta_{b',a'}} & X(b')X(a'). \end{array}$$

(2) A *lax functor* from $\mathbf{B}$ to $\mathbf{C}$ is a colax functor from $\mathbf{B}$ to $\mathbf{C}^{\mathrm{co}}$ (see Notation 2.8).

(3) A *pseudofunctor* from $\mathbf{B}$ to $\mathbf{C}$ is a colax functor with all η_i and $\theta_{b,a}$ 2-isomorphisms.

(4) We define a 2-category $\overleftarrow{\mathrm{Colax}}(\mathbf{B}, \mathbf{C})$ having all the colax functors $\mathbf{B} \to \mathbf{C}$ as the objects as follows.

1-morphisms. Let $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ be colax functors from $\mathbf{B}$ to $\mathbf{C}$. A 1-*morphism* (called a *left transformation*) from X to X' is a pair (F, ψ) of data

- a family $F := (F(i))_{i\in\mathbf{B}_0}$ of 1-morphisms $F(i)\colon X(i) \to X'(i)$ in $\mathbf{C}$; and
- a family $\psi := (\psi(a))_{a\in\mathbf{B}_1}$ of 2-morphisms $\psi(a)\colon X'(a)F(i) \Rightarrow F(j)X(a)$

$$\begin{array}{ccc} X(i) & \xrightarrow{F(i)} & X'(i) \\ {\scriptstyle X(a)}\downarrow & \overset{\psi(a)}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle X'(a)} \\ X(j) & \xrightarrow[F(j)]{} & X'(j) \end{array}$$

in $\mathbf{C}$ indexed by $a\colon i \to j$ in $\mathbf{B}_1$ that satisfies the following three conditions:

(0) for each $\alpha\colon a \Rightarrow b$ in $\mathbf{B}(i, j)$ the following is commutative:

$$\begin{array}{ccc} X'(a)F(i) & \xRightarrow{X'(\alpha)F(i)} & X'(b)F(i) \\ {\scriptstyle \psi(a)}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \psi(b)} \\ F(j)X(a) & \xRightarrow[F(j)X(\alpha)]{} & F(j)X(b), \end{array} \tag{7.11}$$

thus ψ gives a family of natural transformations of functors:

$$\begin{array}{ccc} \mathbf{B}(i,j) & \xrightarrow{\quad X' \quad} & \mathbf{C}(X'(i), X'(j)) \\ {\scriptstyle X}\downarrow & \overset{\psi_{ij}}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle \mathbf{C}(F(i), X'(j))} \\ \mathbf{C}(X(i), X(j)) & \xrightarrow[\mathbf{C}(X(i), F(j))]{} & \mathbf{C}(X(i), X'(j)) \end{array} \qquad (i, j \in \mathbf{B}_0),$$

(a) For each $i \in \mathbf{B}_0$ the following is commutative:

$$\begin{array}{ccc} X'(\mathbb{1}_i)F(i) & \xRightarrow{\psi(\mathbf{1}_i)} & F(i)X(\mathbb{1}_i) \\ {\scriptstyle \eta'_i F(i)}\Big\Downarrow & & \Big\Downarrow{\scriptstyle F(i)\eta_i} \\ \mathbb{1}_{X'(i)}F(i) & = & F(i)\mathbb{1}_{X(i)} \end{array} \quad ; \text{ and}$$

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k$ in $\mathbf{B}_1$ the following is commutative:

$$\begin{array}{ccccc} X'(ba)F(i) & \xRightarrow{\theta'_{b,a}F(i)} & X'(b)X'(a)F(i) & \xRightarrow{X'(b)\psi(a)} & X'(b)F(j)X(a) \\ {\scriptstyle \psi(ba)}\Downarrow & & & & \Downarrow{\scriptstyle \psi(b)X(a)} \\ F(k)X(ba) & & \xRightarrow[F(k)\,\theta_{b,a}]{} & & F(k)X(b)X(a). \end{array}$$

2-morphisms. Let $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ be colax functors from $\mathbf{B}$ to $\mathbf{C}$, and (F, ψ), (F', ψ') 1-morphisms from X to X'. A 2-*morphism* from (F, ψ) to (F', ψ') is a family $\zeta = (\zeta(i))_{i\in\mathbf{B}_0}$ of 2-morphisms $\zeta(i) \colon F(i) \Rightarrow F'(i)$ in $\mathbf{C}$ indexed by $i \in \mathbf{B}_0$ such that the following is commutative for all $a \colon i \to j$ in $\mathbf{B}_1$:

$$\begin{array}{ccc} X'(a)F(i) & \xRightarrow{X'(a)\zeta(i)} & X'(a)F'(i) \\ {\scriptstyle \psi(a)}\Downarrow & & \Downarrow{\scriptstyle \psi'(a)} \\ F(j)X(a) & \xRightarrow{\zeta(j)X(a)} & F'(j)X(a). \end{array}$$

Composition of 1-morphisms. Let $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ and $X'' = (X'', \eta'', \theta'')$ be colax functors from $\mathbf{B}$ to $\mathbf{C}$, and let $(F, \psi) \colon X \to X'$, $(F', \psi') \colon X' \to X''$ be 1-morphisms. Then the composite $(F', \psi')(F, \psi)$ of (F, ψ) and (F', ψ') is a 1-morphism from X to X'' defined by

$$(F', \psi')(F, \psi) := (F'F, \psi' \circ \psi),$$

where $F'F := ((F'(i)F(i))_{i\in\mathbf{B}_0}$ and for each $a \colon i \to j$ in $\mathbf{B}$, $(\psi' \circ \psi)(a) := F'(j)\psi(a) \circ \psi'(a)F(i)$ is the pasting of the diagram

$$\begin{array}{ccccc} X(i) & \xrightarrow{F(i)} & X'(i) & \xrightarrow{F'(i)} & X''(i) \\ {\scriptstyle X(a)}\downarrow & \swarrow\!\!\swarrow{\scriptstyle \psi(a)} & \downarrow{\scriptstyle X'(a)} & \swarrow\!\!\swarrow{\scriptstyle \psi'(a)} & \downarrow{\scriptstyle X''(a)} \\ X(j) & \xrightarrow[F(j)]{} & X'(j) & \xrightarrow[F'(j)]{} & X''(j). \end{array}$$

Remark 7.2. (1) Note that a (strict) 2-functor from $\mathbf{B}$ to $\mathbf{C}$ is a pseudofunctor with all η_i and $\theta_{b,a}$ identities.

(2) By regarding the category I as a 2-category with all 2-morphisms identities, the definition (1) of colax functors above coincides with Definition 2.3.

(3) When $\mathbf{B} = I$, the definition (4) of $\overleftarrow{\mathrm{Colax}}(\mathbf{B}, \mathbf{C})$ above coincides with that of $\overleftarrow{\mathrm{Colax}}(I, \mathbf{C})$ given before.

Definition 7.3. We denote by $\Bbbk$-**dgAb** the 2-subcategory of $\Bbbk$-**Cat** consisting of the dg abelian $\Bbbk$-categories (= abelian $\Bbbk$-categories with dg-structures), the dg $\Bbbk$-functors between them, and the natural transformations between those functors.

(1) Since $\Bbbk$-**dgCat** is a 2-category,

$$\mathrm{dgMod}' := \Bbbk\text{-}\mathbf{dgCat}((\text{-})^{\mathrm{op}}, \mathrm{dgMod}\,\Bbbk) \colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{dgAb}^{\mathrm{coop}}$$

is a 2-functor, which we can regard as a contravariant lax functor

$$\mathrm{dgMod}' := \Bbbk\text{-}\mathbf{dgCat}((\text{-})^{\mathrm{op}}, \mathrm{dgMod}\,\Bbbk) \colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{dgAb}.$$

(2) We define a pseudofunctor $\mathrm{dgMod} \colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{dgAb}$ as follows.

- For each $\mathscr{C} \in \Bbbk\text{-}\mathbf{dgCat}_0$ we set $\mathrm{dgMod}\,\mathscr{C} := \mathrm{dgMod}'\,\mathscr{C}$.
- For each $F \colon \mathscr{C} \to \mathscr{C}'$ in $\Bbbk\text{-}\mathbf{dgCat}_1$ we set $\mathrm{dgMod}\,F := \text{-} \otimes_{\mathscr{C}} \overline{F} \colon \mathrm{dgMod}\,\mathscr{C} \to \mathrm{dgMod}\,\mathscr{C}'$, where $\overline{F}$ is the dg $\mathscr{C}$-$\mathscr{C}'$-bimodule defined by $\overline{F}(y,x) := \mathscr{C}'(y, F(x))$ for all $x \in \mathscr{C}_0$, $y \in \mathscr{C}'_0$, which we sometimes write as $\overline{F} := \mathscr{C}'(?, F(\text{-}))$.
- For each $\alpha \colon F \Rightarrow G$ in $\Bbbk\text{-}\mathbf{dgCat}_2$ (with $F, G \colon \mathscr{C} \to \mathscr{C}'$ in $\Bbbk\text{-}\mathbf{dgCat}_1$) we define $\mathrm{dgMod}\,\alpha \colon \mathrm{dgMod}\,F \Rightarrow \mathrm{dgMod}\,G$ by setting
$$(\mathrm{dgMod}\,\alpha)x := \mathscr{C}'(?, \alpha x) \colon \mathscr{C}'(?, Fx) \Rightarrow \mathscr{C}'(?, Gx)$$
for all $x \in \mathscr{C}_0$.
- For each $\mathscr{C} \in \Bbbk\text{-}\mathbf{dgCat}$ we define $\eta_{\mathscr{C}} \colon \mathrm{dgMod}\,\mathbb{1}_{\mathscr{C}} \Rightarrow \mathbb{1}_{\mathrm{dgMod}\,\mathscr{C}}$ by setting
$$\eta_{\mathscr{C}} M \colon M \otimes_{\mathscr{C}} \mathscr{C}(?, \text{-}) \to M$$
to be the canonical isomorphisms for all $M \in \mathrm{dgMod}\,\mathscr{C}$.
- For each pair of dg functors $\mathscr{C} \xrightarrow{F} \mathscr{C}' \xrightarrow{G} \mathscr{C}''$ in $\Bbbk\text{-}\mathbf{dgCat}$ we define
$$\theta_{G,F} \colon \mathrm{dgMod}\,GF \Rightarrow \mathrm{dgMod}\,G \circ \mathrm{dgMod}\,F$$
as the inverse of the canonical isomorphism
$$\text{-} \otimes_{\mathscr{C}} \mathscr{C}''(?, GF(\text{-})) \Rightarrow \text{-} \otimes_{\mathscr{C}} \mathscr{C}'(?, F(\text{-})) \otimes_{\mathscr{C}'} \mathscr{C}''(?, G(\text{-})).$$

It is straightforward to check that this defines a pseudofunctor.

(3) Denote by $\Bbbk\text{-}\mathbf{dgModCat}$ the 2-subcategory of $\Bbbk\text{-}\mathbf{dgAb}$ consisting of the following:

- objects: $\mathrm{dgMod}\,\mathscr{C}$ with $\mathscr{C} \in \Bbbk\text{-}\mathbf{dgCat}_0$,
- 1-morphisms: dg functors between objects having exact right adjoints, and
- 2-morphisms: all dg natural transformations between those dg functors.

Then note that the pseudofunctor $\mathrm{dgMod} \colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{dgAb}$ defined above can be seen as a pseudofunctor $\Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{dgModCat}$.

(4) For each $\mathscr{M} \in \Bbbk\text{-}\mathbf{dgAb}_0$, we denote by $\mathscr{K}_p(\mathscr{M})$ the full subcategory of the homotopy category $\mathscr{K}(\mathscr{M})$ of $\mathscr{M}$ consisting of the *homotopically projective* objects M, i.e., objects M such that $\mathscr{K}(\mathscr{M})(M, A) = 0$ for all acyclic objects A. We also define $\sigma_{\mathscr{M}} \colon \mathscr{K}_p(\mathscr{M}) \to \mathscr{K}(\mathscr{M})$ and $Q_{\mathscr{M}} \colon \mathscr{K}(\mathscr{M}) \to \mathscr{D}(\mathscr{M})$ to be the inclusion functor and the quotient functor, respectively. Then the composite $\mathbf{j}_{\mathscr{M}} := Q_{\mathscr{M}} \circ \sigma_{\mathscr{M}} \colon \mathscr{K}_p(\mathscr{M}) \to \mathscr{D}(\mathscr{M})$ has a left adjoint $\mathbf{p}_{\mathscr{M}} \colon \mathscr{D}(\mathscr{M}) \to \mathscr{K}_p(\mathscr{M})$ such that the unit of the adjoint is the identity: $\mathbf{p}_{\mathscr{M}}\mathbf{j}_{\mathscr{M}} = \mathbb{1}_{\mathscr{K}_p(\mathscr{M})}$, and the counit $\varepsilon_{\mathscr{M}} \colon \mathbf{j}_{\mathscr{M}}\mathbf{p}_{\mathscr{M}} \Rightarrow \mathbb{1}_{\mathscr{D}(\mathscr{M})}$ is a natural isomorphism having the form $\varepsilon_{\mathscr{M}} = (Q_{\mathscr{M}}(\eta_{\mathscr{M},M}))_{M \in \mathscr{D}(\mathscr{M})_0}$, where $\eta_{\mathscr{M},M} \colon \sigma_{\mathscr{M}}\mathbf{p}_{\mathscr{M}} M \to M$ is a quasi-isomorphism in $\mathscr{K}(\mathscr{M})$ for all $M \in (\mathscr{K}\mathscr{M})_0$. In particular, both $\mathbf{p}_{\mathscr{M}}$ and $\mathbf{j}_{\mathscr{M}}$ are equivalences and quasi-inverses to each other. Note that $\eta_{\mathscr{M},M}$ above also induces a natural quasi-isomorphism $\eta_{\mathscr{M}} \colon \sigma_{\mathscr{M}} \circ \mathbf{p}_{\mathscr{M}} \circ Q_{\mathscr{M}} \Rightarrow \mathbb{1}_{\mathscr{K}(\mathscr{M})}$ by setting $\eta_{\mathscr{M}} := (\eta_{\mathscr{M},M} : \sigma_{\mathscr{M}} \circ \mathbf{p}_{\mathscr{M}} \circ Q_{\mathscr{M}} M = \mathbf{p}_{\mathscr{M}} M \to M)_{M \in \mathscr{K}(\mathscr{M})_0}$. When $\mathscr{M} = \mathrm{dgMod}\,\mathscr{C}$ for some $\mathscr{C} \in \Bbbk\text{-}\mathbf{dgCat}_0$, we set $\sigma_{\mathscr{C}} := \sigma_{\mathscr{M}}, Q_{\mathscr{C}} := Q_{\mathscr{M}}, \mathbf{j}_{\mathscr{C}} := \mathbf{j}_{\mathscr{M}}, \mathbf{p}_{\mathscr{C}} := \mathbf{p}_{\mathscr{M}}, \eta_{\mathscr{C}} := \eta_{\mathscr{M}}, \varepsilon_{\mathscr{C}} := \varepsilon_{\mathscr{M}}$ for short.

(5) We can define a pseudofunctor $\mathscr{D}\colon \Bbbk\text{-}\mathbf{dgModCat} \to \Bbbk\text{-}\mathbf{Tri}$ as follows.

(a) For each $\mathrm{dgMod}\,\mathscr{C}$ in $\Bbbk\text{-}\mathbf{dgModCat}_0$ with $\mathscr{C} \in \Bbbk\text{-}\mathbf{dgCat}$ we set $\mathscr{D}(\mathrm{dgMod}\,\mathscr{C})$ to be the derived category of $\mathrm{dgMod}\,\mathscr{C}$.

(b) For each dg functor $F\colon \mathrm{dgMod}\,\mathscr{C} \to \mathrm{dgMod}\,\mathscr{C}'$ in $\Bbbk\text{-}\mathbf{dgModCat}_1$, F naturally induces a functor $\mathscr{K}F\colon \mathscr{K}(\mathrm{dgMod}\,\mathscr{C}) \to \mathscr{K}(\mathrm{dgMod}\,\mathscr{C}')$, which restricts to a functor $\mathscr{K}_pF\colon \mathscr{K}_p(\mathrm{dgMod}\,\mathscr{C}) \to \mathscr{K}_p(\mathrm{dgMod}\,\mathscr{C}')$ because F has an exact right adjoint. Then we set $\mathscr{D}F$ to be the left derived functor $\mathbf{L}F\colon \mathscr{D}(\mathrm{dgMod}\,\mathscr{C}) \to \mathscr{D}(\mathrm{dgMod}\,\mathscr{C}')$ of F, which is defined as the composite $\mathbf{L}F := \mathbf{j}_{\mathscr{C}'} \circ \mathscr{K}_pF \circ \mathbf{p}_{\mathscr{C}}$.

(c) For each dg natural transformation $\alpha\colon F \Rightarrow F'$ in $\Bbbk\text{-}\mathbf{dgModCat}_2$ with dg functors $F, F'\colon \mathrm{dgMod}\,\mathscr{C} \to \mathrm{dgMod}\,\mathscr{C}'$ in $\Bbbk\text{-}\mathbf{dgModCat}_1$, α naturally induces a natural transformation $\mathscr{K}_p\alpha\colon \mathscr{K}_pF \Rightarrow \mathscr{K}_pF'$. Then we define $\mathscr{D}\alpha := \mathbf{j}_{\mathscr{C}'} \circ \mathscr{K}_p\alpha \circ \mathbf{p}_{\mathscr{C}}$.

(d) We define $\mathscr{D}_{\mathrm{dgMod}\,\mathscr{C}}\colon \mathscr{D}(\mathbb{1}_{\mathrm{dgMod}\,\mathscr{C}})(= \mathbf{j}_{\mathscr{C}}\mathbf{p}_{\mathscr{C}}) \Rightarrow \mathbb{1}_{\mathscr{D}(\mathrm{dgMod}\,\mathscr{C})}$ by $\mathscr{D}_{\mathrm{dgMod}\,\mathscr{C}} := \varepsilon_{\mathscr{C}}$.

(e) Note that for any composable morphisms $\mathrm{dgMod}\,\mathscr{C} \xrightarrow{F} \mathrm{dgMod}\,\mathscr{C}' \xrightarrow{F'} \mathrm{dgMod}\,\mathscr{C}''$ in $\Bbbk\text{-}\mathbf{dgModCat}_1$ we have $\mathbf{L}(F' \circ F) = \mathbf{L}F' \circ \mathbf{L}F$ because $\mathbf{p}_{\mathscr{C}'}\mathbf{j}_{\mathscr{C}'} = \mathbb{1}_{\mathscr{K}_p(\mathrm{dgMod}\,\mathscr{C}')}$. We then define $\mathscr{D}_{F',F}\colon \mathbf{L}(F' \circ F) \Rightarrow \mathbf{L}F' \circ \mathbf{L}F$ as the identity $\mathbb{1}_{\mathbf{L}(F' \circ F)}$.

