WEIGHTED LOCAL HARDY SPACES WITH VARIABLE EXPONENTS

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Abstract. This paper defines local weighted Hardy spaces with variable exponent. Local Hardy spaces permit atomic decomposition, which is one of the main themes in this paper. A consequence is that the atomic decomposition is obtained for the functions in the Lebesgue spaces with exponentially decaying exponent. As an application, we obtain the boundedness of singular integral operators, the Littlewood–Paley characterization and wavelet decomposition.

Key words: variable exponent, Hardy space, local Muckenhoupt weight, atomic decomposition, wavelet, modular inequality
AMS Subject Classification: 42B35, 42C40.

1. Introduction

Motivated by Bui [3] and Tang [55], we define weighted Hardy spaces with variable exponents and obtain some decomposition results for functions in $L^{p(\cdot)}(w)$ as an application. A variable exponent means a positive measurable function $p(\cdot):\mathbb{R}^n\to(0,\infty)$. Here and below the space $L^0(\mathbb{R}^n)$ denotes the linear space of all Lebesgue measurable functions in $\mathbb{R}^n$ and $\mathbb{N}_0\equiv\{0,1,\ldots\}$.

We begin with the definition of weighted Lebesgue spaces with variable exponents. Let $w:\mathbb{R}^n\to[0,\infty)$ be a weight. That is, w is a locally integrable function that satisfies $0<w(x)<\infty$ for almost all $x\in\mathbb{R}^n$. As in [6, 11, 28, 35], for a variable exponent $p(\cdot):\mathbb{R}^n\to(0,\infty)$, the weighted Lebesgue space $L^{p(\cdot)}(w)$ with a variable exponent is defined by

$$L^{p(\cdot)}(w)\equiv\bigcup_{\lambda>0}\{f\in L^0(\mathbb{R}^n)\,:\,\rho^w_{p(\cdot)}(\lambda^{-1}f)<\infty\},$$

where

$$\rho^w_{p(\cdot)}(f)\equiv\|\,|f|^{p(\cdot)}w\,\|_{L^1}.$$

Moreover, for $f\in L^{p(\cdot)}(w)$ the variable Lebesgue quasi-norm $\|\cdot\|_{L^{p(\cdot)}(w)}$ is defined by

$$\|f\|_{L^{p(\cdot)}(w)}\equiv\inf\left(\left\{\lambda>0\,:\,\rho^w_{p(\cdot)}(\lambda^{-1}f)\le1\right\}\cup\{\infty\}\right).$$

If $w\equiv1$, we write $L^{p(\cdot)}(1)=L^{p(\cdot)}(\mathbb{R}^n)$ and $\|\cdot\|_{L^{p(\cdot)}(1)}=\|\cdot\|_{L^{p(\cdot)}}$.

We write $w(E)\equiv\displaystyle\int_E w(x)\mathrm{d}x$ and $m_E(w)=\dfrac{w(E)}{|E|}$ for a weight w and a measurable set E. We postulate on w and $p(\cdot)$ the following conditions: As for the weight w, we assume that $w\in A^{\rm loc}_\infty$. The weight w is an $A^{\rm loc}_\infty$-*weight*, if $0<w<\infty$ almost everywhere, and $[w]_{A^{\rm loc}_\infty}\equiv\displaystyle\sup_{Q\in\mathcal{Q},|Q|\le1}m_Q(w)\exp(-m_Q(\log w))<\infty$, where in the sequel $\mathcal{Q}$ stands for the set of all compact cubes whose edges are parallel to the coordinate axes. The quantity $[w]_{A^{\rm loc}_\infty}$ is referred to as the $A^{\rm loc}_\infty$-*constant*. For the variable exponent $p(\cdot)$, consider two classes: The class $\mathcal{P}_0$ consists of all $p(\cdot):\mathbb{R}^n\to(0,\infty)$ such that

$$0<p_-\equiv\operatorname{essinf}_{x\in\mathbb{R}^n}p(x)\le p_+\equiv\operatorname{esssup}_{x\in\mathbb{R}^n}p(x)<\infty, \tag{1.1}$$

while the subclass $\mathcal{P}$ of $\mathcal{P}_0$ collects all exponents $p(\cdot)$ satisfying $p_- > 1$. We consider the local log-Hölder continuity condition and the log-Hölder-type decay condition at infinity. Recall that $p(\cdot)$ satisfies the local log-Hölder continuity condition (denoted by $p(\cdot) \in \mathrm{LH}_0$) if

$$|p(x)-p(y)| \le \frac{c_*}{\log(|x-y|^{-1})} \quad \text{for} \quad |x-y| \le \frac{1}{2},\ x,y \in \mathbb{R}^n, \tag{1.2}$$

while the exponent $p(\cdot)$ satisfies the log-Hölder-type decay condition at infinity (denoted by $p(\cdot) \in \mathrm{LH}_\infty$) if

$$|p(x)-p_\infty| \le \frac{c^*}{\log(e+|x|)} \quad \text{for} \quad x \in \mathbb{R}^n. \tag{1.3}$$

Here c_*, c^* and p_∞ are positive constants independent of x and y.

Based on the paper by Tang [55], who followed the idea of Feffereman and Stein [16], we define local weighted Hardy spaces with variable exponents using grand maximal functions. To this end, we recall the definition of grand maximal functions by Tang. Let $L \in \mathbb{N}_0$. The set $\mathcal{P}_L(\mathbb{R}^n)^\perp$ denotes the set of all $f \in L^0(\mathbb{R}^n)$ for which $\langle\cdot\rangle^L f \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} x^\alpha f(x)\mathrm{d}x = 0$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le L$. By convention, we define $\mathcal{P}_{-1}(\mathbb{R}^n)^\perp = L^1(\mathbb{R}^n)$. Such a function f satisfies the *moment condition of order* L. In this case, we write $f \perp \mathcal{P}_L(\mathbb{R}^n)$. If $f \perp \mathcal{P}_L(\mathbb{R}^n)$ for all $L \in \mathbb{N}_0$, we write $f \perp \mathcal{P}(\mathbb{R}^n)$.