It is straightforward to check that this defines a pseudofunctor.

Definition 7.4. We denote by $\Bbbk$-**Tri** the 2-category of the triangulated $\Bbbk$-categories, the triangle $\Bbbk$-functors between them, and the natural transformations between those functors.

(1) A 2-functor $\mathscr{K}_p\colon \Bbbk\text{-}\mathbf{add\text{-}dg} \to \Bbbk\text{-}\mathbf{Tri}$ is canonically defined by setting $\mathscr{K}_p(\mathscr{M})$ to be the homotopy category of homotopically projective dg $\mathscr{M}$-modules for all $\mathscr{M} \in \Bbbk\text{-}\mathbf{add\text{-}dg}$. Then the composite pseudofunctor $\mathscr{K}_p \circ \mathrm{dgMod}\colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{Tri}$ is "equivalent" to $\mathscr{D} \circ \mathrm{dgMod}\colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{Tri}$.

(2) A 2-functor $\mathscr{K}_p^{\mathrm{b}}\colon \Bbbk\text{-}\mathbf{add\text{-}dg} \to \Bbbk\text{-}\mathbf{Tri}$ is canonically defined by setting $\mathscr{K}_p^{\mathrm{b}}(\mathscr{M})$ to be the smallest full triangulated subcategory of $\mathscr{K}_p(\mathscr{M})$ closed under isomorphisms, and containing the representable functors $\mathscr{M}(\text{-}, M)$ with $M \in \mathscr{M}_0$, for all $\mathscr{M} \in \Bbbk\text{-}\mathbf{add\text{-}dg}$.

(3) Then the composite pseudofunctor $\mathrm{per} := \mathscr{K}_p^{\mathrm{b}} \circ \mathrm{dgMod}\colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{Tri}$ turns out to be a subpseudofunctor of $\mathscr{K}_p \circ \mathrm{dgMod}\colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{Tri}$. We call $\mathrm{per}(\mathscr{C}) = \mathscr{K}_p^{\mathrm{b}}(\mathrm{dgMod}\,\mathscr{C})$ the perfect derived category of $\mathscr{C}$, and often regarded as a subcategory of $\mathscr{D}(\mathrm{dgMod}\,\mathscr{C})$ by the equivalence $\mathbf{j}_{\mathscr{C}}\colon \mathscr{K}_p(\mathrm{dgMod}\,\mathscr{C}) \to \mathscr{D}(\mathrm{dgMod}\,\mathscr{C})$. Then recall that the objects of $\mathrm{per}(\mathscr{C})$ are the compact objects of $\mathscr{D}(\mathrm{dgMod}\,\mathscr{C})$.

We cite the following theorem from [8], which is a useful tool to define new colax functors from an old one by composing with pseudofunctors.

Theorem 7.5. *Let* $\mathbf{B}, \mathbf{C}$ *and* $\mathbf{D}$ *be 2-categories and* $V\colon \mathbf{C} \to \mathbf{D}$ *a pseudofunctor. Then the obvious correspondence*

$$\overleftarrow{\mathrm{Colax}}(\mathbf{B}, V)\colon \overleftarrow{\mathrm{Colax}}(\mathbf{B}, \mathbf{C}) \to \overleftarrow{\mathrm{Colax}}(\mathbf{B}, \mathbf{D})$$

turns out to be a pseudofunctor.

Corollary 7.6. *Let $X\colon I \to \Bbbk\text{-}\mathbf{dgCat}$ be a colax functor. Then the following are colax functors again.*

$$\operatorname{dgMod} X := \operatorname{dgMod} \circ X\colon I \to \Bbbk\text{-}\mathbf{dgAb},$$
$$\mathscr{D}(\operatorname{dgMod} X) := \mathscr{D} \circ \operatorname{dgMod} \circ X\colon I \to \Bbbk\text{-}\mathbf{Tri}, \text{ and}$$
$$\operatorname{per} X := \operatorname{per} \circ X\colon I \to \Bbbk\text{-}\mathbf{Tri}$$

Remark 7.7. Let $X = (X, X_i, X_{b,a}) \in \overleftarrow{\operatorname{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$.

(1) A more precise description of the *dg module colax functor*

$$\operatorname{dgMod} X := \operatorname{dgMod} \circ X = (\operatorname{dgMod} X, \operatorname{dgMod} X_i, \operatorname{dgMod} X_{b,a})\colon I \to \Bbbk\text{-}\mathbf{dgModCat}$$

of X is given as follows.

- for each $i \in I_0$, $(\operatorname{dgMod} X)(i) = \operatorname{dgMod}(X(i))$; and
- for each $a\colon i \to j$ in I the functor $(\operatorname{dgMod} X)(a)\colon (\operatorname{dgMod} X)(i) \to (\operatorname{dgMod} X)(j)$ is given by $(\operatorname{dgMod} X)(a) = \text{-} \otimes_{X(i)} \overline{X(a)}$, where $\overline{X(a)}$ is a dg $X(i)$-$X(j)$-bimodule defined by

$$\overline{X(a)}(x, y) := X(j)(y, X(a)(x))$$

for all $x \in X(i)_0$ and $y \in X(j)_0$.

(2) A more precise description of the colax functor $\mathscr{D}(\operatorname{dgMod} X)\colon I \to \Bbbk\text{-}\mathbf{Tri}$ which is called the *derived dg module colax functor* of X is as follows.

- for each $i \in I_0$, $\mathscr{D}(\operatorname{dgMod} X)(i) = \mathscr{D}(\operatorname{dgMod}(X(i)))$; and
- For each $a\colon i \to j$ in I, $\mathscr{D}(\operatorname{dgMod} X)(a)\colon \mathscr{D}(\operatorname{dgMod} X)(i) \to \mathscr{D}(\operatorname{dgMod} X)(j)$ is given by

$$\text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}\colon \mathscr{D}(\operatorname{dgMod} X(i)) \to \mathscr{D}(\operatorname{dgMod} X(j)).$$

Note that by the remark in Definition 7.4 (3), $\operatorname{per}(X)$ is a colax subfunctor of $\mathscr{D}(\operatorname{dgMod} X)$.

Remark 7.8. Let $\mathscr{C} \in \Bbbk\text{-}\mathbf{dgCat}_0$. Then it is obvious by definitions that

$$\Delta(\operatorname{per}(\mathscr{C})) = \operatorname{per}(\Delta(\mathscr{C})).$$

Proposition 7.9. *The pseudofunctor* per *preserves I-precoverings, that is, if $(F, \psi)\colon X \to \Delta(\mathscr{C})$ is an I-precovering in $\overleftarrow{\operatorname{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$ with $\mathscr{C} \in \Bbbk\text{-}\mathbf{dgCat}_0$, then so is*

$$\operatorname{per}(F, \psi)\colon \operatorname{per}(X) \to \Delta(\operatorname{per}(\mathscr{C}))$$

in $\overleftarrow{\operatorname{Colax}}(I, \Bbbk\text{-}\mathbf{Tri})$.

Proof. Let $i, j \in I_0$ and $M \in (\operatorname{per} X(i))_0, N \in (\operatorname{per} X(j))_0$. It suffices to show that $\operatorname{per}(F, \psi)$ induces an isomorphism

$$\operatorname{per}(F, \psi)^{(1)}_{M,N}\colon \coprod_{a \in I(i,j)} \operatorname{per} X(j)(M \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}, N) \to \operatorname{per} \mathscr{C}(M \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{F(i)}, N \overset{\mathbf{L}}{\otimes}_{X(j)} \overline{F(j)}).$$

By assumption, (F, ψ) induces an isomorphism

$$(F, \psi)^{(1)}_{x,y}\colon \coprod_{a \in I(i,j)} X(j)(X(a)x, y) \to \mathscr{C}(F(i)x, F(j)y)$$

for all $x \in X(i)_0, y \in X(j)_0$. We first show the following.

Claim. *There exists an isomorphism*

$$\mathbf{R}\operatorname{Hom}_{\mathscr{C}}(\overline{F(i)}, N \overset{\mathbf{L}}{\otimes}_{X(j)} \overline{F(j)}) \to \coprod_{a \in I(i,j)} \mathbf{R}\operatorname{Hom}_{X(j)}(\overline{X(a)}, N)).$$

Indeed, this is given by the composite of the following isomorphisms:

$$\begin{aligned}
\mathbf{R}\operatorname{Hom}_{\mathscr{C}}(\overline{F(i)}, N \overset{\mathbf{L}}{\otimes}_{X(j)} \overline{F(j)}) &= \mathbf{R}\operatorname{Hom}_{\mathscr{C}}(\mathscr{C}(?, F(i)(\text{-})), N \overset{\mathbf{L}}{\otimes}_{X(j)} \mathscr{C}(?, F(j)(\cdot))) \\
&\overset{\text{(a)}}{\to} N \overset{\mathbf{L}}{\otimes}_{X(j)} \mathscr{C}(F(i)(\text{-}), F(j)(\cdot)) \\
&\overset{\text{(b)}}{\to} N \overset{\mathbf{L}}{\otimes}_{X(j)} \coprod_{a \in I(i,j)} X(j)(X(a)(\text{-}), (\cdot)) \\
&\overset{\text{(c)}}{\to} \coprod_{a \in I(i,j)} N \overset{\mathbf{L}}{\otimes}_{X(j)} X(j)(X(a)(\text{-}), (\cdot)) \\
&\overset{\text{(d)}}{\to} \coprod_{a \in I(i,j)} N(X(a)(\text{-}), (\cdot)) \\
&\overset{\text{(e)}}{\to} \coprod_{a \in I(i,j)} \mathbf{R}\operatorname{Hom}_{X(j)}(X(j)(?, X(a)(\text{-})), N) \\
&= \coprod_{a \in I(i,j)} \mathbf{R}\operatorname{Hom}_{X(j)}(\overline{X(a)}, N),
\end{aligned}$$

where (a) is obtained by the Yoneda lemma, (b) is an isomorphism induced from $((F, \psi)^{(1)}_{\text{-},\cdot})^{-1}$, (c) is the natural isomorphism induced by the cocontinuity of the tensor product, (d) comes from the property of the tensor product, and (e) is given by the Yoneda lemma. Now, it is not hard to verify the commutativity of the following diagram:

$$\begin{array}{ccc}
\coprod_{a \in I(i,j)} \operatorname{per} X(j)(M \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}, N) & \xrightarrow{\operatorname{per}(F,\psi)^{(1)}_{M,N}} & \operatorname{per} \mathscr{C}(M \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{F(i)}, N \overset{\mathbf{L}}{\otimes}_{X(j)} \overline{F(j)}) \\
\text{(a)} \downarrow \simeq & & \text{(b)} \downarrow \simeq \\
\coprod_{a \in I(i,j)} \operatorname{per} X(i)(M, \mathbf{R}\operatorname{Hom}_{X(j)}(\overline{X(a)}, N)) & & \operatorname{per} X(i)(M, \mathbf{R}\operatorname{Hom}_{\mathscr{C}}(\overline{F(i)}, N \overset{\mathbf{L}}{\otimes}_{X(j)} \overline{F(j)})) \\
\text{(c)} \downarrow \simeq & \swarrow_{\text{(d)}}^{\simeq} & \\
\operatorname{per} X(i)(M, \coprod_{a \in I(i,j)} \mathbf{R}\operatorname{Hom}_{X(j)}(\overline{X(a)}, N)), & &
\end{array}$$

where the isomorphisms (a) and (b) are given by adjoints, and (c) is the natural morphism, which is an isomorphism because M is compact, and (d) is an isomorphism given by the claim above. Hence $\mathrm{per}(F,\psi)^{(1)}_{M,N}$ is an isomorphism. □

Definition 7.10 (Quasi-equivalences [26])**.** Let $\mathscr{A}, \mathscr{B}$ be small dg categories and $E\colon \mathscr{A} \to \mathscr{B}$ a dg functor. Then E is called a *quasi-equivalence* if

1. The restriction $E_{X,Y}\colon \mathscr{A}(X,Y) \to \mathscr{B}(E(X),E(Y))$ of E to $\mathscr{A}(X,Y)$ is a quasi-isomorphism for all $X, Y \in \mathscr{A}_0$; and
2. The induced functor $H^0(E) : H^0(\mathscr{A}) \to H^0(\mathscr{B})$ is an equivalence.

Definition 7.11. Let $\mathscr{A}$ be a small dg category, and $\mathscr{T}$ a full subcategory of $\mathscr{D}(\mathrm{dgMod}\,\mathscr{A})$. Then $\mathscr{T}$ is called a *tilting dg subcategory* for $\mathscr{A}$, if

1. $\mathscr{T}_0 \subseteq \mathrm{per}(\mathscr{A})(\subseteq \mathscr{K}_p(\mathrm{dgMod}\,\mathscr{A}))$, i,e, every $T \in \mathscr{T}_0$ is a compact object in $\mathscr{D}(\mathrm{dgMod}\,\mathscr{A})$.
2. $\mathrm{thick}\,\mathscr{T} = \mathrm{per}(\mathscr{A})$, $\mathrm{thick}\,\mathscr{T}$ is the smallest full triangulated subcategory of $\mathscr{D}(\mathrm{dgMod}\,\mathscr{A})$ closed under direct summands that contains $\mathscr{T}$.

We cite the following from [24, Theorem 8.1] without a proof.

Theorem 7.12. *Let $\mathscr{A}$ and $\mathscr{C}$ be small dg categories. Consider the following conditions.*

1. *There is a dg functor $H : \mathrm{dgMod}\,\mathscr{C} \to \mathrm{dgMod}\,\mathscr{A}$ such that $\mathbf{L}H : \mathscr{D}(\mathrm{dgMod}\,\mathscr{C}) \to \mathscr{D}(\mathrm{dgMod}\,\mathscr{A})$ is an equivalence.*
2. *$\mathscr{C}$ is quasi-equivalent to a tilting dg subcategory for $\mathscr{A}$.*
3. *There exists a dg category $\mathscr{B}$ and dg functors*

$$\mathrm{dgMod}\,\mathscr{C} \xrightarrow{G} \mathrm{dgMod}\,\mathscr{B} \xrightarrow{F} \mathrm{dgMod}\,\mathscr{A}$$

such that $\mathbf{L}G$ and $\mathbf{L}F$ are equivalences.

Then

(a) (1) *implies* (2).
(b) (2) *implies* (3).

8. Quasi-equivalences and derived equivalences

The following statement is stated in [26] without a proof as a remark for [26, Lemma 3.10]. For completeness, we give a proof of it in this section.

Theorem 8.1. *Let $E : \mathscr{A} \to \mathscr{B}$ be a quasi-equivalence between dg categories $\mathscr{A}$ and $\mathscr{B}$. Then $\text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} \overline{E}\colon \mathscr{D}(\mathrm{dgMod}\,\mathscr{A}) \to \mathscr{D}(\mathrm{dgMod}\,\mathscr{B})$ is an equivalence of triagulated categories, where $\overline{E}$ is an $\mathscr{A}$-$\mathscr{B}$-bimodule defined by $\overline{E} := \mathscr{B}(?, E(\text{-}))$. In particular, $\mathscr{A}$ and $\mathscr{B}$ are derived equivalent.*

For the proof we prepare the following three lemmas.

Lemma 8.2. *Let $\mathscr{D}$ and $\mathscr{D}'$ be triangulated categories, and $F : \mathscr{D} \to \mathscr{D}'$ and $G : \mathscr{D}' \to \mathscr{D}$ triangle functors. Assume that the following conditions are satisfied*

1. *F is fully faithful,*

(2) *G is a right adjoint to F, and*
(3) *$G(X) = 0$ implies $X = 0$ for all objects X of $\mathscr{D}'$.*

Then F is an equivalence.

Proof. We denote the unit and the counit of the adjoint by $\eta : \mathbb{1}_{\mathscr{D}} \Rightarrow G \circ F$ and by $\varepsilon : F \circ G \Rightarrow \mathbb{1}_{\mathscr{D}'}$, respectively. Let $D \in \mathscr{D}'$, and take a distinguished triangle

$$FG(D) \xrightarrow{\varepsilon_D} D \to Y \to FG(D)[1]$$

in $\mathscr{D}'$. Apply the functor G to get

$$GFG(D) \xrightarrow{G(\varepsilon_D)} G(D) \to G(Y) \to G(D)[1].$$

Since F is fully faithful, $\eta : \mathbb{1} \Rightarrow G \circ F$ is an isomorphism. In particular, $\eta_{G(D)}$ is an isomorphism. Then the equality $G(\varepsilon_D)\eta_{G(D)} = \mathbb{1}_{G(D)}$ yields a commutative diagram with triangle rows:

$$\begin{array}{ccccccc}
GFG(D) & \xrightarrow{G(\varepsilon_D)} & G(D) & \longrightarrow & G(Y) & \longrightarrow & G(D)[1] \\
{\scriptstyle \eta_{G(D)}^{-1}}\downarrow & & \| & & \downarrow & & \downarrow \\
G(D) & \xrightarrow[\mathbb{1}_{G(D)}]{} & G(D) & \longrightarrow & G(Y) & \longrightarrow & G(D)[1]
\end{array}.$$

Thus $G(Y) = 0$. Therefore, $Y = 0$ and $FG(D) \cong D$. Hence F is an equivalence. $\square$

Lemma 8.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be dg categories, and N a dg $\mathscr{A}$-$\mathscr{B}$-bimodule. Assume that*
(1) *the dg module $N(\text{-}, A)$ is compact in $\mathscr{D}(\mathscr{B})$ for all $A \in \mathscr{A}$,*
(2) *The canonical morphism $\alpha_{Y,Z,k} \colon H^k(\mathscr{A}(Y, Z)) \to \mathrm{Hom}_{\mathscr{D}(\mathscr{B})}(N(\text{-}, Y), N(\text{-}, Z)[k])$ is an isomorphism for all $Y, Z \in \mathscr{A}$ and for all $k \in \mathbb{Z}$.*

Then $\text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N$ is fully faithful.

Proof. We know that $(\text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N, \mathbf{R}\mathrm{Hom}_{\mathscr{B}}(N, \text{-}))$ is an adjoint pair, say with the usual unit η. Therefore to show that $\text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N$ is fully faithful, it suffices to show the following.

Claim. *For each $M \in \mathscr{D}(\mathscr{A})$, $\eta_M : M \to \mathbf{R}\mathrm{Hom}_{\mathscr{B}}(N, M \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N)$ is an isomorphism in $\mathscr{D}(\mathscr{A})$.*

To show this, let $\mathscr{C}$ be the full subcategory of $\mathscr{D}(\mathscr{A})$ formed by those objects M such that η_M is an isomorphism. To show the claim we have only to show that $\mathscr{C} = \mathscr{D}(\mathscr{A})$. As is easily seen $\mathscr{C}$ is a triangulated subcategory of $\mathscr{D}(\mathscr{A})$. Therefore it suffices to show the following two facts:
(i) $\mathscr{A}(\text{-}, A) \in \mathscr{C}$ for all $A \in \mathscr{A}$; and
(ii) $\mathscr{C}$ is closed under small coproducts.

(i) Let $A \in \mathscr{A}$. We show that $\mathscr{A}(\text{-}, A) \in \mathscr{C}$, namely that

$$\eta_{\mathscr{A}(\text{-},A)} : \mathscr{A}(\text{-}, A) \to \mathbf{R}\mathrm{Hom}_{\mathscr{B}}(N, \mathscr{A}(\text{-}, A) \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N) \cong \mathbf{R}\mathrm{Hom}_{\mathscr{B}}(N, N(\text{-}, A))$$

is an isomorphism in $\mathscr{D}(\mathscr{A})$. It suffices to show that

$$\eta_{\mathscr{A}(\text{-},A)} : \mathscr{A}(\text{-}, A) \to \mathbf{R}\mathrm{Hom}_{\mathscr{B}}(N, N(\text{-}, A))$$

is a quasi-isomorphism. For each $A' \in \mathscr{A}$ and $k \in \mathbb{Z}$ we have the following commutative diagram:

$$\begin{array}{ccc} H^k(\mathscr{A}(A', A)) & \xrightarrow{H^k(\eta_{\mathscr{A}(A',A)})} & H^k(\mathbf{R}\mathrm{Hom}_{\mathscr{B}}(N(\text{-}, A'), N(\text{-}, A)) \\ & {\scriptstyle \alpha_{A',A,k}} \searrow \quad \swarrow {\scriptstyle \beta_{A',A,k}} & \\ & \mathrm{Hom}_{\mathscr{D}(\mathscr{B})}(N(\text{-}, A'), N(\text{-}, A)[k]) & \end{array},$$

where $\beta_{A',A,k}$ is the canonical isomorphism. Since $\alpha_{A',A,k}$ is an isomorphism by the assumption (2), $H^k(\eta_{\mathscr{A}(A',A)})$ turns out to be an isomorphism, which shows (i).

(ii) Let I be a small set and let $M_i \in \mathscr{C}$ for all $i \in I$. We have the following commutative diagram with canonical morphisms in $\mathscr{D}(\mathscr{A})$:

$$\begin{array}{ccc} \bigoplus_{i\in I} M_i & \xrightarrow{\eta_{\bigoplus_{i\in I} M_i}} & \mathbf{R}\mathrm{Hom}_{\mathscr{B}}(N, (\bigoplus_{i\in I} M_i) \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N) \\ \| & & \wr\uparrow\text{(a)} \\ \| & & \mathbf{R}\mathrm{Hom}_{\mathscr{B}}(N, \bigoplus_{i\in I}(M_i \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N) \\ \| & & \wr\uparrow\text{(b)} \\ \bigoplus_{i\in I} M_i & \xrightarrow[\bigoplus_{i\in I}\eta_{M_i}]{\sim} & \bigoplus_{i\in I} \mathbf{R}\mathrm{Hom}_{\mathscr{B}}(N, M_i \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N) \end{array},$$

where (a) is an isomorphism because $\text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N$ is a left adjoint and preserves small coproducts, and (b) is an isomorphism by the assumption (1). Thus

$$\eta_{\bigoplus_{i\in I} M_i} : \bigoplus_{i\in I} M_i \to \mathbf{R}\mathrm{Hom}_{\mathscr{B}}(N, \bigoplus_{i\in I} M_i \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N)$$

is an isomorphism, and hence we have $\bigoplus_{i\in I} M_i \in \mathscr{C}$. As a consequence, $\mathscr{C}$ is closed under small coproducts. □

Lemma 8.4. *Let $\mathscr{A}$ and $\mathscr{B}$ be dg categories and $E\colon \mathscr{A} \to \mathscr{B}$ a quasi-equivalence. Then for each right $\mathscr{B}$-module M the following holds:*

$$\mathbf{R}\mathrm{Hom}_{\mathscr{B}}(\mathscr{B}(\text{-}, E(\text{-})), M) = 0 \textit{ implies } M = 0.$$

Proof. Let M be a $\mathscr{B}$-module, and assume that $\mathbf{R}\mathrm{Hom}_{\mathscr{B}}(\mathscr{B}(\text{-}, E(\text{-})), M) = 0$. Take any $B \in \mathscr{B}$. It is enough to show that $M(B) = 0$. Since $H^0(E) : H^0(\mathscr{A}) \to H^0(\mathscr{B})$ is an equivalence (the condition (2) in Definition 7.10), there exists an object $A \in \mathscr{A}$, such that $E(A) = H^0(E)(A) \cong B$ in $H^0(\mathscr{B})$. Then by the functor $H^0(\mathscr{B}) \to \mathscr{D}(\mathscr{B}), X \mapsto \mathscr{B}(\text{-}, X)$ we have $\mathscr{B}(\text{-}, E(A)) \cong \mathscr{B}(\text{-}, B)$ in $\mathscr{D}(\mathscr{B})$. Hence by the dg Yoneda lemma we have

$$M(B) \cong \mathbf{R}\mathrm{Hom}_{\mathscr{B}}(\mathscr{B}(\text{-}, B), M) \cong \mathbf{R}\mathrm{Hom}_{\mathscr{B}}(\mathscr{B}(\text{-}, E(A)), M) = 0,$$

as required. □

Proof of Theorem 8.1. Define a dg $\mathscr{A}$-$\mathscr{B}$-bimodule N by $N(B,A) := \mathscr{B}(B, E(A))$ $(A \in \mathscr{A}, B \in \mathscr{B})$. Then N satisfies the condition (1) in Lemma 8.3, and by the assumption (in particular, by the condition (1) in Definition 7.10) N also satisfies the condition (2) in Lemma 8.3. Therefore $F := \text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N \colon \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{B})$ is fully faithful by Lemma 8.3. Moreover $G := \mathbf{R}\mathrm{Hom}_{\mathscr{B}}(N, \text{-})$ is a right adjoint to F and satisfies the condition (3) in Lemma 8.2 by the assumption and Lemma 8.4. Hence F is an equivalence between $\mathscr{D}(\mathscr{A})$ and $\mathscr{D}(\mathscr{B})$ by Lemma 8.2. □

9. Derived equivalences of dg module colax functors

In this section, we define necessary terminologies such as 2-quasi-isomorphisms for 2-morphisms, quasi-equivalences for 1-morphisms, and the derived 1-morphism $\mathbf{L}\overline{(F,\psi)} \colon \mathscr{D}(\mathrm{dgMod}\, X) \to \mathscr{D}(\mathrm{dgMod}\, X')$ of a 1-morphism $(F,\psi)\colon X \to X'$ between colax functors, and show the fact that the derived 1-morphism of a quasi-equivalence between colax functors X, X' turns out to be an equivalence between derived dg module colax functors of X, X'. Finally, we give definitions of tilting subfunctors and of derived equivalences.

Definition 9.1. Let $\mathbf{C}$ be a 2-category and $(F,\psi)\colon X \to X'$ a 1-morphism in the 2-category $\overleftarrow{\mathrm{Colax}}(I,\mathbf{C})$. Then (F,ψ) is called *I-equivariant* if for each $a \in I_1$, $\psi(a)$ is a 2-isomorphism in $\mathbf{C}$.

We cite the following without a proof.

Lemma 9.2 ([7]). *Let $\mathbf{C}$ be a 2-category and $(F,\psi)\colon X \to X'$ a 1-morphism in the 2-category $\overleftarrow{\mathrm{Colax}}(I,\mathbf{C})$. Then (F,ψ) is an equivalence in $\overleftarrow{\mathrm{Colax}}(I,\mathbf{C})$ if and only if*

(1) *For each $i \in I_0$, $F(i)$ is an equivalence in $\mathbf{C}$; and*
(2) *For each $a \in I_1$, $\psi(a)$ is a 2-isomorphism in $\mathbf{C}$ (namely, (F,ψ) is I-equivariant).*

To define the notion of 2-quasi-isomorphisms in $\Bbbk$-$\mathbf{dgCat}$, we need the following statement.

Lemma 9.3. *Let $G, G'\colon \mathscr{C} \to \mathscr{C}'$ be 1-morphisms and $\alpha\colon G \Rightarrow G'$ a 2-morphism in the 2-category $\Bbbk$-$\mathbf{dgCat}$. We define a $\mathscr{C}$-$\mathscr{C}'$-bimodule $\overline{G}$ by $\overline{G} := \mathscr{C}'(?, G(\cdot))$; and consider the morphism*

$$\overline{\alpha} := \mathscr{C}'(?, \alpha(\cdot))\colon \overline{G} \Rightarrow \overline{G'},$$

of $\mathscr{C}$-$\mathscr{C}'$-bimodules, and also the morphism

$$\overline{\alpha}^{\wedge} := \mathbf{R}\mathrm{Hom}_{\mathscr{C}'}(\overline{\alpha}, \mathscr{C}'(?, \text{-})) = \mathbf{R}\mathrm{Hom}_{\mathscr{C}'}(\mathscr{C}'(?, \alpha(\cdot)), \mathscr{C}'(?, \text{-})) = \mathscr{C}'(\alpha(\cdot), \text{-})\colon \overline{G'}^{\wedge} \Rightarrow \overline{G}^{\wedge}$$

of $\mathscr{C}'$-$\mathscr{C}$-bimodules. Then the following are equivalent.

(1) $\text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{C}} \overline{\alpha}\colon \text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{C}} \overline{G} \Rightarrow \text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{C}} \overline{G'}\colon \mathscr{D}(\mathrm{dgMod}\,\mathscr{C}) \to \mathscr{D}(\mathrm{dgMod}\,\mathscr{C}')$ *is a 2-isomorphism in $\Bbbk$-$\mathbf{Tri}$.*
(2) $\mathscr{C}'(\text{-}, \alpha_x)\colon \mathscr{C}'(\text{-}, G(x)) \to \mathscr{C}'(\text{-}, G'(x))$ *is a quasi-isomorphism in $\mathrm{dgMod}\,\mathscr{C}'$ for all $x \in \mathscr{C}_0$.*
(3) $\overline{\alpha}^{\wedge} \overset{\mathbf{L}}{\otimes}_{\mathscr{C}} \text{-}\colon \overline{G'}^{\wedge} \overset{\mathbf{L}}{\otimes}_{\mathscr{C}} \text{-} \Rightarrow \overline{G}^{\wedge} \overset{\mathbf{L}}{\otimes}_{\mathscr{C}} \text{-}\colon \mathscr{D}(\mathrm{dgMod}\,\mathscr{C}^{\mathrm{op}}) \to \mathscr{D}(\mathrm{dgMod}\,\mathscr{C}'^{\mathrm{op}})$ *is a 2-isomorphism in $\Bbbk$-$\mathbf{Tri}$.*

(4) $\mathscr{C}'(\alpha_x, \text{-})\colon \mathscr{C}'(G(x), \text{-}) \to \mathscr{C}'(G'(x), \text{-})$ *is a quasi-isomorphism in* $\operatorname{dgMod}\mathscr{C}'^{\mathrm{op}}$.

Proof. (1) ⇒ (2). Let $x \in \mathscr{C}_0$. Note that we have $\mathscr{C}(\text{-}, x) \overset{\mathbf{L}}{\otimes}_{\mathscr{C}} \overline{\alpha} \cong \mathscr{C}'(\text{-}, \alpha_x)$, which is an isomorphism in $\mathscr{D}(\operatorname{dgMod}\mathscr{C}')$ if and only if it is a quasi-isomorphism in $\operatorname{dgMod}\mathscr{C}'$. Hence (2) follows from (1) by applying (1) to the representable functor $\mathscr{C}(\text{-}, x)$.

(2) ⇒ (1). Let $\mathscr{U}$ be the full subcategory of $\mathscr{D}(\operatorname{dgMod}\mathscr{C})$ consisting of objects M satisfying the condition that $M \overset{\mathbf{L}}{\otimes}_{\mathscr{C}} \overline{\alpha}\colon M \overset{\mathbf{L}}{\otimes}_{\mathscr{C}} \overline{G} \to M \overset{\mathbf{L}}{\otimes}_{\mathscr{C}} \overline{G'}$ is an isomorphism. Then by (2) we have $\mathscr{C}(\text{-}, x) \in \mathscr{U}$ for all $x \in \mathscr{C}_0$. Here, it is easy to show that $\mathscr{U}$ is a triangulated subcategory of $\mathscr{D}(\operatorname{dgMod}\mathscr{C})$ and that $\mathscr{U}$ is closed under isomorphisms and direct sums with small index sets. Therefore we have $\mathscr{U} = \mathscr{D}(\operatorname{dgMod}\mathscr{C})$, which means that (1) holds.

(2) ⇒ (4). Assume that $\mathscr{C}'(\text{-}, \alpha_x)\colon \mathscr{C}'(\text{-}, Gx) \to \mathscr{C}'(\text{-}, G'x)$ is a quasi-isomorphism in $\operatorname{dgMod}\mathscr{C}'$. Then it is an isomorphism in $\mathscr{D}(\operatorname{dgMod}\mathscr{C}')$. We set $\operatorname{Hom}_{\mathscr{C}'}(\cdot, \text{-}) := (\operatorname{dgMod}\mathscr{C}')(\cdot, \text{-})$. Then the functor

$$\mathbf{R}\operatorname{Hom}_{\mathscr{C}'}(\cdot, \mathscr{C}'(\text{-}, ?))\colon \mathscr{D}(\operatorname{dgMod}\mathscr{C}') \to \mathscr{D}(\operatorname{dgMod}\mathscr{C}'^{\mathrm{op}})$$

(the variable is at $\cdot$) sends it to an isomorphism

$$\begin{aligned}\mathbf{R}\operatorname{Hom}_{\mathscr{C}'}(\mathscr{C}'(\text{-}, \alpha_x), \mathscr{C}'(\text{-}, ?))\colon\ & \mathbf{R}\operatorname{Hom}_{\mathscr{C}'}(\mathscr{C}'(\text{-}, G'(x)), \mathscr{C}'(\text{-}, ?)) \\ & \to \mathbf{R}\operatorname{Hom}_{\mathscr{C}'}(\mathscr{C}'(\text{-}, G(x)), \mathscr{C}'(\text{-}, ?)),\end{aligned}$$

in $\mathscr{D}(\operatorname{dgMod}\mathscr{C}'^{\mathrm{op}})$, which is isomorphic to

$$\begin{aligned}\operatorname{Hom}_{\mathscr{C}'}(\mathscr{C}'(\text{-}, \alpha_x), \mathscr{C}'(\text{-}, ?))\colon\ & \operatorname{Hom}_{\mathscr{C}'}(\mathscr{C}'(\text{-}, G'(x)), \mathscr{C}'(\text{-}, ?)) \\ & \to \operatorname{Hom}_{\mathscr{C}'}(\mathscr{C}'(\text{-}, G(x)), \mathscr{C}'(\text{-}, ?)),\end{aligned}$$

and by the Yoneda lemma, it is also isomorphic to

$$\mathscr{C}'(\alpha_x, ?)\colon \mathscr{C}'(G'(x), ?) \to \mathscr{C}'(G(x), ?)$$

and is an isomorphism in $\mathscr{D}(\operatorname{dgMod}\mathscr{C}'^{\mathrm{op}})$. As a consequence, $\mathscr{C}'(\alpha_x, ?)$ is a quasi-isomorphism in $\operatorname{dgMod}\mathscr{C}'^{\mathrm{op}}$.

(4) ⇒ (2). This is proved in the same way as in the converse direction.

(3) ⇔ (4). The same proof for the equivalence (1) ⇔ (2) works also for this case. □

Definition 9.4. Let $G, G'\colon \mathscr{C} \to \mathscr{C}'$ be 1-morphisms and $\alpha\colon G \Rightarrow G'$ a 2-morphism in the 2-category $\Bbbk$-**dgCat**. Then α is called a 2-*quasi-isomorphism* in $\Bbbk$-**dgCat** if one of the statements (1), ..., (4) in Lemma 9.3 holds.

Remark 9.5. We can use the condition (2) above to check whether α is a 2-quasi-equivalence. Once it is checked, we can use the property (1).

Definition 9.6. Let $(F, \psi)\colon X \to X'$ be a 1-morphism in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then (F, ψ) is called a *quasi-equivalence* if

(1) For each $i \in I_0$, $F(i) : X(i) \to X'(i)$ is a quasi-equivalence; and
(2) For each $a \in I_1$, $\psi(a)$ is a 2-quasi-isomorphism (see Definition 9.4).

See the diagram below to understand the situation:

$$
\begin{array}{ccc}
X(i) & \xrightarrow[\text{q-eq}]{F(i)} & X'(i) \\
{\scriptstyle X(a)}\downarrow & \overset{\psi(a)}{\underset{\text{2-qis}}{\swarrow\!\!\swarrow}} & \downarrow{\scriptstyle X'(a)} \\
X(j) & \xrightarrow[F(j)]{\text{q-eq}} & X'(j).
\end{array}
$$

The following is an analogue of the "left derived functor" of "$(\overline{F},\overline{\psi})$".

Definition 9.7. Let $(F,\psi)\colon X \to X'$ be a 1-morhism in $\overleftarrow{\mathrm{Colax}}(I,\Bbbk\text{-}\mathbf{dgCat})$. Then we define a 1-morphism

$$\mathbf{L}\overline{(F,\psi)}\colon \mathscr{D}(\mathrm{dgMod}\,X) \to \mathscr{D}(\mathrm{dgMod}\,X')$$

in $\overleftarrow{\mathrm{Colax}}(I,\Bbbk\text{-}\mathbf{Tri})$ by $\mathbf{L}\overline{(F,\psi)} := \mathscr{D}(\mathrm{dgMod}(F,\psi)) = \overleftarrow{\mathrm{Colax}}(I,\mathscr{D}\circ\mathrm{dgMod})((F,\psi))$. The explicit form of $\mathbf{L}\overline{(F,\psi)} := (((\mathbf{L}\overline{F})(i))_{i\in I_0}, ((\mathbf{L}\overline{\psi})(a))_{a\in I_1})$ is given as follows.

Let $i \in I_0$, and consider the dg $X(i)$-$X'(i)$-bimodule $\overline{F(i)}$ defined by $\overline{F(i)}(?,\text{-}) = X'(i)(?,F(i)(\text{-}))$. Then this defines a triangle functor

$$(\mathbf{L}\overline{F})(i) := \text{-}\overset{\mathbf{L}}{\otimes}_{X(i)}\overline{F(i)}\colon \mathscr{D}(\mathrm{dgMod}\,X(i)) \to \mathscr{D}(\mathrm{dgMod}\,X'(i))$$

between triangulated categories.