Let $N \in \mathbb{N}_0$, which will be specified shortly. Denote by $B(r)$ the open ball centered at the origin of radius $r > 0$. The set $\mathcal{D}(\mathbb{R}^n)$ consists of all infinitely differentiable functions defined on $\mathbb{R}^n$ whose support is compact. Following [55, p.457], we write

$$\begin{aligned}
\mathcal{D}_N^0(\mathbb{R}^n) &\equiv \{\varphi \in \mathcal{D}(\mathbb{R}^n) \setminus \mathcal{P}_0(\mathbb{R}^n)^\perp \,:\, |\partial^\alpha \varphi| \le \chi_{B(1)}, |\alpha| \le N+1\},\\
\mathcal{D}_N(\mathbb{R}^n) &\equiv \{\varphi \in \mathcal{D}(\mathbb{R}^n) \setminus \mathcal{P}_0(\mathbb{R}^n)^\perp \,:\, |\partial^\alpha \varphi| \le \chi_{B(2^{3n+30})}, |\alpha| \le N+1\}.
\end{aligned}$$

Let $f \in \mathcal{D}'(\mathbb{R}^n)$ and N be large enough. For $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $t > 0$, write $\varphi_t \equiv t^{-n}\varphi(t^{-1}\cdot)$. Define three local grand maximal operators by

$$\begin{aligned}
\mathcal{M}_N^0 f(x) &\equiv \sup\{|\varphi_t * f(x)| \,:\, 0 < t < 1, \varphi \in \mathcal{D}_N^0(\mathbb{R}^n)\},\\
\overline{\mathcal{M}}_N^0 f(x) &\equiv \sup\{|\varphi_t * f(x)| \,:\, 0 < t < 1, \varphi \in \mathcal{D}_N(\mathbb{R}^n)\},\\
\mathcal{M}_N f(x) &\equiv \sup\{|\varphi_t * f(z)| \,:\, |z-x| < t < 1, \varphi \in \mathcal{D}_N(\mathbb{R}^n)\}.
\end{aligned}$$

It is obvious that $\mathcal{M}_N^0 f \le \overline{\mathcal{M}}_N^0 f \le \mathcal{M}_N f$ for any $N \in \mathbb{N}_0$.

We recall the notion of the class $A_p^{\rm loc}$ of weights. Let $1 < p < \infty$. A weight w belongs to $A_p^{\rm loc}$ if

$$[w]_{A_p^{\rm loc}} \equiv \sup_{Q \in \mathcal{Q}, |Q| \le 1} m_Q(w) m_Q(w^{-\frac{1}{p-1}})^{p-1} < \infty.$$

The quantity $[w]_{A_p^{\rm loc}}$ is referred to as the $A_p^{\rm loc}$-*constant*. Using the technique employed by Rychkov [47], we obtain $A_\infty^{\rm loc} = \bigcup_{q>1} A_q^{\rm loc}$. We set

$$q_w \equiv \inf\{p \in [1,\infty) \,:\, w \in A_p^{\rm loc}\}$$

for $w \in A_\infty^{\rm loc}$. Similar to Tang [55, p. 458], we set

$$N_{p(\cdot),w} \equiv 2 + \left[n\left(\frac{q_w}{\min(1,p_-)} - 1\right)\right].$$

Herein we assume

$$N \ge N_{p(\cdot),w}. \tag{1.4}$$

Here and below we use the following convention on the notation $\lesssim$ and $\gtrsim$. Let $A, B \ge 0$. Then $A \lesssim B$ and $B \gtrsim A$ mean that there exists a constant $C > 0$ such that $A \le CB$, where C depends only on the parameters of importance. The symbol $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$ happen simultaneously, while $A \simeq B$ means that there exists a constant $C > 0$ such that $A = CB$.

The following theorem is the starting point of this paper.

Theorem 1.1. *Let $w \in A_\infty^{\rm loc}$. Also let $p(\cdot) \in \mathcal{P}_0 \cap {\rm LH}_0 \cap {\rm LH}_\infty$ and $N \in \mathbb{N}$ satisfy* (1.4)*. Then*

$$\|\mathcal{M}_N^0 f\|_{L^{p(\cdot)}(w)} \le \|\overline{\mathcal{M}}_N^0 f\|_{L^{p(\cdot)}(w)} \le \|\mathcal{M}_N f\|_{L^{p(\cdot)}(w)} \lesssim \|\mathcal{M}_N^0 f\|_{L^{p(\cdot)}(w)}$$

for all $f \in \mathcal{D}'(\mathbb{R}^n)$.

Note that Rychkov [47] proved Theorem 1.1 when $p(\cdot)$ is a constant exponent. Based on Theorem 1.1, we define weighted Hardy spaces with variable exponents.

Definition 1.2. Let $w \in A_\infty^{\rm loc}$. Also let $p(\cdot) \in \mathcal{P}_0 \cap {\rm LH}_0 \cap {\rm LH}_\infty$ and $N \in \mathbb{N}$ satisfy (1.4). Then the weighted local Hardy spaces $h^{p(\cdot)}(w) = h^{p(\cdot),N}(w)$ with variable exponents is the set of all $f \in \mathcal{D}'(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{h^{p(\cdot)}(w)} \equiv \|\mathcal{M}_N f\|_{L^{p(\cdot)}(w)}$$

is finite.