Next let $a\colon i \to j$ be a morphism in I. Then $\psi(a)\colon X'(a)F(i) \Rightarrow F(j)X(a)$ induces a morphism of $X'(j)$-$X(i)$-bimodules $\overline{\psi(a)}\colon \overline{X'(a)F(i)} \Rightarrow \overline{F(j)X(a)}$, where

$$
\begin{aligned}
\overline{X'(a)F(i)} &:= X'(j)(\text{-},X'(a)F(i)(?))\\
\overline{F(j)X(a)} &:= X'(j)(\text{-},F(j)X(a)(?)), \text{and}\\
\overline{\psi(a)} &:= X'(j)(\text{-},\psi(a)(?)),
\end{aligned}
$$

which induces the diagram

$$
\begin{array}{ccc}
\text{-}\otimes_{X(i)}\overline{F(i)}\otimes_{X'(i)}\overline{X'(a)} & \overset{*}{\Rrightarrow} & \text{-}\otimes_{X(i)}\overline{X(a)}\otimes_{X(j)}\overline{F(j)} \\
{\scriptstyle\sim}\Downarrow & & \Downarrow{\scriptstyle\sim} \\
\text{-}\otimes_{X(i)}\overline{X'(a)F(i)} & \xRightarrow[\text{-}\otimes_{X(i)}\overline{\psi(a)}]{} & \text{-}\otimes_{X(i)}\overline{F(j)X(a)}
\end{array}
\tag{9.12}
$$

of 2-morphisms in $\Bbbk$-**dgModCat**. As the unique 2-morphism making this diagram commutative, we define a 2-morphism $*$ in the diagram

$$
\begin{array}{ccc}
\mathrm{dgMod}\,X(i) & \xrightarrow{\text{-}\otimes\overline{F(i)}} & \mathrm{dgMod}\,X'(i) \\
{\scriptstyle \text{-}\otimes\overline{X(a)}=\mathrm{dgMod}\,X(a)}\downarrow & \overset{*}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle \text{-}\otimes\overline{X'(a)}=\mathrm{dgMod}\,X'(a)} \\
\mathrm{dgMod}\,X(j) & \xrightarrow{\text{-}\otimes\overline{F(j)}} & \mathrm{dgMod}\,X'(j).
\end{array}
$$

The pseudofunctor $\mathscr{D}$ sends the diagram (9.12) to the diagram

$$\begin{array}{ccc}
-\overset{\mathbf{L}}{\otimes}_{X(i)}\overline{F(i)}\overset{\mathbf{L}}{\otimes}_{X'(i)}\overline{X'(a)} & \overset{(\mathbf{L}\overline{\psi})(a)}{=\!=\!=\!\Rightarrow} & -\overset{\mathbf{L}}{\otimes}_{X(i)}\overline{X(a)}\overset{\mathbf{L}}{\otimes}_{X(j)}\overline{F(j)} \\
{\scriptstyle\sim}\Downarrow & & \Downarrow{\scriptstyle\sim} \\
-\overset{\mathbf{L}}{\otimes}_{X(i)}\overline{X'(a)F(i)} & \underset{-\overset{\mathbf{L}}{\otimes}_{X(i)}\overline{\psi(a)}}{\Longrightarrow} & -\overset{\mathbf{L}}{\otimes}_{X(i)}\overline{F(j)X(a)}
\end{array}$$

in $\Bbbk$-**Tri**. As the unique 2-morphism making this diagram commutative, the 2-morphism

$$(\mathbf{L}\overline{\psi})(a)\colon \mathbf{L}\overline{X'(a)}\mathbf{L}\overline{F(i)} \Rightarrow \mathbf{L}\overline{F(j)}\mathbf{L}\overline{X(a)}$$

is given, which is expressed in the diagram

$$\begin{array}{ccc}
\mathscr{D}(\mathrm{dgMod}\,X(i)) & \xrightarrow{\mathbf{L}\overline{F(i)}} & \mathscr{D}(\mathrm{dgMod}\,X'(i)) \\
{\scriptstyle \mathbf{L}\overline{X(a)}=\mathscr{D}(\mathrm{dgMod}\,X(a))}\downarrow & \overset{(\mathbf{L}\overline{\psi})(a)}{\Swarrow} & \downarrow{\scriptstyle \mathbf{L}\overline{X'(a)}=\mathscr{D}(\mathrm{dgMod}\,X'(a))} \\
\mathscr{D}(\mathrm{dgMod}\,X(j)) & \xrightarrow{\mathbf{L}\overline{F(j)}} & \mathscr{D}(\mathrm{dgMod}\,X'(j)).
\end{array}$$

The following says that a quasi-equivalence between colax functors induces a derived equivlence between them, which will be important for our main result.

Proposition 9.8. *Let $(F,\psi)\colon X \to X'$ be a quasi-equivalence in $\overleftarrow{\mathrm{Colax}}(I,\Bbbk\text{-}\mathbf{dgCat})$. Then $\mathbf{L}\overline{(F,\psi)}\colon \mathscr{D}(\mathrm{dgMod}\,X) \to \mathscr{D}(\mathrm{dgMod}\,X')$ is an equivalence in $\overleftarrow{\mathrm{Colax}}(I,\Bbbk\text{-}\mathbf{Tri})$.*

Proof. Let $i \in I_0$. Then since $F(i)\colon X(i) \to X'(i)$ is a quasi-equivalence, we have

$$(\mathbf{L}\overline{F})(i) := -\overset{\mathbf{L}}{\otimes}_{X(i)}\overline{F}(i)\colon \mathscr{D}(\mathrm{dgMod}\,X(i)) \to \mathscr{D}(\mathrm{dgMod}\,X'(i))$$

is an equivalence of triangulated categories by Theorem 8.1.

Let $a\colon i \to j$ be a morphism in I. Then since

$$\psi(a)\colon X'(a)F(i) \Rightarrow F(j)X(a)$$

is a 2-quasi-isomorphism, we have

$$(\mathbf{L}\overline{\psi})(a) := \mathbf{L}(\overline{\psi(a)})\colon \mathbf{L}\overline{X'(a)}\mathbf{L}\overline{F(i)} \Rightarrow \mathbf{L}\overline{F(j)}\mathbf{L}\overline{X(a)}$$

is a 2-isomorphism by definition. It is not hard to verify that

$$\mathbf{L}\overline{(F,\psi)} := (((\mathbf{L}\overline{F})(i))_{i\in I_0}, ((\mathbf{L}\overline{\psi})(a))_{a\in I_1})\colon \mathscr{D}(\mathrm{dgMod}\,X) \to \mathscr{D}(\mathrm{dgMod}\,X')$$

is a 1-morphism in $\overleftarrow{\mathrm{Colax}}(I,\Bbbk\text{-}\mathbf{Tri})$. Then by Lemma 9.2, $\mathbf{L}\overline{(F,\psi)}$ is an equivalence in $\overleftarrow{\mathrm{Colax}}(I,\Bbbk\text{-}\mathbf{Tri})$. □

A dg $\Bbbk$-category $\mathscr{A}$ is called $\Bbbk$-*projective* (resp. $\Bbbk$-*flat*) if $\mathscr{A}(x,y)$ are dg projective (resp. flat) $\Bbbk$-modules for all $x, y \in \mathscr{A}_0$.

Definition 9.9. Let $X\colon I \to \Bbbk\text{-}\mathbf{dgCat}$ be a colax functor.

(1) X is called $\Bbbk$-*projective* (resp. $\Bbbk$-*flat*) if $X(i)$ are $\Bbbk$-projective (resp. $\Bbbk$-flat) for all $i \in I_0$.

(2) A colax subfunctor $\mathscr{T}$ of $\mathrm{per}(X)$ is called *tilting* if for each $i \in I_0$, $\mathscr{T}(i) \subseteq \mathscr{D}(\mathrm{dgMod}\, X(i))$ is a tilting subcategory for $X(i)$ (see Definition 7.11).

(3) A tilting colax subfunctor $\mathscr{T}$ of $\mathrm{per}(X)$ with an I-equivariant inclusion $(\sigma, \rho)\colon \mathscr{T} \hookrightarrow \mathrm{per}(X)$ is called a *tilting colax functor* for X (see the diagram for (σ, ρ) below).

$$\begin{array}{ccc} \mathscr{T}(i) & \xhookrightarrow{\sigma(i)} & \mathrm{per}(X(i)) \\ {\scriptstyle \mathscr{T}(a)}\downarrow & \overset{\sim}{\underset{\rho(a)}{\swarrow\!\!\swarrow}} & \downarrow{\scriptstyle \mathrm{per}(X(a))} \\ \mathscr{T}(j) & \xhookrightarrow[\sigma(j)]{} & \mathrm{per}(X(j)). \end{array}$$

Definition 9.10. Let $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then X and X' are said to be *derived equivalent* if $\mathscr{D}(\mathrm{dgMod}\, X))$ and $\mathscr{D}(\mathrm{dgMod}\, X'))$ are equivalent in the 2-category $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Tri})$. Note by Lemma 9.2 that this is the case if and only if there exists a 1-morphism $(\mathbf{F}, \boldsymbol{\psi}) : \mathscr{D}(\mathrm{dgMod}\, X)) \to \mathscr{D}(\mathrm{dgMod}\, X'))$ in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Tri})$ such that

(1) For each $i \in I_0$, $\mathbf{F}(i) : \mathscr{D}(\mathrm{dgMod}\, X(i)) \to \mathscr{D}(\mathrm{dgMod}\, X'(i))$ is a triangle equivalence in $\Bbbk$-**Tri**; and
(2) For each $a \in I_1$, $\boldsymbol{\psi}(a)$ is a 2-isomorphism in $\Bbbk$-**Tri** (i.e., $(\mathbf{F}, \boldsymbol{\psi})$ is I-equivariant).

Definition 9.11. Let $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then X and X' are said to be *standardly derived equivalent* if there exists a 1-morphism $(\mathbf{F}, \boldsymbol{\psi}) : \mathscr{D}(\mathrm{dgMod}\, X)) \to \mathscr{D}(\mathrm{dgMod}\, X'))$ in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Tri})$ such that

(1) For each $i \in I_0$, $\mathbf{F}(i) = \text{-} \overset{\mathbf{L}}{\otimes}_{X(i)} Y(i) : \mathscr{D}(\mathrm{dgMod}\, X(i)) \to \mathscr{D}(\mathrm{dgMod}\, X'(i))$ is a triangle equivalence in $\Bbbk$-**Tri**, where $\mathbf{F}(i)$ is induced by dg bimodule ${}_{X(i)}Y(i)_{X'(i)}$; and
(2) For each $a \in I_1$, $\boldsymbol{\psi}(a)$ is a 2-isomorphism in $\Bbbk$-**Tri** (i.e., $(\mathbf{F}, \boldsymbol{\psi})$ is I-equivariant).

10. Derived equivalences of Grothendieck constructions

First we cite the statement [24, Theorem 8.2] in k-flat case.

Theorem 10.1 (Keller). *Let $\mathscr{A}$ and $\mathscr{B}$ be small dg $\Bbbk$-categories and assume that $\mathscr{A}$ is $\Bbbk$-flat. Then the following are equivalent.*

(1) *There exists a $\mathscr{B}$-$\mathscr{A}$-bimodue Y such that $\text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{B}} Y : \mathscr{D}(\mathscr{B}) \to \mathscr{D}(\mathscr{A})$ is a derived equivalence.*
(2) *$\mathscr{B}$ is quasi-equivalent to a tilting dg subcategory for $\mathscr{A}$.*

The derived equivalence of the form $\text{-} \overset{\mathbf{L}}{\otimes}_{\mathscr{B}} Y$ above is called a *standard derived equivalence*, and if such Y exists $\mathscr{A}$ and $\mathscr{B}$ are said to be *standardly derived equivalent.*

Definition 10.2. Let $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then a *zigzag chain of quasi-equivalences* between X and X' is a chain of 1-morphisms of the form

$$X =: X_0 \xleftarrow{(F_1,\psi_1)} X_1 \xrightarrow{(F_2,\psi_2)} \cdots \xleftarrow{(F_{n-1},\psi_{n-1})} X_{n-1} \xrightarrow{(F_n,\psi_n)} X_n := X'$$

in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$ with n even ≥ 2, where (F_i, ψ_i) are quasi-equivalences for all $i = 1, \dots, n$. Note that a quasi-equivalence $X \xrightarrow{(F_2,\psi_2)} X'$ is also regarded as a zigzag chain of quasi-equivalences by setting $n = 2$ and (F_1, ψ_1) to be the identity 1-morphism.

The following is the dg case of the main theorem in [7] that gives a generalization of the Morita type theorem characterizing derived equivalences of categories by Rickard [33] and Keller [24] in our setting.

Theorem 10.3. *Let $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. Consider the following conditions:*

(1) *X and X' are derived equivalent.*

(1_p) *$\mathrm{per}(X)$ and $\mathrm{per}(X')$ are equivalent in the 2-category $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Tri})$; and*

(2) *There exists a tilting colax functor $\mathscr{T}$ for X such that there exists a zigzag chain of quasi-equivalences between X' and $\mathscr{T}$ in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. For example when a zigzag chain of quasi-equivalences is given by a sigle quasi-equivalence $X' \to \mathscr{T}$, we have the following diagram for all morphism $a\colon i \to j$ in I:*

$$\begin{array}{ccccc} X'(i) & \xrightarrow{\text{q-eq}} & \mathscr{T}(i) & \hookrightarrow & \mathrm{per}(X(i)) \\ {\scriptstyle X'(a)}\downarrow & \overset{\text{2-qis}}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle \mathscr{T}(a)} & \overset{\sim}{\underset{\rho(a)}{\swarrow\!\!\!\swarrow}} & \downarrow{\scriptstyle \mathrm{per}(X(a))} \\ X'(j) & \xrightarrow[\text{q-eq}]{} & \mathscr{T}(j) & \hookrightarrow & \mathrm{per}(X(j)). \end{array}$$

Then (1) *implies* (1_p),*and* (1_p) *implies* (2).

Proof. (1) $\Rightarrow$ (1_p). Assume that there exists an equivalence $(\mathbf{F}, \boldsymbol{\psi}) : \mathscr{D}(\mathrm{dgMod}\, X') \to \mathscr{D}(\mathrm{dgMod}\, X)$ in the 2-category $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{Tri})$. Then for each $i \in I_0$,

$$\mathbf{F}(i) : \mathscr{D}(\mathrm{dgMod}\, X'(i)) \to \mathscr{D}(\mathrm{dgMod}\, X(i))$$

is a triangle equivalence, and for each morphism $a \in I(i, j)$ with $i, j \in I_0$, $\boldsymbol{\psi}(a)$ in the diagram below is an 2-isomorphism:

$$\begin{array}{ccc} \mathscr{D}(\mathrm{dgMod}\, X'(i)) & \xrightarrow{\mathbf{F}(i)} & \mathscr{D}(\mathrm{dgMod}\, X(i)) \\ {\scriptstyle \mathscr{D}(\mathrm{dgMod}\, X'(a))}\downarrow & \overset{\boldsymbol{\psi}(a)}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle \mathscr{D}(\mathrm{dgMod}\, X(a))} \\ \mathscr{D}(\mathrm{dgMod}\, X'(j)) & \xrightarrow{\mathbf{F}(j)} & \mathscr{D}(\mathrm{dgMod}\, X(j)) \end{array}$$

For each $i \in I_0$ recall that $\{C^\wedge \mid C \in X'(i)_0\}$ is a set of small generators for $\mathscr{D}(\mathrm{dgMod}\, X'(i))$. Consequently, $\{\mathbf{F}(i)(C^\wedge) \mid C \in X'(i)_0\}$ forms a set of small generators for $\mathscr{D}(\mathrm{dgMod}\, X(i))$. By Keller's result we know that $\mathrm{per}\, \mathscr{C}$ coincides with the set of all compact objects in $\mathscr{D}(\mathrm{dgMod}\, \mathscr{C})$ for all dg categories $\mathscr{C}$. Then by noting that a dense functor sends compact objects to compact objects, we have

$$\mathbf{F}(i)((\mathrm{per}\, X'(i))_0) \subseteq (\mathrm{per}\, X(i))_0,$$

and this induces the following strictly commutative diagram:

$$\begin{array}{ccc} \operatorname{per} X'(i) & \xrightarrow{\mathbf{F}(i)} & \operatorname{per} X(i) \\ \downarrow & & \downarrow \\ \mathscr{D}(\operatorname{dgMod} X'(i)) & \xrightarrow{\mathbf{F}(i)} & \mathscr{D}(\operatorname{dgMod} X(i)). \end{array}$$

Thus the induced functor $\mathbf{F}(i)\colon \operatorname{per} X(i) \to \operatorname{per} X'(i)$ is an equivalence of triangulated categories. Since representable functors $C^\wedge := X(i)(\text{-}, C)$ $(C \in X(i)_0)$ are sent to representable functors $X(j)(\text{-}, X(a)(C))$ by the derived tensor functor $\text{-} \overset{\mathbf{L}}{\otimes} \overline{X(a)} = \mathscr{D}(\operatorname{dgMod} X(a))$, we see that

$$\mathscr{D}(\operatorname{dgMod} X(a))((\operatorname{per} X(i))_0) \subseteq (\operatorname{per} X(j))_0,$$

and hence $\psi(a)$ in the diagaram

$$\begin{array}{ccc} \operatorname{per} X'(i) & \xrightarrow{\mathbf{F}(i)} & \operatorname{per} X(i) \\ {\scriptstyle \mathscr{D}(\operatorname{dgMod} X(a))}\downarrow & \overset{\boldsymbol{\psi}(a)}{\swarrow\!\!\!\!\swarrow} & \downarrow{\scriptstyle \mathscr{D}(\operatorname{dgMod} X'(a))} \\ \operatorname{per} X'(j) & \xrightarrow{\mathbf{F}(j)} & \operatorname{per} X(j) \end{array}$$

is defined, and is an isomorphism for each $a \in I(i, j)$ with $i, j \in I_0$. Therefore $(\mathbf{F}, \boldsymbol{\psi})$ induces an equivalence $\operatorname{per} X' \to \operatorname{per} X$, and thus (1) implies (1_p).

$(1_p) \Rightarrow (2)$. For each $i \in I_0$, we set $\mathscr{T}(i)$ to be the full subcategory of $\operatorname{per}(X'(i))$ with $\mathscr{T}(i)_0 = \{D \in (\operatorname{per} X'(i))_0 \mid D \cong \mathbf{F}(i)(C^\wedge),\ \text{for some } C \in X(i)_0\}$. Then for each $a \in I(i, j)$ with $i, j \in I_0$ we have $\mathscr{D}(\operatorname{dgMod} X(a))(\mathscr{T}(i)_0) \subseteq \mathscr{T}(j)_0$ because for each $C \in X(i)_0$ we have

$$\begin{aligned} \mathscr{D}(\operatorname{dgMod} X'(a))(\mathbf{F}(i)(C^\wedge)) &\cong \mathbf{F}(j)\mathscr{D}(\operatorname{dgMod} X(a))(C^\wedge) \\ &= \mathbf{F}(j)(C^\wedge \overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}) \cong \mathbf{F}(j)((X(a)(C))^\wedge). \end{aligned}$$

We have the following diagram

$$\begin{array}{ccccc} X'(i) & \xrightarrow{\mathrm{Y}} & \operatorname{per}(X'(i)) & \xrightarrow{\mathbf{F}(i)} & \operatorname{per}(X(i)) \\ {\scriptstyle X'(a)}\downarrow & \overset{\zeta_i}{\swarrow\!\!\!\!\swarrow} & \downarrow{\scriptstyle \operatorname{per}(X'(a))} & \underset{\boldsymbol{\psi}(a)}{\swarrow\!\!\!\!\swarrow} & \downarrow{\scriptstyle \operatorname{per}(X(a))} \\ X'(j) & \xrightarrow[\mathrm{Y}]{} & \operatorname{per}(X'(j)) & \xrightarrow[\mathbf{F}(j)]{} & \operatorname{per}(X(j)) \end{array}$$

Pasting of this yields the following diagram:

$$\begin{array}{ccc} X'(i) & \xrightarrow{\mathbf{F}(i)\circ \mathrm{Y}} & \mathscr{T}(i) \\ {\scriptstyle X(a)}\downarrow & \underset{\phi(a)}{\swarrow\!\!\!\!\swarrow} & \downarrow{\scriptstyle \mathscr{T}(a)} \\ X'(j) & \xrightarrow[\mathbf{F}(j)\circ \mathrm{Y}]{} & \mathscr{T}(j), \end{array}$$

where $\phi(a) := (\mathbf{F}(j) \circ \zeta_i) \bullet (\boldsymbol{\psi}(a) \circ Y)$. Then $X(i)$ is quasi-equivalent to the full dg subcategory $\mathscr{T}(i)$ of $\operatorname{per}(X'(i))$, which forms a generator for $\mathscr{D}(\operatorname{dgMod} X'(i))$. Since

$\boldsymbol{\psi}(a) : \mathrm{per}(X'(a))\mathbf{F}(i) \Rightarrow \mathbf{F}(j)\,\mathrm{per}(X(a))$ is a 2-isomorphism by assummption, the statement (1_p) implies (2). □

Proposition 10.4. *Let $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. Assume that $(F, \psi)\colon X \to X'$ in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$ is a quasi-equivalence. Then $\mathrm{Gr}(F, \psi)\colon \mathrm{Gr}(X) \to \mathrm{Gr}(X')$ is a quasi-equivalence.*

Proof. Let $X = (X, X_i, X_{b,a})$ and $X' = (X', X'_i, X'_{b,a})$ be objects in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$, and let $(F, \psi)\colon X \to X'$ be a 1-morphism in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. Recall that for each ${}_ix, {}_jy \in \mathrm{Gr}(X)_0$, we have

$$\mathrm{Gr}(X)({}_ix, {}_jy) := \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y).$$

Then a 1-morphism

$$\mathrm{Gr}(F, \psi)\colon\ \mathrm{Gr}(X) \to \mathrm{Gr}(X')$$

in $\Bbbk$-**dgCat** is defined as follows.

- For each ${}_ix \in \mathrm{Gr}(X)_0$, $\mathrm{Gr}(F, \psi)({}_ix) := {}_i(F(i)x)$.
- For each ${}_ix, {}_jy \in \mathrm{Gr}(X)_0$ and each $f = (f_a)_{a \in I(i,j)} \in \mathrm{Gr}(X)({}_ix, {}_jy)$, we set $\mathrm{Gr}(F, \psi)(f) := (F(j)f_a \circ \psi(a)x)_{a \in I(i,j)}$, where each entry is the composite of

$$X'(a)F(i)x \xrightarrow{\psi(a)x} F(j)X(a)x \xrightarrow{F(j)f_a} F(j)y.$$

Then we have the following

$$\mathrm{Gr}(F, \psi)\colon\ \mathrm{Gr}(X)({}_ix, {}_jy) \to \mathrm{Gr}(X')(\mathrm{Gr}(F, \psi)({}_ix), \mathrm{Gr}(F, \psi)({}_jy))$$

$$\mathrm{Gr}(F, \psi)\colon\ \mathrm{Gr}(X)({}_ix, {}_jy) := \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) \to \bigoplus_{a \in I(i,j)} X'(j)(X'(a)F(i)x, F(j)y)$$

Assume that $(F, \psi) : X \to X'$ is a quasi-equivalence, that is

1. For each $i \in I_0$, $F(i) : X(i) \to X'(i)$ is a quasi-equivalence; and
2. For each $a \in I_1$, $\psi(a)$ is a 2-quasi-isomorphism.

Claim 1. *Let ${}_ix, {}_jy \in \mathrm{Gr}(X)_0$. Then the restriction*

$$\mathrm{Gr}(F, \psi)_{{}_ix, {}_jy}\colon\ \mathrm{Gr}(X)({}_ix, {}_jy) \to \mathrm{Gr}(X')(\mathrm{Gr}(F, \psi)({}_ix), \mathrm{Gr}(F, \psi)({}_jy))$$

of $\mathrm{Gr}(F, \psi)$ to $\mathrm{Gr}(X)({}_ix, {}_jy)$ is a quasi-isomorphism.

Indeed, note first that the domain and the codomain of $\mathrm{Gr}(F, \psi)_{{}_ix, {}_jy}$ have the following form:

$$\mathrm{Gr}(F, \psi)_{{}_ix, {}_jy}\colon \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) \to \bigoplus_{a \in I(i,j)} X'(j)(X'(a)F(i)x, F(j)y)$$

We have to show that for each $k \in \mathbb{Z}$,

$$H^k(\mathrm{Gr}(X))({}_ix, {}_jy) \xrightarrow{H^k(\mathrm{Gr}(F,\psi)_{{}_ix, {}_jy})} H^k\,\mathrm{Gr}(X')(\mathrm{Gr}(F, \psi)({}_ix), \mathrm{Gr}(F, \psi)({}_jy))$$

is an isomorphism. Since we have the commutative diagram

$$
\begin{array}{ccc}
H^k(\mathrm{Gr}(X)({}_ix, {}_jy)) & \xrightarrow{H^k(\mathrm{Gr}(F,\psi)_{{}_ix,{}_jy})} & H^k(\mathrm{Gr}(X')(\mathrm{Gr}(F,\psi)({}_ix), \mathrm{Gr}(F,\psi)({}_jy))) \\
\downarrow\cong & & \downarrow\cong \\
\bigoplus_{a\in I(i,j)} H^k(X(j)(X(a)x, y)) & & \bigoplus_{a\in I(i,j)} H^k(X'(j)(X'(a)F(i)x, F(j)y)) \\
& \searrow^{\bigoplus_{a\in I(i,j)} H^k(F(j))} \quad \nearrow_{\bigoplus_{a\in I(i,j)} H^k(X'(j)(\psi(a)_x, F(j)y))} & \\
& \bigoplus_{a\in I(i,j)} H^k(X'(j)(F(j)X(a)x, F(j)y) &
\end{array}
\tag{10.13}
$$

By the assumption, $H^k(F(j)) : H^k(X(j))(X(a)x, y) \to H^k X'(j)(F(j)X(a)x, F(j)y)$ is an isomorphism, and therefore so is $\bigoplus_{a\in I(i,j)} H^k(F(j))$.

Let $a\colon i \to j$ be a morphism in I. Then since

$$\psi(a)\colon X'(a)F(i) \Rightarrow F(j)X(a)$$

is a 2-quasi-isomorphism, we have

$$(\mathbf{L}\overline{\psi})(a)\colon \mathbf{L}\overline{X'(a)}\mathbf{L}\overline{F(i)} \Rightarrow \mathbf{L}\overline{F(j)}\mathbf{L}\overline{X(a)}$$

is a 2-isomorphism

By the following diagram

$$
\begin{array}{ccc}
\mathscr{D}(\mathrm{dgMod}\, X(i)) & \xrightarrow{\mathbf{L}\overline{F(i)}} & \mathscr{D}(\mathrm{dgMod}\, X'(i)) \\
{\scriptstyle \mathbf{L}\overline{X(a)} = \mathscr{D}(\mathrm{dgMod}\, X(a))}\downarrow & \swarrow\!\!\!\!\swarrow_{(\mathbf{L}\overline{\psi})(a)} & \downarrow{\scriptstyle \mathbf{L}\overline{X'(a)} = \mathscr{D}(\mathrm{dgMod}\, X'(a))} \\
\mathscr{D}(\mathrm{dgMod}\, X(j)) & \xrightarrow{\mathbf{L}\overline{F(j)}} & \mathscr{D}(\mathrm{dgMod}\, X'(j)).
\end{array}
$$

we have an isomorphism

$$\mathbf{L}(\overline{\psi(a)})((\text{-}, x))\colon \mathbf{L}\overline{X'(a)}\mathbf{L}\overline{F(i)}((\text{-}, x)) \xrightarrow{\sim} \mathbf{L}\overline{F(j)}\mathbf{L}\overline{X(a)}((\text{-}, x)),$$

that is, an isomorphism

$$\mathbf{L}(\overline{\psi(a)})((\text{-}, x)) = (\text{-}, \psi(a)(x))\colon X'(j)(\text{-}, X'(a)(F(i)(x))) \xrightarrow{\sim} X'(j)(\text{-}, F(j)(X(a)(x)))$$

in $\mathscr{D}(\mathrm{dgMod}\, X'(j))$. Since we have a commutative diagram

$$\begin{array}{ccc}
\mathscr{D}(\mathrm{dgMod}\, X'(j))(X'(j)(\text{-}, F(j)X(a)(x)), X'(j)(\text{-}, F(j)(y)[k]) & \xrightarrow{\mathscr{D}(\mathrm{dgMod}\, X'(j))(X'(j)(\text{-},\psi(a)_x), X'(j)(\text{-},F(j)(y)[k])} & \mathscr{D}(\mathrm{dgMod}\, X'(j))(X'(j)(\text{-}, X'(a)F(i)(x), X'(j)(\text{-}, F(j)(y)[k]) \\
\downarrow \cong & & \downarrow \cong \\
H^k(X'(j)(F(j)X(a)(x), F(j)(y)) & \xrightarrow{H^k(X'(j)(\psi(a)_x, F(j)y))} & H^k(X'(j)(X'(a)F(i)(x), F(j)y)
\end{array}$$

with the vertical canonical isomorphism, we see that

$$\begin{aligned}H^k(X'(j)(\psi(a)_x, F(j)y))\colon H^k(X'(j)(F(j)X(a)(x), F(j)(y))\\ \to H^k(X'(j)(X'(a)F(i)(x), F(j)y)\end{aligned}$$

is an isomorphism, and hence so is $\bigoplus_{a\in I(i,j)} H^k(X'(j)(\psi(a)_x, F(j)y))$. Therefore, we conclude that $H^k(\mathrm{Gr}(F,\psi)_{{}_ix,{}_jy})$ is an isomorphim by the commutative diagram (10.13). Hence it follows that $\mathrm{Gr}(F,\psi)_{{}_ix,{}_jy}$ is a quasi-isomorphism for all ${}_ix$ and ${}_jy$.

Next we show the following:

Claim 2. $H^0(\mathrm{Gr}(X)) \xrightarrow{H^0(\mathrm{Gr}(\mathrm{Gr}(F,\psi))} H^0(\mathrm{Gr}(X'))$ *is an equivalence.*

By Claim 1 for $k = 0$, we have that

$$\bigoplus_{a\in I(i,j)} H^0(X(j)(X(a)x, y)) \xrightarrow{H^0(\mathrm{Gr}(F,\psi)_{{}_ix,{}_jy})} \bigoplus_{a\in I(i,j)} H^0((X'(j)(X'(a)F(i)x, F(j)y))$$

is bijective for all ${}_ix$ and ${}_jy$. Thus,

$$H^0(\mathrm{Gr}(F,\psi))\colon H^0(\mathrm{Gr}(X)) \to H^0(\mathrm{Gr}(X'))$$

is fully faithful. It only remains to show that it is dense. By the definition of Grothendieck construction, we have

$$H^0(\mathrm{Gr}(X'))_0 = H^0(\bigsqcup_{i\in I_0} X'(i)_0) = \bigsqcup_{i\in I_0} H^0(X'(i))_0 = \bigsqcup_{i\in I_0} X'(i)_0.$$

For any ${}_ix' \in \bigsqcup_{i\in I_0} X'(i)_0$ with $i \in I_0$ and $x' \in X'(i)_0$, note that

$$H^0(X(i)) \xrightarrow{H^0(F(i))} H^0(X'(i))$$

is dense by (1) above. Thus there exists $x \in X(i)_0$ such that $y := F(i)(x) = H^0(F(i)(x)) \cong x'$ in $H^0(X'(i))$. Thus there exists $f : x' \xrightarrow{\sim} y$ in $H^0(X'(i))$. Since

$$H^0(\mathrm{Gr}(F,\psi))({}_ix) = \mathrm{Gr}(F,\psi)({}_ix) = {}_iF(i)(x) = {}_iy,$$

it suffices to show that ${}_iy \cong {}_ix'$ in $H^0(\mathrm{Gr}(X'))$. Noting that

$$H^0(\mathrm{Gr}(X'))({}_ix', {}_iy) = H^0(\mathrm{Gr}(X')({}_ix', {}_iy)) = H^0(\bigoplus_{a\in I(i,i)} (X'(i)(X'(a)x', y)) = \bigoplus_{a\in I(i,i)} H^0((X'(i)(X'(a)x', y)), \text{ and}$$

$$H^0(\mathrm{Gr}(X'))({}_iy, {}_ix') = H^0(\mathrm{Gr}(X')({}_iy, {}_ix')) = H^0(\bigoplus_{a\in I(i,i)} (X'(i)(X'(a)y, x')) = \bigoplus_{a\in I(i,i)} H^0((X'(i)(X'(a)y, x')),$$

we can take elements

$$(\delta_{b,\mathbf{1}_i} f^{-1} \circ X'_i(y))_{b\in I(i,i)} \in \bigoplus_{a\in I(i,i)} H^0((X'(i)(X'(a)y, x')), \text{and}$$
$$(\delta_{a,\mathbf{1}_i} f \circ X'_i(x))_{a\in I(i,i)} \in \bigoplus_{a\in I(i,i)} H^0((X'(i)(X'(a)x', y)),$$

where entries are of the following forms

$$X'(\mathbb{1}_i)y \xrightarrow{X'_i(y)} y \xrightarrow{f^{-1}} x', \quad X'(\mathbb{1}_i)x' \xrightarrow{X'_i(x')} x' \xrightarrow{f} y,$$

respectively. A direct calculation shows that

$$(\delta_{b,\mathbf{1}_i} f^{-1} \circ X'_i(y))_{b\in I(i,i)} \circ (\delta_{a,\mathbf{1}_i} f \circ X'_i(x'))_{a\in I(i,i)} = \mathbb{1}_{{}_ix'},$$
$$(\delta_{a,\mathbf{1}_i} f \circ X'_i(x'))_{a\in I(i,i)} \circ (\delta_{b,\mathbf{1}_i} f^{-1} \circ X'_i(y))_{b\in I(i,i)} = \mathbb{1}_{{}_iy}$$

Then we have ${}_iy \cong {}_ix'$ in $H^0(\mathrm{Gr}(X'))$. Therefore $H^0(\mathrm{Gr}(F, \psi))$ is dense. □

The following is our main result in this paper.

Theorem 10.5. *Let* $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. *Assume that* X *is* $\Bbbk$*-flat and that there exists a tilting colax functor* $\mathscr{T}$ *for* X *such that there exists a zigzag chain of* quasi-equivalences *between* $\mathscr{T}$ *and* X' *in* $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$ (*the condition* (2) *in* Theorem 10.3). *Then* $\mathrm{Gr}(X)$ *and* $\mathrm{Gr}(X')$ *are derived equivalent.*

Proof. Note that $\mathrm{Gr}(X)$ is also $\Bbbk$-flat by definition of $\mathrm{Gr}(X)$. Let $\mathscr{T}$ be a tilting colax subfunctor of $\mathrm{per}(X)$ with an I-equivariant inclusion $(\sigma, \rho)\colon \mathscr{T} \hookrightarrow \mathrm{per}(X)$. Put $(P, \phi) := (P_X, \phi_X)$ for short. Let $\mathscr{T}'$ be the full subcategory of $\mathrm{per}(\mathrm{Gr}(X))$ (which is a subcategory of $\mathscr{K}_p(\mathrm{dgMod}\,\mathrm{Gr}(X))$) consisting of the objects $\mathrm{per}(P(i))(U)$ with $i \in I_0$ and $U \in \mathscr{T}(i)_0$, which is called the *gluing* of $\mathscr{T}(i)$'s.