Note that the norm $\|\cdot\|_{h^{p(\cdot)}}$ is independent of N in the sense that different choices of N satisfying (1.4) yield equivalent norms.

The main purpose of this note is to investigate equivalent norms of $h^{p(\cdot)}(w)$. Among others, we are interested in the characterization by means of atoms and their related norms.

Definition 1.3. Let $w \in A_\infty^{\rm loc}$, $q \in (0,\infty]$ and $L \in \mathbb{N}_0 \cup \{-1\}$. Also let $p(\cdot) \in \mathcal{P}_0 \cap {\rm LH}_0 \cap {\rm LH}_\infty$ and $N \in \mathbb{N}$ satisfy (1.4).

(1) If $q > \max(p_+, q_w)$ and $L \ge \left[n\left(\frac{q_w}{\min(1,p_-)} - 1\right)\right]$, then a triplet $(p(\cdot), q, L)$ is called admissible.
(2) Let Q be a cube with $|Q| < 1$. A function $a \in L^q(w)$ is a $(p(\cdot), q, L)_w$-atom supported on Q if $a \in \mathcal{P}_L(\mathbb{R}^n)^\perp$, a is supported on Q and satisfies $\|a\|_{L^q(w)} \le w(Q)^{\frac{1}{q}}$.
(3) Let Q be a cube with $|Q| = 1$. A function $a \in L^q(w)$ is a $(p(\cdot), q, L)_w$-atom supported on Q if a is supported on Q and satisfies $\|a\|_{L^q(w)} \le w(Q)^{\frac{1}{q}}$.
(4) Assume $w(\mathbb{R}^n) < \infty$, or equivalently, $1 \in L^1(w)$. Then we say that a function a is a single $(p(\cdot), q)_w$-atom if $\|a\|_{L^q(w)} \le w(\mathbb{R}^n)^{\frac{1}{q}}$.

If $w = 1$, then subscript w is omitted in these notions.

For example, χ_Q is a $(p(\cdot), \infty, -1)_w$-atom for any cube Q.

Unlike [55], we assume that the volume of the cubes for $(p(\cdot), q, L)_w$-atoms is less than or equal to 1.

We remark that a single $(p(\cdot), q)_w$-atom does not have to belong to $\mathcal{P}_L(\mathbb{R}^n)^\perp$. For example, 1 is a single $(p(\cdot), q)_w$-atom.

Definition 1.4. Let $w \in A_\infty^{\rm loc}$, $p(\cdot) \in \mathcal{P}_0 \cap {\rm LH}_0 \cap {\rm LH}_\infty$, $v \in (0, p_-) \cap [0,1]$ and $q \in (0,\infty]$. Let $L \in \mathbb{Z}$ satisfy $L \ge \left[n\left(\frac{q_w}{\min(1,p_-)} - 1\right)\right]$. Then the weighted atomic local Hardy space

$h^{p(\cdot),q,L;v}(w)$ is defined as the set of all $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfying that $f = \sum\limits_{j=0}^{\infty} \lambda_j a_j$, where a_0 is a single $(p(\cdot),q)_w$-atom, each $a_j, j \in \mathbb{N}$ is a $(p(\cdot),q,L)_w$-atom supported on a cube Q_j, $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$ and

$$\mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^{\infty};\{Q_j\}_{j=1}^{\infty}) \equiv \left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^v \chi_{Q_j} \right)^{\frac{1}{v}} \right\|_{L^{p(\cdot)}(w)} < \infty.$$

The norm $\|f\|_{h^{p(\cdot),q,L;v}(w)}$ is defined as the infimum of $|\lambda_0| + \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^{\infty};\{Q_j\}_{j=1}^{\infty})$ over all expressions of f above.

We will show that $h^{p(\cdot)}(w)$ can be characterized via atoms by proving that, with an equivalece of norms, the weighted atomic local Hardy space $h^{p(\cdot),q,L;v}(w)$ coincides with $h^{p(\cdot)}(w)$ as long as $q > \max(q_w, p_+)$, $v \in (0, p_-) \cap [0,1]$ and $L \geq \left[n\left(\frac{q_w}{v} - 1\right)\right]$.

Theorem 1.5. *Let $p(\cdot) \in \mathcal{P}_0 \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and $N \in \mathbb{N}$ satisfy* (1.4). *Let $v \in (0,p_-) \cap [0,1]$. Also let $w \in A_\infty^{\mathrm{loc}}$ and $L \in \mathbb{Z}$ satisfy*

$$N \geq L \geq \left[n\left(\frac{q_w}{v} - 1\right)\right]. \tag{1.5}$$

Suppose that a parameter q satisfies

$$\max(q_w, p_+) < q \leq \infty. \tag{1.6}$$

Then $h^{p(\cdot),q,L;v}(w) \cong h^{p(\cdot)}(w)$ with an equivalence of norms.

Let $p(\cdot)$ be a variable exponent with $1 < p_- \leq p_+ < \infty$. A locally integrable weight w is an $A_{p(\cdot)}^{\mathrm{loc}}$*-weight*, if $0 < w < \infty$ almost everywhere and

$$[w]_{A_{p(\cdot)}^{\mathrm{loc}}} \equiv \sup_{Q \in \mathcal{Q}, |Q| \leq 1} \frac{1}{|Q|} \|\chi_Q\|_{L^{p(\cdot)}(w)} \|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} < \infty, \tag{1.7}$$

where $\sigma \equiv w^{-\frac{1}{p(\cdot)-1}}$ and $p'(\cdot)$ is the dual exponent given by $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$. If $w \in A_{p(\cdot)}^{\mathrm{loc}}$, then weighted local Hardy spaces with a variable exponent and weighted Lebesgue spaces with variable exponent coincide as given in the following theorem.