We now show that $\mathscr{T}'$ is a tilting subcategory of $\mathrm{per}(\mathrm{Gr}(X))$. For a triangulated category $\mathscr{U}$ and a class of objects $\mathscr{V}$ in $\mathscr{U}$ denote by $\mathrm{thick}\,\mathscr{V}$ the smallest thick subcategory of $\mathscr{U}$ containing $\mathscr{V}$. Then for each $i \in I_0$ and $x \in X(i)$ we have

$$\begin{aligned}\mathrm{per}(P(i))(X(i)(\text{-}, x)) &\cong X(i)(\text{-}, x)) \otimes_{X(i)} \overline{P(i)}\\ &= X(i)(\text{-}, x)) \otimes_{X(i)} \mathrm{Gr}(X)(\text{-}, P(i)(?))\\ &\cong \mathrm{Gr}(X)(\text{-}, P(i)(x)) = \mathrm{Gr}(X)(\text{-}, {}_ix).\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{Gr}(X)(\text{-}, {}_i x) &\cong \mathrm{per}(P(i))(X(i)(\text{-}, x)) \\ &\in \mathrm{per}(P(i))(\mathrm{thick}\,\mathscr{T}(i)) \\ &\subseteq \mathrm{thick}\{\mathrm{per}(P(i))(U) \mid U \in \mathscr{T}(i)\} \\ &\subseteq \mathrm{thick}\,\mathscr{T}'.\end{aligned}$$

Therefore, thick $\mathscr{T}' = \mathrm{per}(\mathrm{Gr}(X))$, and hence $\mathscr{T}'$ is a tilting subcategory of $\mathrm{per}(\mathrm{Gr}(X))$, as desired. Hence $\mathrm{Gr}(X)$ and $\mathscr{T}'$ are derived equivalent by Keller's Theorem [24, Theorem 8.2] because $\mathrm{Gr}(X)$ is $\Bbbk$-flat. Let (F, ψ) be the restriction of $\mathrm{per}((P, \phi))$ to $\mathscr{T}$. Then by construction $(F, \psi)\colon \mathscr{T} \to \Delta(\mathscr{T}')$ is a dense functor, and it is an I-precovering because so is

$$\mathrm{per}((P, \phi))\colon \mathrm{per}(X) \to \Delta(\mathrm{per}(\mathrm{Gr}(X)))$$

by Proposition 7.9. Thus (F, ψ) is an I-covering, which shows that $\mathscr{T}' \simeq \mathrm{Gr}(\mathscr{T})$ by Corollary 6.3. Since there exists a zigzag chain of quasi-equivalences between $\mathscr{T}$ and X' in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$, we have a zigzag chain of quasi-equivalences between $\mathrm{Gr}(\mathscr{T})$ and $\mathrm{Gr}(X')$ in $\Bbbk$-**dgCat** by Proposition 10.4, and hence they are derived equivalent by Theorem 8.1. As a consequence, $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$ are derived equivalent. □

The following is immediate from Theorems 10.3 and 10.5.

Corollary 10.6. *Let $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. If X and X' are derived equivalent, then so are $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$.* □

For the special case that $I = G$ is a group, which has a unique object $*$, the theorem above have the form below.

Definition 10.7. Let $\mathscr{A}$ and $\mathscr{B}$ be dg categories with G-actions.

(1) A tilting dg subcategory $\mathscr{T}$ for $\mathscr{A}$ is called *G-equivariant* if there exists a G-equivariant inclusion $(\sigma, \rho)\colon \mathscr{T} \to \mathrm{per}(\mathscr{A})$.
(2) $\mathscr{A}$ and $\mathscr{B}$ are said to be *G-quasi-equivalent* if there exists a quasi-equivalence $(F, \phi)\colon \mathscr{A} \to \mathscr{B}$ in $\overleftarrow{\mathrm{Colax}}(G, \Bbbk\text{-}\mathbf{dgCat})$.

Corollary 10.8. *Let $\mathscr{A}$ and $\mathscr{B}$ be dg categories with G-actions, and assume that $\mathscr{B}$ is G-quasi-equivalent to a G-equivariant tilting dg subcategory for $\mathscr{A}$. Then the orbit categories $\mathscr{A}/G$ and $\mathscr{B}/G$ are derived equivalent.*

The following is easy to verify.

Lemma 10.9. *Let C, C' be in $\Bbbk$-**dgCat**. If C and C' are derived equivalent, then so are $\Delta(C)$ and $\Delta(C')$.* □

Corollary 10.6 together with the lemma above and Example 5.2 gives us a unified proof of the following fact.

Theorem 10.10. *Assume that $\Bbbk$ is a field and that dg $\Bbbk$-algebras A and A' are derived equivalent. Then the following pairs are derived equivalent as well:*

(1) *dg path categories AQ and $A'Q$ for any quiver Q;*
(2) *incidence dg categories AS and $A'S$ for any poset S; and*

(3) *monoid dg algebras AG and $A'G$ for any monoid G.*

□

Theorem 10.3 together with Theorem 5.6 in [7] suggests us that the following would be true.

Conjecture 10.11. *Let $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$. Consider the following conditions as in the theorem above.*

(1) *X and X' are derived equivalent.*
(2) *There exists a tilting colax functor $\mathscr{T}$ for X such that $\mathscr{T}$ and X' are quasi-equivalent in $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbf{dgCat})$.*

If X' is $\Bbbk$-projective, then (2) implies (1).

We may even conjecture that the statement (2) implies the following (call this Conjecture 10.11′):

(1_s) X and X' are standardly derived equivalent.

11. Examples

Remark 11.1. Let G be a group, which we regard as a groupoid with only one object $*$. Let (Q, W) be a quiver with potentials. Regard the complete Ginzburg dg algebra $\widehat{\Gamma}(Q, W)$ as a dg category with only one object, and a G-action on it as a functor $X_{Q,W} \colon G \to \Bbbk\text{-}\mathbf{dgCat}$ with $X_{Q,W}(*) = \widehat{\Gamma}(Q, W)$. Then $\mathrm{Gr}(X_{Q,W})$ is nothing but the orbit category $\widehat{\Gamma}(Q, W)/G$, which is also equivalent to the skew group dg algebra $\widehat{\Gamma}(Q, W) * G$, and is calculated as $\widehat{\Gamma}(Q_G, W_G)$ up to Morita equivaleces in the case that G is a finite group in [30] (see also [20] for the finite abelian case). Therefore in this case note that $\mathrm{Gr}(X_{Q,W})$ is calculated as $\widehat{\Gamma}(Q_G, W_G)$ up to Morita equivalences. category whose set of objects is given by Q_0 (resp. $(Q_G)_0$).

11.1. **Mutations, the complete Ginzburg dg algebras and derived equivalences.** In the example below we will use the constructions of mutations and the Ginzburg dg algebras, and a "tilting" bimodule given by Keller–Yang. To make it easy to understand these examples, we recall these constructions and fix our notations.

11.1.1. *Mutations.* Let Q be a quiver. A path in Q is said to be *cyclic* if its source and target coincide. A potential on Q is an element of the closure $\mathrm{Pot}(\Bbbk Q)$ of the subspace of $\Bbbk Q$ generated by all non-trivial cyclic paths in Q. We say that two potentials are *cyclically equivalent* if their difference is in the closure of the subspace generated by the differences $a_1 \cdots a_s - a_2 \cdots a_s a_1$ for all cycles $a_1 \cdots a_s$ in Q.

The complete path algebra $\widehat{\Bbbk Q}$ is the completion of the path algebra $\Bbbk Q$ with respect to the ideal generated by the arrows of Q. Let $\mathfrak{m}$ be the ideal of $\widehat{\Bbbk Q}$ generated by the arrows of Q. A *quiver with potential* is a pair (Q, W) of a quiver Q and a potential W of Q such that W is in $\mathfrak{m}^2$ and no two cyclically equivalent cyclic paths appear in the decomposition of W.

A quiver with potential is called *trivial* if its potential is a linear combination of cyclic paths of length 2 and its Jacobian algebra is the product of copies of the base field $\Bbbk$. A quiver with potential is called *reduced* if $\partial_a W$ is contained in $\mathfrak{m}^2$ for all arrows a of Q.

Let (Q', W') and (Q'', W'') be two quivers with potentials such that Q' and Q'' have the same set of vertices. Their direct sum, denoted by $(Q', W') \oplus (Q'', W'')$, is the new quiver with potential (Q, W), where Q is the quiver whose vertex set is the same as the vertex set of Q' (and Q'') and whose arrow set is the disjoint union of the arrow set of Q' and the arrow set of Q'', and $W = W' + W''$.

Two quivers with potentials (Q, W) and (Q', W') are *right-equivalent* if Q and Q' have the same set of vertices and there exists an algebra isomorphism $\phi : \Bbbk Q \to \Bbbk Q'$ whose restriction on vertices is the identity map and $\phi(W)$ and W' are cyclically equivalent. Such an isomorphism ϕ is called a right-equivalence.

For any quiver with potential (Q, W), there exist a trivial quiver with potential $(Q_{\mathrm{tri}}, W_{\mathrm{tri}})$ and a reduced quiver with potential $(Q_{\mathrm{red}}, W_{\mathrm{red}})$ such that (Q, W) is right-equivalent to the direct sum $(Q_{\mathrm{tri}}, W_{\mathrm{tri}}) \oplus (Q_{\mathrm{red}}, W_{\mathrm{red}})$. Furthermore, the right-equivalence class of each of $(Q_{\mathrm{tri}}, W_{\mathrm{tri}})$ and $(Q_{\mathrm{red}}, W_{\mathrm{red}})$ is uniquely determined by the right equivalence class of (Q, W). We call $(Q_{\mathrm{tri}}, W_{\mathrm{tri}})$ and $(Q_{\mathrm{red}}, W_{\mathrm{red}})$ the *trivial part* and the *reduced part* of (Q, W), respectively.

Definition 11.2. Let (Q, W) be a quiver with potential, and i a vertex of Q. Assume the following conditions:

(1) the quiver Q has no loops;
(2) the quiver Q does not have 2-cycles at i;
(3) no cyclic path occurring in the expansion of W starts and ends at i.

Note that under the condition (1), any potential is cyclically equivalent to a potential satisfying (3). We define a new quiver with potential $\widetilde{\mu_i}(Q, W) = (Q', W')$ as follows. The new quiver Q' is obtained from Q by the following procedure:

Step 1: For each arrow β with target i and each arrow α with source i, add a new arrow $[\alpha\beta]$ from the source of β to the target of α .

Step 2: Replace each arrow α with source or target i with an arrow α^* in the opposite direction.

The new potential W' is the sum of two potentials W'_1 and W'_2, where the potential W'_1 is obtained from W by replacing each composition $\alpha\beta$ by $[\alpha\beta]$, where β is an arrow with target i, and the potential W'_2 is given by

$$W'_2 = \sum_{\alpha,\beta \in Q_1} [\alpha\beta]\beta^*\alpha^*,$$

where the sum ranges over all pairs of arrows α and β such that β ends at i and α starts at i. It is easy to see that $\widetilde{\mu_i}(Q, W)$ satisfies (1), (2) and (3). We define $\mu_i(Q, W)$ as the reduced part of $\widetilde{\mu_i}(Q, W)$, and call μ_i the *mutation* at the vertex i.

11.1.2. *The complete Ginzburg dg algebras.*

Definition 11.3. Let (Q, W) be a quiver with potential. The *complete Ginzburg dg algebra* $\widehat{\Gamma}(Q, W)$ is constructed as follows [18]: Let $\widetilde{Q}$ be the graded quiver with the same vertices as Q and whose arrows are

- the arrows of Q (they all have degree 0),
- an arrow $\overline{\alpha} : j \to i$ of degree -1 for each arrow $\alpha : i \to j$ of Q,
- a loop $t_i : i \to i$ of degree -2 for each vertex i of Q.

The underlying graded algebra of $\widehat{\Gamma}(Q, W)$ is the completion of the graded path algebra $k\widetilde{Q}$ in the category of graded vector spaces with respect to the ideal generated by the arrows of $\widetilde{Q}$. Thus, the n-th component of $\widehat{\Gamma}(Q, W)$ consists of elements of the form $\sum_p \lambda_p p$ with $\lambda_p \in \Bbbk$, where p runs over all paths of degree n. The differential of $\widehat{\Gamma}(Q, W)$ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule

$$d(uv) = d(u)v + (-1)^p ud(v),$$

for all homogeneous u of degree p and all v, and takes the following values on the arrows of $\widetilde{Q}$:

- $da = 0$ for each arrow a of Q,
- $d(\overline{a}) = \partial_a W$ for each arrow a of Q,
- $d(t_i) = e_i(\sum_a [a, a^*])e_i$ for each vertex i of Q, where e_i is the trivial path at i and the sum is taken over the set of arrows of Q.

Remark 11.4. We regard the complete Ginzburg dg algebra $\widehat{\Gamma}(Q, W)$ as a dg category as follows.

- The objects are the vertices of $\widetilde{Q}$ (namely the vertices of Q).
- $\widehat{\Gamma}(Q, W)(i, j) := e_j\widehat{\Gamma}(Q, W)e_i$ for all objects i, j.
- The composition is given by the multiplication of $\widehat{\Gamma}(Q, W)$.
- The grading and the differential are naturally defined from those of the dg algebra structure.

The following lemma is an easy consequence of the definition (cf. [27, Lemma 2.8]).

Lemma 11.5. *Let (Q, W) be a quiver with potential. Then the Jacobian algebra* $\operatorname{Jac}(Q, W)$ *is the 0-th cohomology of the complete Ginzburg dg algebra $\widehat{\Gamma}(Q, W)$, i.e.*

$$\operatorname{Jac}(Q, W) = H^0(\widehat{\Gamma}(Q, W)).$$

11.1.3. *Derived equivalences.* Let (Q, W) be a quiver with potential and i a fixed vertex of Q. We assume (1), (2) and (3) as above. Write $\widetilde{\mu}_i(Q, W) = (Q', W')$. Let $\Gamma = \widehat{\Gamma}(Q, W)$ and $\Gamma' = \widehat{\Gamma}(Q', W')$ be the complete Ginzburg dg algebras associated to (Q, W) and (Q', W'), respectively. We set $P_j = e_j\Gamma$ and $P'_j = e_j\Gamma'$ for all vertices j of Q.

We cite the following from [27, Theorem 3.2] without a proof.

Theorem 11.6. *There is a triangle equivalence*

$$F : \mathscr{D}(\operatorname{dgMod}\Gamma') \to \mathscr{D}(\operatorname{dgMod}\Gamma)$$

which sends the P'_j to P_j for $j \neq i$, and sends P'_i to the cone T_i over the morphism

$$P_i \to \bigoplus_{\alpha\in Q_1, s(\alpha)=i} P_{t(\alpha)}$$
$$a \mapsto \sum_{\alpha\in Q_1, s(\alpha)=i} e_{t(\alpha)}\alpha a,$$

The functor F restricts to triangle equivalences from $\operatorname{per}(\Gamma')$ *to* $\operatorname{per}(\Gamma)$ *and from* $\mathscr{D}_{fd}(\Gamma')$ *to* $\mathscr{D}_{fd}(\Gamma)$.

The proof is based on a construction of a Γ'-Γ-bimodule T, and F is defined by $F := (\text{-}) \overset{\mathbf{L}}{\otimes}_\Gamma T \colon \mathscr{D}(\Gamma') \to \mathscr{D}(\Gamma)$. We recall the construction of T by Keller-Yang below. As a right Γ-module, let T be the direct sum of T_i and P_j for all $j \in Q_0$ with $j \neq i$. A left Γ'-module structure on T will be defined in the next proposition. To this end we define a map $f\colon \{e_j \mid j \in Q_0\} \cup (\widetilde{Q'})_1 \to \operatorname{End}_\Gamma(T)$ as follows. First, we set $f(e_j) := f_j : T_j \to T_j$ to be the identity map for all $j \in Q_0$.

We denote by λ_a the left multiplication $x \mapsto ax$ by a below when this makes sense, and by $e_{\Sigma i}$ the unique idempotent in Γ such that $e_{\Sigma i}\Gamma = \Sigma P_i = P_i[1]$, the shift of P_i, for all $i \in Q_0$.

Let $\alpha \in Q_1$ with $s(\alpha) = i$. Then define $f_{\alpha^*} : T_{t(\alpha)} \to T_i$ of degree 0 as the cannonical embedding $T_{t(\alpha)} = P_{t(\alpha)} \hookrightarrow T_i$, that is,

$$f_{\alpha^*} := \lambda_{e_{t(\alpha)}} : T_{t(\alpha)} \to T_i, \quad a \mapsto e_{t(\alpha)}a.$$

Define also the morphsim $f_{\overline{\alpha^*}} : T_i \to T_{t(\alpha)}$ of degree -1 by

$$f_{\overline{\alpha^*}}((e_{\Sigma i})a_i + \sum_{\rho\in Q_1, s(\rho)=i} e_{t(\rho)}a_\rho) = -\alpha t_i a_i - \sum_{\rho\in Q_1, s(\rho)=i} \alpha\overline{\rho}a_\rho$$

Let $\beta \in Q_1$ with $t(\beta) = i$. Then define the morphism $f_{\beta^*} : T_i \to T_{s(\beta)}$ of degree 0 by

$$f_{\beta^*}((e_{\Sigma i})a_i + \sum_{\rho\in Q_1, s(\rho)=i} e_{t(\rho)}a_\rho) = -\overline{\beta}a_i - \sum_{\rho\in Q_1, s(\rho)=i} (\partial_{\rho\beta}W)a_\rho.$$

Define also the morphsim $f_{\overline{\beta^*}} : T_{s(\beta)} \to T_i$ of degree -1 as the composite of the morphism $\lambda_{e_{\Sigma i}\beta}\colon T_{s(\beta)} \to \Sigma P_i$ and the cannonical embedding $\Sigma P_i \hookrightarrow T_i$, that is,

$$f_{\overline{\beta^*}} := \lambda_{e_{\Sigma i}\beta} : T_{s(\beta)} \to T_i, \quad a \mapsto e_{\Sigma i}\beta a.$$

Let $\alpha, \beta \in Q_1$ with $s(\alpha) = i, t(\beta) = i$. Then define

$$f_{[\alpha\beta]} := \lambda_{\alpha\beta} : T_{s(\beta)} \to T_{t(\alpha)}, \quad a \mapsto \alpha\beta a.$$

and

$$f_{\overline{[\alpha\beta]}} := 0 : T_{t(\alpha)} \to T_{s(\beta)}.$$

Let $\gamma \in Q_1$ be an arrow not incident to i. Then define

$$f_\gamma := \lambda_\gamma : T_{s(\gamma)} \to T_{t(\gamma)}, \quad a \mapsto \gamma a,$$
$$f_{\overline{\gamma}} := \lambda_{\overline{\gamma}} : T_{t(\gamma)} \to T_{s(\gamma)}, \quad a \mapsto \overline{\gamma}a.$$

Let $j \in Q_0$ with $j \neq i$. Then define

$$f_{t'_j} := \lambda_{t_j} : T_j \to T_j, \quad a \mapsto t_j a.$$

It is a morphism of degree -2. Finally, define $f_{t'_i}$ as the linear morphism of degree -2 from T_i to itself given by

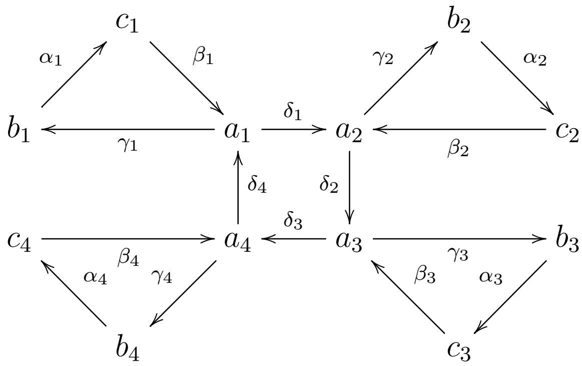
$$f_{t'_i}((e_{\Sigma i})a_i + \sum_{\rho \in Q_1, s(\rho)=i} e_\rho a_\rho) = -e_{\Sigma i}(t_i a_i + \sum_{\rho \in Q_1, s(\rho)=i} \overline{\rho} a_\rho).$$

By [27, Proposition 3.5] we have the following.

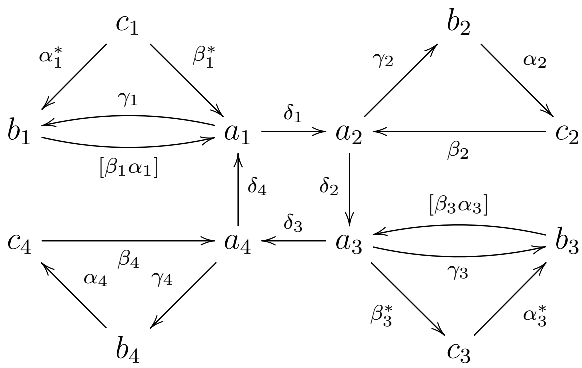
Proposition 11.7. *The map $f\colon \{e_j \mid j \in Q_0\} \cup (\widetilde{Q'})_1 \to \operatorname{End}_\Gamma(T)$ defined above extends to a homomorphism of dg algebras from Γ' to $\operatorname{End}_\Gamma(T)$. In this way, T becomes a left dg Γ'-module, and also a dg Γ'-Γ-bimodule.* □

11.2. **Examples.**

Example 11.8. Let (Q, W) be the quiver with potential given as follows:

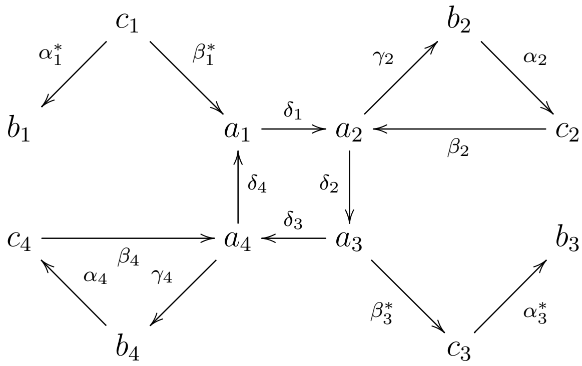


$W = \delta_4\delta_3\delta_2\delta_1 + \sum_{i=1}^{3} \gamma_i\beta_i\alpha_i$. If we do mutations at c_1 and c_3 for (Q, W), we get the following quiver with potential (Q', W')



$W' = \delta_4\delta_3\delta_2\delta_1 + \gamma_1[\beta_1\alpha_1] + \gamma_3[\beta_3\alpha_3] + \gamma_2\beta_2\alpha_2 + \gamma_4\beta_4\alpha_4 + [\beta_1\alpha_1]\alpha_1^*\beta_1^* + [\beta_3\alpha_3]\alpha_3^*\beta_3^*.$

The reduced part $(Q'_{\mathrm{red}}, W'_{\mathrm{red}})$ of (Q', W') is given as follows:

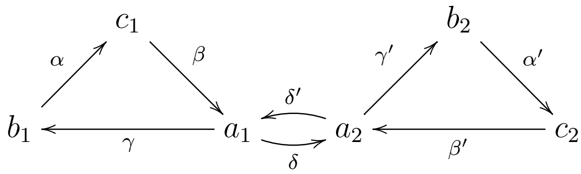


$W'_{\mathrm{red}} = \delta_4\delta_3\delta_2\delta_1 + \gamma_2\beta_2\alpha_2 + \gamma_4\beta_4\alpha_4 + [\beta_1\alpha_1]\alpha_1^*\beta_1^* + [\beta_3\alpha_3]\alpha_3^*\beta_3^*$.

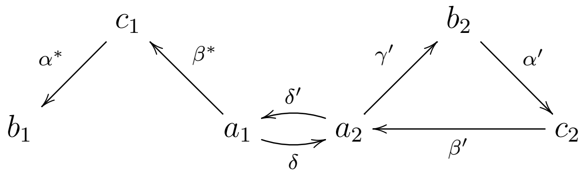
Consider the cyclic group G of order 2 with generator g, and define a G-action on (Q, W) as a unique quiver automorphism induced by the permutation of indexes $i = 1, 2, 3, 4$:

$$i \mapsto i - 2 \pmod 4. \tag{11.14}$$

Then the quiver with potential (Q_G, W_G) is given as follows:

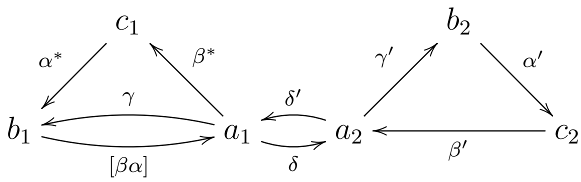


$W_G = (\delta'\delta)^2 + 2\gamma\beta\alpha + 2\gamma'\beta'\alpha'$. Define also a G-action on $(Q'_{\mathrm{red}}, W'_{\mathrm{red}})$ by the same permutation of indexes as (11.14). Then the quiver with potential $((Q'_{\mathrm{red}})_G, (W'_{\mathrm{red}})_G)$ is given as follows:

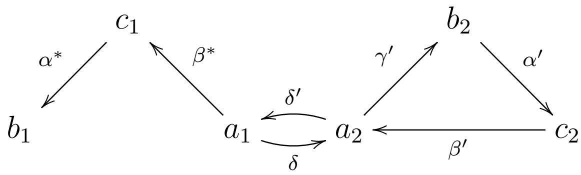


$(W'_{\mathrm{red}})_G = (\delta'\delta)^2 + 2\gamma'\beta'\alpha'$.

If we do mutations at c_1 and c_3 for (Q, W), then we do mutation at c_1 for (Q_G, W_G). Then the reduced part of $\mu_{c_1}(Q_G, W_G)$ coincides with $((Q'_{\mathrm{red}})_G, (W'_{\mathrm{red}})_G)$. Indeed, the quiver with potential $\mu_{c_1}(Q_G, W_G)$ is the following

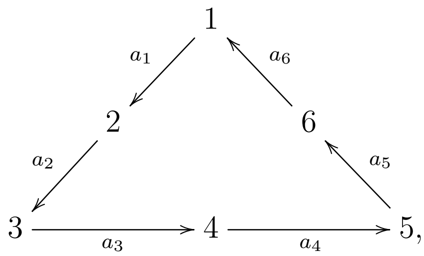


$\mu_{c_1}(W_G) = (\delta'\delta)^2 + 2\gamma[\beta\alpha] + 2\gamma'\beta'\alpha' + 2[\beta\alpha]\alpha^*\beta^*$. The potential is not reductive, so we have the following quiver with potential



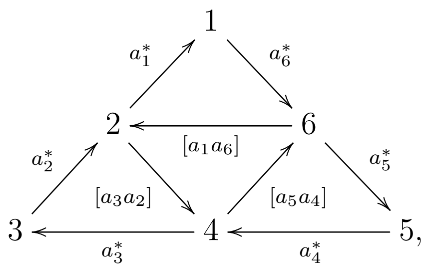
$\mu_{c_1}(W_G) = (\delta'\delta)^2 + 2\gamma'\beta'\alpha'$. Hence by Keller-Yang's result [27, Theorem 3.2 (b)] the Ginzburg dg algebras of (Q_G, W_G) and $((Q'_{\rm red})_G, (W'_{\rm red})_G)$ are derived equivalent. On the other hand, by Remark 11.1 we know that $\mathrm{Gr}(X_{Q,W})$ is Morita equivalent to $\widehat{\Gamma}(Q_G, W_G)$, and $\mathrm{Gr}(X_{Q',W'})$ is Morita equivalent to $\widehat{\Gamma}(Q'_G, W'_G)$, and which is isomorphic to $\widehat{\Gamma}((Q'_{\rm red})_G, (W'_{\rm red})_G)$ by Keller–Yang [27, Lemma 2.9] because (Q', W') and $(Q'_{\rm red}, W'_{\rm red})$ are right-equivalent. As a consequence, $\mathrm{Gr}(X_{Q,W})$ and $\mathrm{Gr}(X_{Q',W'})$ are derived equivalent. The same conclusion can be obtained from our result Corollary 10.8 as in the next example.

Example 11.9. Let (Q, W) be the quiver with potential given as follows:



$$W = a_5a_4a_3a_2a_1a_6.$$

Let $I = \{1, 3, 5\}$. Mizuno [32] defined successive mutation $\mu_I(Q, W) = \mu_5 \circ \mu_3 \circ \mu_1(Q, W) = (Q', W')$ given by the quiver with potential as follows:



$$W' = [a_1a_6]a_6^*a_1^* + [a_3a_2]a_2^*a_3^* + [a_5a_4]a_4^*a_5^* + [a_5a_4][a_3a_2][a_1a_6].$$

By [32, Theorem 1.1], the Jacobian algebras $\mathrm{Jac}(Q, W)$ and $\mathrm{Jac}(Q', W')$ are derived equivalent.

(1) Consider the cyclic group G of order 3 with generator g, and define the action of g on (Q, W) by $i \mapsto i - 2$ and $a_i \mapsto a_{i-2}$ (modulo 6). Therefore, we have

$$Ga_1 = \{a_1, a_5, a_3\}, Ga_2 = \{a_2, a_6, a_4\}.$$

In this case (Q_G, W_G) is the quiver with potential given as follows:

$$1 \underset{\alpha}{\overset{\beta}{\rightleftarrows}} 2$$

$$W_G = (\beta\alpha)^3.$$

(2) Next we define the action of g on (Q', W') by

$$i \mapsto i-2,\ a_i^* \mapsto a_{i-2}^*,\ \text{and } [a_i a_{i+5}] \mapsto [a_{i-2}a_{i+3}] \pmod 6$$

for all $i = 1, \ldots, 6$.

Therefore, we have

$$Ga_1^* = \{a_1^*, a_5^*, a_3^*\}, Ga_2^* = \{a_2^*, a_6^*, a_4^*\}.$$

In this case (Q'_G, W'_G) is the quiver with potential given as follows:

$$G1 \underset{Ga_2^*}{\overset{Ga_1^*}{\rightleftarrows}} G2 \circlearrowright G[a_6a_1]$$

$$W'_G = 3G[a_6a_1]G(a_1^*)G(a_6^*) + G([a_6a_1])^3.$$

Here the Jacobian algebras $\mathrm{Jac}(Q_G, W_G)$ and $\mathrm{Jac}(Q'_G, W'_G)$ are representation-finite, selfinjective algebras, and by the main theorem in [3], they are derived equivalent because their derived equivalence types are the same. By Keller-Yang's result [27], the complete Ginzburg dg algebras $\widehat{\Gamma}(Q, W)$ and $\widehat{\Gamma}(Q', W')$ are derived equivalent as dg algebras. By using Corollary 10.8, we will show that $\widehat{\Gamma}(Q, W)/G$ and $\widehat{\Gamma}(Q', W')/G$ are derived equivalent as dg algebras. Therefore the complete Ginzburg dg algebras $\widehat{\Gamma}(Q_G, W_G)$ and $\widehat{\Gamma}(Q'_G, W'_G)$ are derived equivalent as dg algebras by Remark 11.1. We set $\Gamma^{(1)} := \widehat{\Gamma}(\mu_1(Q, W)), \Gamma^{(2)} := \widehat{\Gamma}(\mu_3 \circ \mu_1(Q, W)), \Gamma' := \widehat{\Gamma}(\mu_5 \circ \mu_3 \circ \mu_1(Q, W)) = \widehat{\Gamma}(Q', W')$. Then Keller–Yang's theorem (Theorem 11.6) gives us the following derived equivalences F_3, F_2, F_1 defined as $(\text{-}) \overset{\mathbf{L}}{\otimes}_{\Gamma'} T^{(3)}$, $(\text{-}) \overset{\mathbf{L}}{\otimes}_{\Gamma^{(2)}} T^{(2)}$, $(\text{-}) \overset{\mathbf{L}}{\otimes}_{\Gamma^{(1)}} T^{(1)}$ using the dg bimodules $T^{(3)}, T^{(2)}, T^{(1)}$ constructed as in Proposition 11.7, respectively. These functors send objects as follows:

$$\begin{array}{ccccccc}
\mathscr{D}(\mathrm{dgMod}\,\Gamma') & \xrightarrow{F_3} & \mathscr{D}(\mathrm{dgMod}\,\Gamma^{(2)}) & \xrightarrow{F_2} & \mathscr{D}(\mathrm{dgMod}\,\Gamma^{(1)}) & \xrightarrow{F_1} & \mathscr{D}(\mathrm{dgMod}\,\Gamma) \\
P'_5 & \mapsto & (P_5^{(2)} \to P_6^{(2)}) & \mapsto & (P_5^{(1)} \to P_6^{(1)}) & \mapsto & (P_5 \to P_6) =: T(5) \\
P'_3 & \mapsto & P_3^{(2)} & \mapsto & (P_3^{(1)} \to P_4^{(1)}) & \mapsto & (P_3 \to P_4) =: T(3) \\
P'_1 & \mapsto & P_1^{(2)} & \mapsto & P_1^{(1)} & \mapsto & (P_1 \to P_2) =: T(1) \\
P'_i & \mapsto & P_i^{(2)} & \mapsto & P_i^{(1)} & \mapsto & P_i =: T(i), (i = 2, 4, 6)
\end{array}$$

where $P'_i = e_i\Gamma', P_i^{(2)} = e_i\Gamma^{(2)}, P_i^{(1)} = e_i\Gamma^{(1)}$ for all $i \in Q_0$. Then $F := F_1 \circ F_2 \circ F_3 = (\text{-}) \overset{\mathbf{L}}{\otimes}_{\Gamma'} T^{(3)} \overset{\mathbf{L}}{\otimes}_{\Gamma^{(2)}} T^{(2)} \overset{\mathbf{L}}{\otimes}_{\Gamma^{(1)}} T^{(1)}$ is an equivalence from $\mathscr{D}(\mathrm{dgMod}\,\Gamma')$ to $\mathscr{D}(\mathrm{dgMod}\,\Gamma)$. Here $T^{(3)} \overset{\mathbf{L}}{\otimes}_{\Gamma^{(2)}} T^{(2)} \overset{\mathbf{L}}{\otimes}_{\Gamma^{(1)}} T^{(1)}$ is a dg Γ'-Γ-bimodule and is isomorphic to the direct sum T of the indecomposable objects $T(i), (i = 1, \ldots, 6)$ as a dg right Γ-module, by which we identify these and regard T as a dg Γ'-Γ-bimodule. Let $\mathscr{T}$ be the full subcategory of $\operatorname{per}(\Gamma)$ consisting of $T(1), T(2), \cdots, T(6)$. We show that $\mathscr{T}$ is a desired tilting subcategory for Γ.

Now since g acts on P_i by ${}^gP_i = P_{i-2}, (i = 1, \dots, 6)$ by the G-action in (1) above, we have ${}^gT(i) = T(i-2), (i = 1, \dots, 6)$. On the other hand by the G-action in (2), g acts on P'_i by ${}^gP'_i = P'_{i-2}, (i = 1, \dots, 6)$.

We construct a 1-morphism $(F', \phi)\colon \Gamma' \to \mathscr{T}$ that is a G-quasi-equivalence. To this end we have to construct a quasi-equivalence $F'\colon \Gamma' \to \mathscr{T}$ and a 2-quasi-isomorphism $\phi(a)\colon \mathscr{T}(a) \circ F' \Rightarrow F' \circ a$ in $\Bbbk$-**dgCat** for each $a \in G$ (see Definition 9.6):

$$\begin{array}{ccccc} \Gamma' & \xrightarrow{F'} & \mathscr{T} & \hookrightarrow & \operatorname{per}(\Gamma) \\ {\scriptstyle a}\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle \phi(a)} & \downarrow{\scriptstyle \mathscr{T}(a) = {}^a(\text{-})} & & \downarrow{\scriptstyle {}^a(\text{-})} \\ \Gamma' & \xrightarrow[F']{} & \mathscr{T} & \hookrightarrow & \operatorname{per}(\Gamma) \end{array}$$

(It is trivial that the right square is strictly commutative). We now define F' as follows: First recall the Yoneda embedding $Y\colon \Gamma' \to \operatorname{dgMod}\Gamma'$ is defined by $Y(i) := \Gamma'(\text{-}, i) = e_i\Gamma'$ for all $i \in \Gamma'_0$, and $Y(\mu) := \Gamma'(\text{-}, \mu)$ for all $\mu \in \Gamma'_1$. Let $\alpha_M\colon \Gamma' \overset{\mathbf{L}}{\otimes}_{\Gamma'} M \to M$ be the usual natural isomorphisim for all Γ'-Γ-bimodule M. This yields the isomorphism $e_i\alpha_M\colon e_i\Gamma' \overset{\mathbf{L}}{\otimes}_{\Gamma'} M \to e_iM$ for each $i \in \Gamma'_0$ that is natural in i and in M. Note that the naturality in i means that for each $f\colon i \to j$ in Γ', we have a commutative diagram

$$\begin{array}{ccc} e_i\Gamma' \overset{\mathbf{L}}{\otimes}_{\Gamma'} M & \xrightarrow{e_i\alpha_M} & e_iM \\ {\scriptstyle \Gamma'(\text{-},f)\overset{\mathbf{L}}{\otimes}_{\Gamma'}M}\downarrow & & \downarrow{\scriptstyle M(\text{-},f)} \\ e_j\Gamma' \overset{\mathbf{L}}{\otimes}_{\Gamma'} M & \xrightarrow[e_j\alpha_M]{} & e_jM. \end{array}$$

We then define $F' := F \circ Y\colon \Gamma' \to \operatorname{per}(\Gamma) \subseteq \mathscr{D}(\operatorname{dgMod}\Gamma)$, thus $F'(i) = e_i\Gamma' \overset{\mathbf{L}}{\otimes}_{\Gamma'} T \xrightarrow{e_i\alpha_T} T(i)$ for all $i \in Q_0$, and $F'(\mu) = \Gamma'(\text{-}, \mu) \overset{\mathbf{L}}{\otimes}_{\Gamma'} T \cong \lambda_\mu\colon T(i) \to T(j)$ for all $\mu \in \Gamma'_1(i, j)$ with $i, j \in \Gamma'_0$. Thus we have a commutative diagram

$$\begin{array}{ccc} F'(i) & \xrightarrow{e_i\alpha_T} & T(i) \\ {\scriptstyle F'(\mu)}\downarrow & & \downarrow{\scriptstyle \lambda_\mu} \\ F'(j) & \xrightarrow[e_j\alpha_T]{} & T(j). \end{array}$$

Next we define a 2-quasi-isomorphism $\phi(a)\colon {}^aF' \Rightarrow F'a$ for each $a \in G$. Let $i \in \Gamma'_0$, and $a \in G$. Then the isomorphism $e_i\alpha_T\colon F'(i) \to T(i)$ yields isomomrphisms ${}^a(F'(i)) \xrightarrow{{}^a(e_i\alpha_T)} {}^aT(i) = T(ai)$, and $F'(ai) \xrightarrow{e_{ai}\alpha_T} T(ai)$. Thus we have an isomorphism

$$\phi_i(a) := (e_{ai}\alpha_T)^{-1} \circ {}^a(e_i\alpha_T)\colon {}^a(F'(i)) \to F'(ai).$$

We then define $\phi(a) := (\phi_i(a))_{i \in \Gamma'_0}\colon {}^aF' \Rightarrow F'a$ for all $a \in G$ and $\phi := (\phi(a))_{a \in G}$.