Theorem 1.6. *Let $w \in A_{p(\cdot)}^{\mathrm{loc}}$. Also let $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and $N \in \mathbb{N}$ satisfy* (1.4). *Then $h^{p(\cdot)}(w) = L^{p(\cdot)}(w)$ with an equivalence of norms.*

It may be interesting to compare Theorem 1.6 with [24].

Note that Theorem 1.6 is proved by Rychkov [47] when $p(\cdot)$ is a constant exponent.

Thanks to Theorems 1.5 and 1.6, the following decomposition results on $L^{p(\cdot)}(w)$ is given. Here, the moment condition is not necessary.

Theorem 1.7. *Let $f \in L^0(\mathbb{R}^n)$, $w \in A_{p(\cdot)}^{\mathrm{loc}}$ and and $L \in \mathbb{N}_0 \cup \{-1\}$. Also let $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Assume that the parameters q and q_0 satisfy $q, q_0 > p_+$ and $\sigma = w^{-\frac{1}{p(\cdot)-1}} \in A_{p'(\cdot)/q_0'}$.*

(1) *Suppose $w \in L^1(\mathbb{R}^n)$. Then the following are equivalent:*
 (I) *$f \in L^{p(\cdot)}(w)$.*
 (II) *There exist a single $(p(\cdot),q)_w$-atom a_0 and a collection $\{a_j\}_{j=1}^{\infty} \subset L^0(\mathbb{R}^n)$, where each $a_j, j \in \mathbb{N}$ is a $(p(\cdot),q,L)_w$-atom supported on a cube Q_j with $|Q_j| \leq 1$, and*

a collection $\{\lambda_j\}_{j=0}^{\infty}$ *of complex constants such that* $f = \sum_{j=0}^{\infty} \lambda_j a_j$ *in* $L^{p(\cdot)}(w)$ *and that*

$$|\lambda_0| + \mathcal{A}_{p(\cdot),w,1}(\{\lambda_j\}_{j=1}^{\infty}; \{Q_j\}_{j=1}^{\infty}) = |\lambda_0| + \left\| \sum_{j=1}^{\infty} |\lambda_j| \chi_{Q_j} \right\|_{L^{p(\cdot)}(w)} < \infty.$$

(III) *There exist a single* $(p(\cdot), q_0)$*-atom* a_0 *and a collection* $\{a_j\}_{j=1}^{\infty} \subset L^0(\mathbb{R}^n)$*, where each* $a_j, j \in \mathbb{N}$ *is a* $(p(\cdot), q_0, L)$*-atom supported on a cube* Q_j *with* $|Q_j| \le 1$*, and a collection* $\{\lambda_j\}_{j=0}^{\infty}$ *of complex constants such that* $f = \sum_{j=0}^{\infty} \lambda_j a_j$ *in* $L^{p(\cdot)}(w)$ *and that*

$$|\lambda_0| + \mathcal{A}_{p(\cdot),w,1}(\{\lambda_j\}_{j=1}^{\infty}; \{Q_j\}_{j=1}^{\infty}) = |\lambda_0| + \left\| \sum_{j=1}^{\infty} |\lambda_j| \chi_{Q_j} \right\|_{L^{p(\cdot)}(w)} < \infty.$$

In this case,

$$\|f\|_{L^{p(\cdot)}(w)} \sim \inf \left\{ |\lambda_0| + \mathcal{A}_{p(\cdot),w,1}(\{\lambda_j\}_{j=1}^{\infty}; \{Q_j\}_{j=1}^{\infty}) \right\},$$

where each a_j *and* λ_j *move over all elements in* $L^0(\mathbb{R}^n)$ *and* $\mathbb{C}$ *to satisfy the conditions of* (II) *or* (III).

(2) *Suppose* $w \notin L^1(\mathbb{R}^n)$*. Then the following are equivalent*:

(I) $f \in L^{p(\cdot)}(w)$.

(II) *There exist a collection* $\{a_j\}_{j=1}^{\infty} \subset L^0(\mathbb{R}^n)$*, where each* $a_j, j \in \mathbb{N}$ *is a* $(p(\cdot), q, L)_w$*-atom supported on a cube* Q_j *with* $|Q_j| \le 1$*, and a collection* $\{\lambda_j\}_{j=1}^{\infty}$ *of complex constants such that* $f = \sum_{j=1}^{\infty} \lambda_j a_j$ *in* $L^{p(\cdot)}(w)$ *and that*

$$\mathcal{A}_{p(\cdot),w,1}(\{\lambda_j\}_{j=1}^{\infty}; \{Q_j\}_{j=1}^{\infty}) = \left\| \sum_{j=1}^{\infty} |\lambda_j| \chi_{Q_j} \right\|_{L^{p(\cdot)}(w)} < \infty.$$

(III) *There exist a collection* $\{a_j\}_{j=1}^{\infty} \subset L^0(\mathbb{R}^n)$*, where each* $a_j, j \in \mathbb{N}$ *is a* $(p(\cdot), q_0, L)$*-atom supported on a cube* Q_j *with* $|Q_j| \le 1$*, and a collection* $\{\lambda_j\}_{j=1}^{\infty}$ *of complex constants such that* $f = \sum_{j=1}^{\infty} \lambda_j a_j$ *in* $L^{p(\cdot)}(w)$ *and that*

$$\mathcal{A}_{p(\cdot),w,1}(\{\lambda_j\}_{j=1}^{\infty}; \{Q_j\}_{j=1}^{\infty}) = \left\| \sum_{j=1}^{\infty} |\lambda_j| \chi_{Q_j} \right\|_{L^{p(\cdot)}(w)} < \infty.$$

In this case,

$$\|f\|_{L^{p(\cdot)}(w)} \sim \inf \mathcal{A}_{p(\cdot),w,1}(\{\lambda_j\}_{j=1}^{\infty}; \{Q_j\}_{j=1}^{\infty}),$$

where each a_j *and* λ_j *move over all elements in* $L^0(\mathbb{R}^n)$ *and* $\mathbb{C}$ *to satisfy the conditions of* (II) *or* (III).

We verify that q satisfying all the requirements in Theorem 1.7 exists (see Lemma 2.8).

The implication (I) $\Longrightarrow$ (II)/(III) is included in Theorems 1.5 and 1.6. We prove the implication (II)/(III) $\Longrightarrow$ (I).

Note that Theorem 1.7 overlaps with the previous studies on weighted Lebesgue and Hardy spaces ($w \in A_p, p(\cdot)$ is constant) [53, Chapter IIIV], Lebesgue spaces with variable exponents ($w = 1$), [43, 48], Musielak–Orlicz spaces with general Young functions ($\Phi(x,t) = t^{p(x)} w(x)$, $w \in A_\infty$) [29, Theorem 3.7] and mixed Lebesgue spaces [46, Theorem 3].

Here, we briefly address the development of Hardy spaces with variable exponents along with some related results. We also compare these works with the results obtained in this paper. Meyer [42] established several equivalent wavelet characterization of $H^1(\mathbb{R}^n)$. Liu [41] developed an equivalent wavelet characterization of the weak Hardy space $H^{1,\infty}(\mathbb{R}^n)$. Wu [59, Theorem 3.2] reported the wavelet characterization of the weighted Hardy space $H^p(\mathbb{R}^n)$ for any $p \in (0,1]$. Later García-Cuerva and Martell [20] characterized $H^p(\mathbb{R}^n)$ for any $p \in (0,1]$ in terms of wavelets without compact supports using the vector-valued Calderón–Zygmund theory. See [36] for the wavelet characterization of dual spaces. Our results are new in the sense that we do not assume $p_+ \le 1$. For example, in [60], D. Yang and S. Yang assumed that φ is of uniformly upper type 1, which corresponds to the assumption $p_+ \le 1$. Note that the normed space of sequences used in this paper differs from the ones in [55, 60]. In fact, as we mentioned, in [55, 60] the authors assumed a condition corresponding to $p_+ \le 1$. In this case, arguing similarly to [29, Theorem 3.12], we learn that the sequence norm $\mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty)$ can be replaced by

$$\mathcal{A}^\dagger_{p(\cdot),w}(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty) \equiv \inf \left\{ \lambda > 0 \,:\, \sum_{j=1}^\infty \int_{Q_j} \left(\frac{|\lambda_j|}{\lambda} \chi_{Q_j}(x) \right)^{p(x)} w(x) \mathrm{d}x \le 1 \right\}.$$

Several approaches define Hardy spaces based on general Banach lattices or characterize them in terms of wavelets. However, most require that the underlying Banach lattices be rearrangement invariant or the Hardy–Littlewood maximal operator be bounded there. For example, see [49, 50, 51, 54, 58] for example. We remark that our results do not fall under the scope of [58] since F. Wang, D. Yang and S. Yang assumed the boundedness property of the Hardy–Littlewood maximal operator. Nevertheless, the present paper is based on the idea of [55, 58]. Since the variable exponents and weights distort the function spaces strongly, we can not hope for such a situation. Therefore, an alternative approach using the local Hardy–Littlewood maximal operator, given by (2.1), is necessary.

The rest of this paper is organized as follows. Section 2 collects preliminary facts. Section 3 discusses the fundamental properties of function spaces. We prove Theorems 1.1 and 1.5 in Sections 4 and 5, respectively. Section 6 is oriented to the applications of Theorems 1.1 and 1.5. We mainly discuss the boundedness property of singular integral operators. Section 6 has some commonality with [25]. Section 7 is devoted to the Littlewood–Paley characterization of $h^{p(\cdot)}(w)$, which is a further application of the results in Section 6. Further examples and the relations to other function spaces are provided in Section 9. Section 8 considers wavelet characterization. In Section 10, we compare the definition of weighted Lebesgue spaces with variable exponents. There are many attempts to extend the classical Muckenhoupt class to the setting of variable exponents inspired by the works [8, 9, 13]. For example, see [5, 6, 10, 12, 14]. Here we consider the local counterpart of the work [14] and compare it with the results in [45]. We remark that [14] is a preprint. So, we gave details for the facts related [14]. However, our results related to [14] are essentially minor modifications of [14].

2. Preliminaries

Many tools are necessary to establish our results. First, we recall the notion of generalized dyadic grids in Section 2.1. Section 2.2 collects norm inequalities. We establish some boundedness properties of the weighted maximal operator in Section 2.3. Section 2.4 refines the openness property obtained by Hytönen and Pérez [27]. Section 2.5 is oriented to the boundedness of operators including their vector-valued boundedness. To establish the theory of the atomic decomposition, we depend on the boundedness properties of some operators, which are adapted to our class of weights. Thus, we will carefully collect the results on the boundedness of operators. To develop the atomic decomposition theory, we consider the power of the normed spaces. This is necessary since $p_- \le 1$. In Section 2.6 we transform our results obtained in the previous sections to consider the case of $w \in A_\infty^{\rm loc}$. Finally, keeping in mind that our characterization of $h^{p(\cdot)}(w)$ includes the one by the Littlewood–Paley operators, in Section 2.8 we recall some important inequalities obtained by Rychkov [47].