Claim 1. *The pair* (F', ϕ) *is a* 1*-morphism* $\Gamma' \to \mathscr{T}$ (see Definition 2.4).

Indeed, because $F'(i)$ is clearly a dg-functor, it suffices to show that $\phi(a)$ is a 2-morphism in $\Bbbk$-**dgCat** for each $a \in G$. Namely, we have to show the commutativity of the diagram

$$\begin{array}{ccc}
{}^{a}F'(u) & \xrightarrow{\phi_u(a)} & F'(au) \\
{\scriptstyle {}^{a}F'(\mu)}\downarrow & & \downarrow{\scriptstyle F'(a\mu)} \\
{}^{a}F'(v) & \xrightarrow[\phi_v(a)]{} & F'(av)
\end{array}$$

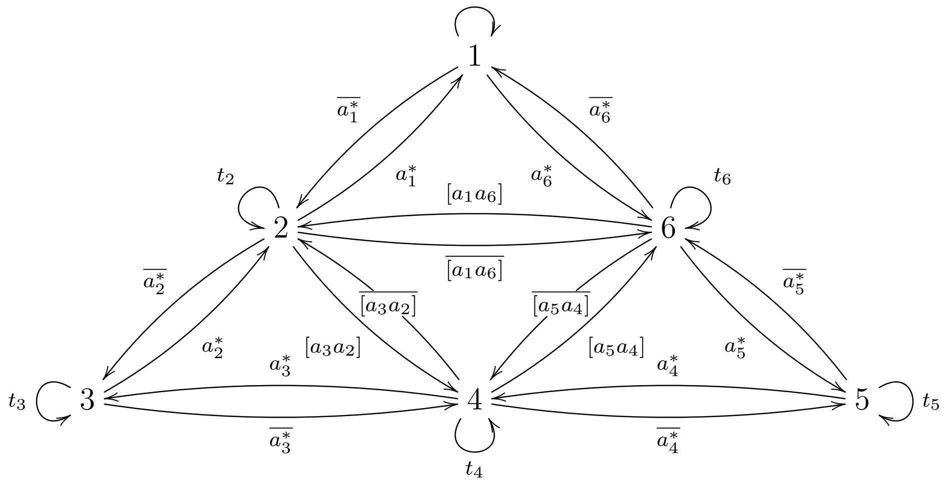
for all $\mu\colon u \to v$ in Γ'_1 and $a \in G$. It suffices to show the commutativity of this only for $a = g$ and for all $\mu \in \widetilde{Q'}_1$. Therefore finally we have only to show the commutativity of the diagram

$$\begin{array}{ccccccc}
{}^{g}F'(u) & \xrightarrow{{}^{g}(e_u\alpha_T)} & {}^{g}T(u) & = & T(gu) & \xleftarrow{e_{a_u}\alpha_T} & F'(gu) \\
{\scriptstyle {}^{g}F'(\mu)}\downarrow & & \downarrow{\scriptstyle {}^{g}T(\mu)} & & \downarrow{\scriptstyle T(g\mu)} & & \downarrow{\scriptstyle F'(g\mu)} \\
{}^{g}F'(v) & \xrightarrow[e_{a_v}\alpha_T]{} & {}^{g}T(v) & = & T(gv) & \xleftarrow[e_{a_v}\alpha_T]{} & F'(gv)
\end{array} \tag{11.15}$$

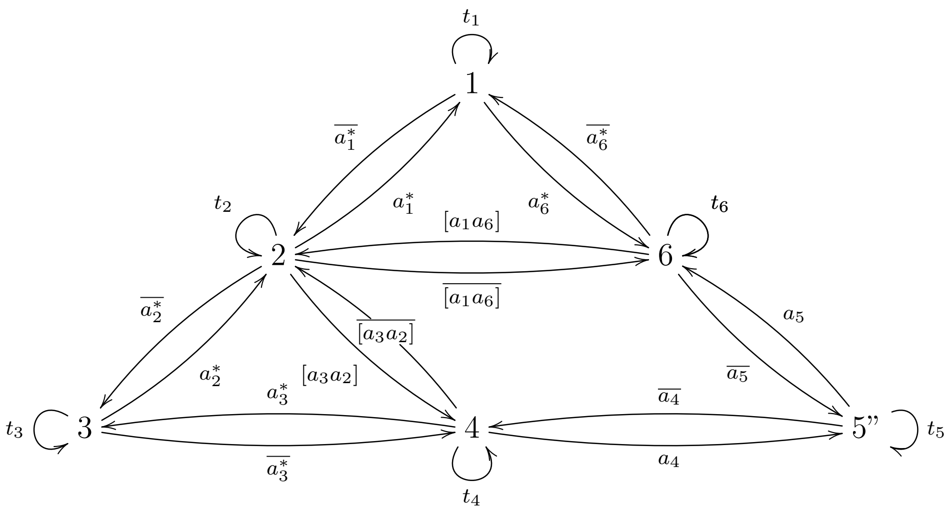
for all $\mu \in \widetilde{Q'}_1$. We check this only for three cases below. The remaining cases are checked similarly, and is left to the reader.

Now the quivers of $\Gamma', \Gamma^{(2)}, \Gamma^{(1)}, \Gamma$ are given as follows:

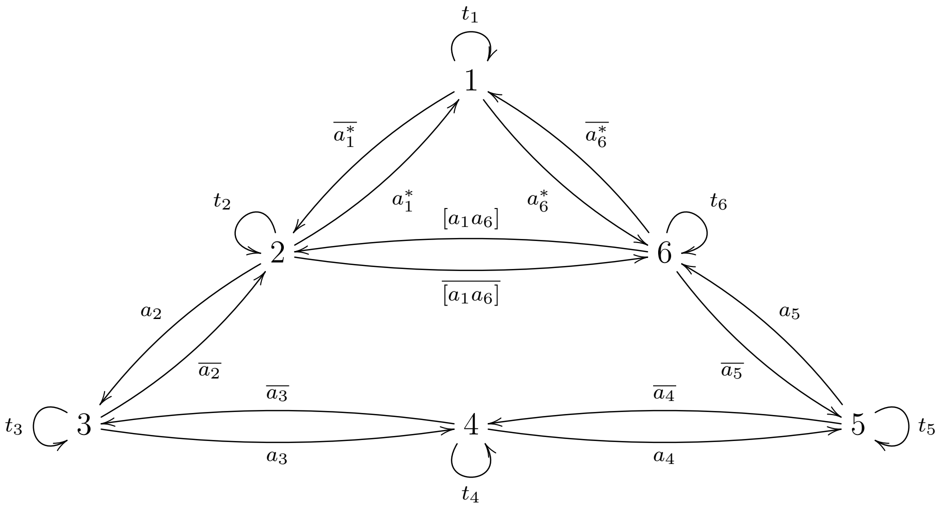
$\Gamma' =$



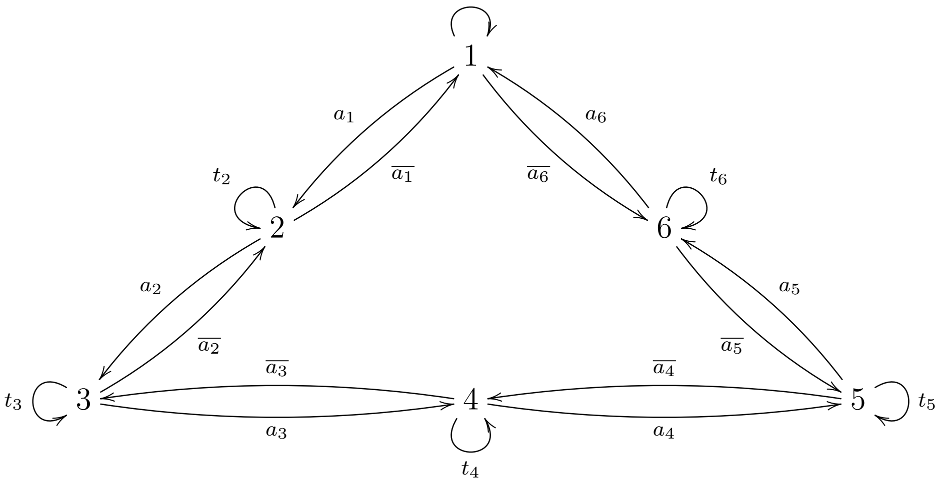
$\Gamma^{(2)} =$



$\Gamma^{(1)} =$



$\Gamma =$



Case 1. $\mu = a_i^* \in \widetilde{Q'}$ for some $i = 1, \dots, 6$, say $i = 1$. Then up to Yoneda embeddings (for the first three correspondences) we have $a_1^* \overset{F_3}{\mapsto} a_1^* \overset{F_2}{\mapsto} a_1^* \overset{F_1}{\mapsto} f_{a_1^*} \overset{g(\text{-})}{\mapsto} f_{a_5^*}$. Since we have commutative diagrams

$$\begin{CD} F'(2) @>{e_2\alpha_T}>> T(2) \\ @V{F'(a_1^*)}VV @VV{f_{a_1^*}}V \\ F'(1) @>>{e_1\alpha_T}> T(1) \end{CD} \quad \text{and} \quad \begin{CD} T(6) @<{e_6\alpha_T}<< F'(6) \\ @V{f_{a_5^*}}VV @VV{F'(a_5^*)}V \\ T(5) @<<{e_5\alpha_T}< F'(5) \end{CD} \; ,$$

we have a commutative diagram:

$$\begin{CD} {}^gF'(2) @>{{}^g(e_2\alpha_T)}>> {}^gT(2) @= T(6) @<{e_{g2}\alpha_T}<< F'(g2) \\ @V{{}^gF'(a_1^*)}VV @VV{{}^gf_{a_1^*}}V @VV{f_{ga_1^*}}V @VV{F'(ga_1^*)}V \\ {}^gF'(1) @>>{{}^g(e_1\alpha_T)}> {}^gT(1) @= T(5) @<<{e_{g1}\alpha_T}< F'(g1), \end{CD}$$

and hence (11.15) is verified in this case.

Case 2. $\mu = \overline{a_i^*} \in \widetilde{Q'}$ for some $i = 1, \dots, 6$, say $i = 1$. Then up to Yoneda embeddings (for the first three correspondences) we have $\overline{a_1^*} \overset{F_3}{\mapsto} \overline{a_1^*} \overset{F_2}{\mapsto} \overline{a_1^*} \overset{F_1}{\mapsto} f_{\overline{a_1^*}} \overset{g(\text{-})}{\mapsto} f_{\overline{a_5^*}}$. Since we have commutative diagrams

$$\begin{CD} F'(1) @>{e_1\alpha_T}>> T(1) \\ @V{F'(\overline{a_1^*})}VV @VV{f_{\overline{a_1^*}}}V \\ F'(2) @>>{e_2\alpha_T}> T(2) \end{CD} \quad \text{and} \quad \begin{CD} T(5) @<{e_5\alpha_T}<< F'(5) \\ @V{f_{\overline{a_5^*}}}VV @VV{F'(\overline{a_5^*})}V \\ T(6) @<<{e_6\alpha_T}< F'(6) \end{CD} \; ,$$

we have a commutative diagram:

$$\begin{CD} {}^gF'(1) @>{{}^g(e_1\alpha_T)}>> {}^gT(1) @= T(5) @<{e_{g1}\alpha_T}<< F'(g1) \\ @V{{}^gF'(\overline{a_1^*})}VV @VV{{}^gf_{\overline{a_1^*}}}V @VV{f_{g\overline{a_1^*}}}V @VV{F'(g\overline{a_1^*})}V \\ {}^gF'(2) @>>{{}^g(e_1\alpha_T)}> {}^gT(2) @= T(6) @<<{e_{g2}\alpha_T}< F'(g2), \end{CD}$$

and hence (11.15) is verified in this case.

Case 3. $\mu = t_i \in \widetilde{Q'}$ for some $i = 1, \dots, 6$, say $i = 1$. Then up to Yoneda embeddings (for the first three correspondences) we have $t_1 \overset{F_3}{\mapsto} t_1 \overset{F_2}{\mapsto} t_1 \overset{F_1}{\mapsto} f_{t'_1} \overset{g(\text{-})}{\mapsto} f_{t'_5}$. Therefore we have ${}^gF'(t_1) = f_{t'_5} = F'(t_5) = F'(gt_1)$. Hence ${}^g(F'(t_1)) = F'(gt_1)$.

Since we have commutative diagrams

$$\begin{CD} F'(1) @>{e_1\alpha_T}>> T(1) \\ @V{F'(t_1)}VV @VV{f_{t'_1}}V \\ F'(1) @>>{e_1\alpha_T}> T(1) \end{CD} \quad \text{and} \quad \begin{CD} T(5) @<{e_5\alpha_T}<< F'(5) \\ @V{f_{t'_5}}VV @VV{F'(t_5)}V \\ T(5) @<<{e_5\alpha_T}< F'(5) \end{CD} \; ,$$

we have a commutative diagram:

$$\begin{array}{ccccccc} {}^gF'(1) & \xrightarrow{{}^g(e_1\alpha_T)} & {}^gT(1) & = & T(5) & \xleftarrow{e_{g1}\alpha_T} & F'(g1) \\ {\scriptstyle {}^gF'(t_1)}\downarrow & & \downarrow{\scriptstyle {}^gf_{t'_1}} & & \downarrow{\scriptstyle f_{gt'_1}} & & \downarrow{\scriptstyle F'(gt_1)} \\ {}^gF'(1) & \xrightarrow[e_{a_1}\alpha_T]{} & {}^gT(1) & = & T(5) & \xleftarrow[e_{g1}\alpha_T]{} & F'(g1), \end{array}$$

and hence (11.15) is verified in this case. We check the conditions (a) and (b) in Definition 2.4.

Verifications of (a): This is equivalent to the equation that $\phi(1) = \mathbb{1}_{F'}$, which follows from the construction of ϕ and the fact that both Γ' and $\mathscr{T}$ have strict G-actions.

Verification of (b): This condition is equivalent to saying that the following diagram is commutative:

$$\begin{array}{ccc} {}^b({}^a(F'(i))) & \xrightarrow{{}^b(\phi_i(a))} & {}^b((F'(ai))) \\ & {\scriptstyle \phi_i(ba)}\searrow & \downarrow{\scriptstyle \phi_{(ai)}(b)} \\ & & F'(bai) \end{array} \tag{11.16}$$

for all $a, b \in G$ and $i \in \Gamma'_0$. By definition of $\phi_i(a)$, the following diagram is commutative:

$$\begin{array}{ccc} {}^a(F'(i)) & \xrightarrow{{}^a(e_i\alpha_T)} & {}^aT(i) \\ {\scriptstyle \phi_i(a)}\downarrow & & \| \\ F'(ai) & \xrightarrow[e_{ai}\alpha_T]{} & T(ai). \end{array}$$

This yields the following commutative diagram:

$$\begin{array}{ccccc} {}^b({}^a(F'(i))) & \xrightarrow[{}^b({}^a(e_i\alpha_T))]{} & {}^b({}^a(T(i))) & \xleftarrow[{}^{ba}(e_i\alpha_T)]{} & {}^{ba}(F'(i)) \\ {\scriptstyle {}^b(\phi_i(a))}\downarrow & & \| & & \\ {}^b(F'(ai)) & \xrightarrow[{}^b(e_{ai}\alpha_T)]{} & {}^b(T(ai)) & & \downarrow{\scriptstyle \phi_i(ba)} \\ {\scriptstyle \phi_{ai}(b)}\downarrow & & \| & & \\ F'(bai) & \xrightarrow{e_{b(ai)}\alpha_T} & T(b(ai)) & \xleftarrow{e_{bai}\alpha_T} & F'(bai), \end{array}$$

which shows the commutativity of the diagram (11.16).

It remains to show that (F', ϕ) is a quasi-equivalence. Namely we have to show the following claims:

Claim 2. *F' is an isomorphism, and hence a quasi-equivalence.*

Indeed, we regard Γ' as a dg category following Remark 11.4. For each $i \in Q_0$, we have $F'(i) = T(i)$. Hence F' is bijective on objects. Moreover, for each $i, j \in Q_0$, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{D}(\mathrm{dgMod}\,\Gamma')(\Gamma'(\text{-}, i), \Gamma'(\text{-}, j)) & \xrightarrow{F} & \mathscr{D}(\mathrm{dgMod}\,\Gamma)(F(i), F(j)) \\ {\scriptstyle Y}\uparrow & & \| \\ \Gamma'(i, j) & \xrightarrow[F']{} & \mathscr{T}(F'(i), F'(j)), \end{array}$$

where Y and F above are bijective. Hence F' above is bijective.

Claim 3. *$\phi(a)$ is a 2-quasi-isomorphism for all $a \in G$, i.e., $\mathscr{T}(\text{-}, \phi_i(a)) \colon \mathscr{T}(\text{-}, {}^aF'(i)) \to \mathscr{T}(\text{-}, F'(ai))$ is a quasi-isomorphism in $\mathrm{dgMod}\,\mathscr{T}$ for all $a \in G$ and $i \in \Gamma'_0$.*

Indeed, by construction $\phi_i(a) \colon {}^aF'(i) \to F'(ai)$ is an isomorphism in $\mathscr{T}$. Therefore $\mathscr{T}(\text{-}, \phi_i(a))$ is an isomorpism in $\mathrm{dgMod}\,\mathscr{T}$, and thus it is a quasi-isomorphism.

As a consequence, $\widehat{\Gamma}(Q_G, W_G)$ and $\widehat{\Gamma}(Q'_G, W'_G)$ are derived equivalent. Note that the quivers with potentials (Q_G, W_G) and (Q'_G, W'_G) are not mutated from each other in this case. Therefore we cannot apply [27, Theorem 3.2] by Keller-Yang to have this derived equivalence.

To give an example of the case that the category I is not a group, we need to give how to compute the Grothendieck construction of a functor $X \colon I \to \Bbbk$-**dgCat** at least. This will be done in the forthcoming paper, which will include such an example.

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