2.1. **Generalized dyadic grids.** Let

$$\mathcal{D}^0_{k,a} \equiv \{2^{-k}[m+a/3, m+a/3+1) \,:\, m \in \mathbb{Z}\}$$

for $k \in \mathbb{Z}$ and $a = 0, 1, 2$. Consider

$$\mathcal{D}_{k,\mathbf{a}} \equiv \{Q_1 \times Q_2 \times \cdots \times Q_n \,:\, Q_j \in \mathcal{D}^0_{k,a_j}, j = 1, 2, \ldots, n\}$$

for $k \in \mathbb{Z}$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \{0,1,2\}^n$. Herein, a (generalized) dyadic grid is in the family $\mathcal{D}_{\mathbf{a}} \equiv \bigcup_{k \in \mathbb{Z}} \mathcal{D}_{k,\mathbf{a}}$ for $\mathbf{a} \in \{0,1,2\}^n$. It is noteworthy that for any cube Q there exists $R \in \bigcup_{\mathbf{a} \in \{0,1,2\}^n} \mathcal{D}_{\mathbf{a}}$ such that $Q \subset R$ and that $|R| \le 6^n |Q|$. Proving the boundedness property of the Hardy–Littlewood maximal operator or the local maximal operator $M^{\rm loc}$ defined by (2.1) below, this property allows us to handle the operator $M^{\mathcal{D}_{\mathbf{a}}}$ generated by $\mathcal{D}_{\mathbf{a}}$ instead of these maximal operators. See (2.3) and (2.4), below. Recall that we can handle $\mathcal{D}_{\mathbf{a}}$ similarly for other values of $\mathbf{a} \in \{0,1,2\}^n$ so that in particular, we consider the dyadic grid $\mathfrak{D} = \mathcal{D}_{(1,1,\ldots,1)}$. Denote by $\mathfrak{D}_k = \mathcal{D}_{k,(1,1,\ldots,1)}$ the set of all cubes in $\mathfrak{D}$ with $\ell(Q) = 2^{-k}$. Here, given a cube Q, it is denoted by $\ell(Q)$, which is the sidelength of Q: $\ell(Q) \equiv |Q|^{1/n}$, where $|Q|$ denotes the volume of cube Q. Two cubes Q_1, Q_2 in $\mathfrak{D}$ may intersect at a point but that the difference set $Q_1 \ominus Q_2 = (Q_1 \setminus Q_2) \cup (Q_2 \setminus Q_1)$ is not empty.

For $f \in L^0(\mathbb{R}^n)$, we define the local (Hardy–Littlewood) maximal operator $M^{\rm loc}$ by

$$M^{\rm loc} f(x) \equiv \sup_{Q \in \mathcal{Q}, |Q| \le 1} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| {\rm d}y \quad (x \in \mathbb{R}^n). \tag{2.1}$$

Note that this definition is analogous to the Hardy–Littlewood maximal operator M defined by

$$Mf(x) \equiv \sup_{Q \in \mathcal{Q}} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| {\rm d}y \quad (x \in \mathbb{R}^n). \tag{2.2}$$

Let $M^{\mathcal{D}_{\mathbf{a}}}$, $\mathbf{a} \in \{0,1,2\}^n$, be the maximal operator generated by grid $\mathcal{D}_{\mathbf{a}}$ given by

$$M^{\mathcal{D}_{\mathbf{a}}} f(x) = \sup_{Q \in \mathcal{D}_{\mathbf{a}}} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| {\rm d}y \quad (x \in \mathbb{R}^n). \tag{2.3}$$

Using the above property of the grid $\mathcal{D}_{\mathbf{a}}$, $\mathbf{a} \in \{0,1,2\}^n$, we prove

$$Mf(x) \le 6^n \sum_{\mathbf{a} \in \{0,1,2\}^n} M^{\mathcal{D}_{\mathbf{a}}} f(x) \tag{2.4}$$

for $f \in L^0(\mathbb{R}^n)$. Once we prove the boundedness property of $M^{\mathcal{D}_{\mathbf{a}}}$, $\mathbf{a} \in \{0,1,2\}^n$ on $L^{p(\cdot)}(w)$, (2.4) yields the one of M. In [45, §4], we also establish that the boundedness property of $M^{\mathcal{D}_{\mathbf{a}}}$, $\mathbf{a} \in \{0,1,2\}^n$ on $L^{p(\cdot)}(w)$ yields the one of $M^{\rm loc}$.

2.2. **Weighted variable Lebesgue spaces.** For any measurable subset $\Omega \subset \mathbb{R}^n$, denote

$$p_+(\Omega) \equiv {\rm esssup}_{x \in \Omega} p(x), \quad p_-(\Omega) \equiv {\rm essinf}_{x \in \Omega} p(x).$$

Let $p(\cdot)$ satisfy $1 \le p(\cdot) \le \infty$. If $p(\cdot) \in {\rm LH}_0$ then $p'(\cdot) \in {\rm LH}_0$. Likewise, if $p(\cdot) \in {\rm LH}_\infty$ then $p'(\cdot) \in {\rm LH}_\infty$. Furthermore, $(p_\infty)' = (p')_\infty$.

Recall the generalized Hölder inequality.

Lemma 2.1 (Generalized Hölder inequality)**.** *Let $p(\cdot):\mathbb{R}^n\to[1,\infty]$ be a variable exponent. Then for all $f\in L^{p(\cdot)}(\mathbb{R}^n)$ and $g\in L^{p'(\cdot)}(\mathbb{R}^n)$,*

$$\|f\cdot g\|_{L^1}\le r_p\|f\|_{L^{p(\cdot)}}\|g\|_{L^{p'(\cdot)}}, \tag{2.5}$$

where

$$r_p\equiv 1+\frac{1}{p_-}-\frac{1}{p_+}=\frac{1}{p_-}+\frac{1}{(p')_-}\le 2. \tag{2.6}$$

Now let us recall some properties for the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$. The first one concerns the norm growth.

Lemma 2.2. [14, Lemma 2.1], [43, Lemma 2.2] *Let $p(\cdot)\in\mathcal{P}\cap\mathrm{LH}_0\cap\mathrm{LH}_\infty$.*

1. *For all cubes Q with $|Q|\le 1$, we have $|Q|^{1/p_-(Q)}\lesssim|Q|^{1/p_+(Q)}$. In particular, we have $|Q|^{1/p_-(Q)}\sim|Q|^{1/p_+(Q)}\sim|Q|^{1/p(z)}\sim\|\chi_Q\|_{L^{p(\cdot)}}$.*
2. *For all cubes Q with $|Q|\ge 1$, we have $\|\chi_Q\|_{L^{p(\cdot)}}\sim|Q|^{1/p_\infty}$.*

Next, consider the modular inequality.

Lemma 2.3. [10, Lemma 2.2], [44, Lemma 2.17] *Let $p(\cdot):\mathbb{R}^n\to[1,\infty)$ be a variable exponent such that $p_+<\infty$. Then given any measurable set Ω and any $f\in L^0(\mathbb{R}^n)$, we have the following:*

1. *If $\|\chi_\Omega f\|_{L^{p(\cdot)}}\le 1$, then $\|\chi_\Omega f\|_{L^{p(\cdot)}}^{p_+(\Omega)}\le\int_\Omega|f(x)|^{p(x)}\mathrm{d}x\le\|\chi_\Omega f\|_{L^{p(\cdot)}}^{p_-(\Omega)}$.*
2. *If $\|\chi_\Omega f\|_{L^{p(\cdot)}}\ge 1$, then $\|\chi_\Omega f\|_{L^{p(\cdot)}}^{p_-(\Omega)}\le\int_\Omega|f(x)|^{p(x)}\mathrm{d}x\le\|\chi_\Omega f\|_{L^{p(\cdot)}}^{p_+(\Omega)}$.*

For a variable exponent $p(\cdot):\mathbb{R}^n\to[1,\infty)$ and $f\in L^0(\mathbb{R}^n)$, $\|f\|_{L^{p(\cdot)}(w)}\le 1$ if and only if $\int_{\mathbb{R}^n}|f(x)|^{p(x)}w(x)\mathrm{d}x\le 1$.

We apply Lemma 2.3 to compare $w(Q)$ and $\|\chi_Q\|_{L^{p(\cdot)}(w)}$.

Remark 2.4. Let Q be a cube. In Lemma 2.3, let $f=w^{\frac{1}{p(\cdot)}}\chi_Q$ to obtain the following equivalence:

$$\|\chi_Q\|_{L^{p(\cdot)}(w)}\le 1\Longleftrightarrow w(Q)\le 1. \tag{2.7}$$

A direct consequence of (2.7) is the following:

(1) If $\|\chi_Q\|_{L^{p(\cdot)}(w)}\le 1$, then

$$\|\chi_Q\|_{L^{p(\cdot)}(w)}^{p_+(Q)}\le w(Q)\le\|\chi_Q\|_{L^{p(\cdot)}(w)}^{p_-(Q)}. \tag{2.8}$$

(2) If $\|\chi_Q\|_{L^{p(\cdot)}(w)}\ge 1$, then

$$\|\chi_Q\|_{L^{p(\cdot)}(w)}^{p_-(Q)}\le w(Q)\le\|\chi_Q\|_{L^{p(\cdot)}(w)}^{p_+(Q)}. \tag{2.9}$$

Finally, recall the localization principle due to Hästo. We state it in a form we use in the present paper.

Lemma 2.5. [23] *Let $p(\cdot)\in\mathcal{P}\cap\mathrm{LH}_0\cap\mathrm{LH}_\infty$ and $k_0\in\mathbb{Z}$. Then*

$$\|f\|_{L^{p(\cdot)}}\sim\left(\sum_{Q\in\mathfrak{D}_{k_0}}(\|\chi_Q f\|_{L^{p(\cdot)}})^{p_\infty}\right)^{\frac{1}{p_\infty}}$$

for all $f\in L^0(\mathbb{R}^n)$.

2.3. **Maximal inequalities.** For $f \in L^0(\mathbb{R}^n)$, we recall the local (Hardy–Littlewood) maximal operator $M^{\rm loc}$ by

$$M^{\rm loc}f(x) \equiv \sup_{Q\in\mathcal{Q},|Q|\le 1} \frac{\chi_Q(x)}{|Q|}\int_Q |f(y)|{\rm d}y \quad (x \in \mathbb{R}^n).$$

We can replace $M^{\rm loc}$ by

$$M^{{\rm loc},R}f(x) \equiv \sup_{Q\in\mathcal{Q},|Q|\le R^n} \frac{\chi_Q(x)}{|Q|}\int_Q |f(y)|{\rm d}y \quad (x \in \mathbb{R}^n).$$

Here $R > 0$. Although $M^{\rm loc}$ is typically defined by (2.1), sometimes replace 1 by R as above. This substitution avoids the self-composition of $M^{\rm loc}$ defined by (2.1). We remark that the definition of the $A^{\rm loc}_{p(\cdot)}$-norm is essentially independent of the cube size restriction. That is, given an exponent $p(\cdot) : \mathbb{R}^n \to (1, \infty)$ with $p_- > 1$, a positive number $R > 0$ and a weight w, we say that $w \in A^{{\rm loc},R}_{p(\cdot)}$ if $[w]_{A^{{\rm loc},R}_{p(\cdot)}} \equiv \sup\limits_{|Q|\le R^n} |Q|^{-1}\|\chi_Q\|_{L^{p(\cdot)}(w)}\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} < \infty$, where $\sigma \equiv w^{-\frac{1}{p(\cdot)-1}}$ as before and the supremum is taken over all cubes $Q \in \mathcal{Q}$ with volumes less than or equal to R^n. Then, $w \in A^{{\rm loc},R}_{p(\cdot)}$ if and only if $w \in A^{\rm loc}_{p(\cdot)}$. For more detail, see [45, Section 3.3]. Furthermore, we have

$$\|M^{{\rm loc},R}f\|_{L^{p(\cdot)}(w)} \lesssim \|f\|_{L^{p(\cdot)}(w)} \tag{2.10}$$

for all $f \in L^{p(\cdot)}(w)$ whenever $R > 0$ and $w \in A^{{\rm loc},1}_{p(\cdot)} = A^{\rm loc}_{p(\cdot)}$. For this reason, we subsume the parameter $R > 0$ like this.

The next lemma is analogous of [10, Theorem 1.5]. In our earlier work [45], we established that the class $A^{\rm loc}_{p(\cdot)}$ is suitable for this maximal operator.

Lemma 2.6. [45, Theorem 1.2] *Let $p(\cdot) \in \mathcal{P} \cap {\rm LH}_0 \cap {\rm LH}_\infty$. Also let w be a weight. Then there exists a constant $D > 0$ such that*

$$\|M^{\rm loc}f\|_{L^{p(\cdot)}(w)} \le D\|f\|_{L^{p(\cdot)}(w)} \tag{2.11}$$

for all $f \in L^{p(\cdot)}(w)$ if and only if $w \in A^{\rm loc}_{p(\cdot)}$.

Define $A^{\rm loc}_1$ as the collection of all weights for which there exists a constant $C > 0$ such that $M^{\rm loc}w \le Cw$. The infimum of such C is called the $A^{\rm loc}_1$-constant and is denoted by $[w]_{A^{\rm loc}_1}$. Remark that a similar remark for $A^{\rm loc}_{p(\cdot)}$ for $R \ge 1$ applies to $A^{\rm loc}_1$. We use the following local reverse Hölder property:

Lemma 2.7. *Let $w \in A^{\rm loc}_1$. If we set*

$$\varepsilon \equiv \frac{1}{2^{n+11}[w]_{A^{\rm loc}_1}} > 0,$$

then $w^{1+\varepsilon} \in A^{\rm loc}_1$.

Proof. This is a local version of [27, Theorem 2.3]. We omit the further details. □

We collect some corollaries from (1.7) and Lemmas 2.6 and 2.7. As a byproduct, we learn that q satisfying all requirements in Theorem 1.7 exists. The next assertions are known for the global Muckenhoupt class. However, the corresponding assertion for local class is missing. So, we supply the proof.

Lemma 2.8. *Let w be a weight and let $p(\cdot) \in \mathcal{P} \cap {\rm LH}_0 \cap {\rm LH}_\infty$.*

(1) *The following are equivalent:*

- $w \in A^{\rm loc}_{p(\cdot)}$.
- $w^{-\frac{1}{p(\cdot)-1}} \in A^{\rm loc}_{p'(\cdot)}$.

(2) *Let $w \in A^{\rm loc}_{p(\cdot)}$. Then there exists $\varepsilon > 0$ such that $1+\varepsilon < p_-$ and that $w \in A^{\rm loc}_{p(\cdot)/(1+\varepsilon)}$.*

Proof.

(1) This is an immediate consequence of (1.7).
(2) Let $f \in L^{p(\cdot)}(w) \setminus \{0\}$. We employ the Rubio de Francia algorithm. Then define

$$F \equiv \sum_{k=0}^{\infty} \frac{(M^{\rm loc})^k f}{2^k D^k}$$

where $D > 0$ is the constant in (2.11), $(M^{\rm loc})^k$ denotes the k-fold composition if $k \geq 1$ and $(M^{\rm loc})^0 f \equiv |f|$. Then $F \leq M^{\rm loc}F \leq 2DF$. Hence, $F \in A_1^{\rm loc}$ and $[F]_{A_1^{\rm loc}} \leq 2D$. Due to Lemma 2.7, there exists a constant $\varepsilon \in (0, p_- - 1)$ which depends on D and $p(\cdot)$ such that $M^{\rm loc}(F^{1+\varepsilon})^{\frac{1}{1+\varepsilon}} \leq 2M^{\rm loc}F$. Thus,

$$\begin{aligned}\|M^{\rm loc}(|f|^{1+\varepsilon})^{\frac{1}{1+\varepsilon}}\|_{L^{p(\cdot)}(w)} &\leq \|M^{\rm loc}(F^{1+\varepsilon})^{\frac{1}{1+\varepsilon}}\|_{L^{p(\cdot)}(w)}\\ &\leq 2\|M^{\rm loc}F\|_{L^{p(\cdot)}(w)}\\ &\leq 4D\|F\|_{L^{p(\cdot)}(w)}\\ &\leq 8D\|f\|_{L^{p(\cdot)}(w)}.\end{aligned}$$

Since $f \in L^{p(\cdot)}(w) \setminus \{0\}$ is arbitrary, it follows from Lemma 2.6 that $M^{\rm loc}$ is bounded on $L^{p(\cdot)/(1+\varepsilon)}(w)$. Hence $w \in A^{\rm loc}_{p(\cdot)/(1+\varepsilon)}$.

□

We use the following monotone property of the class $A^{\rm loc}_{p(\cdot)}$. The proof is postponed until Appendix; see the remark after Corollary 10.3.

Proposition 2.9. *Let $p(\cdot), q(\cdot) \in \mathcal{P} \cap {\rm LH}_0 \cap {\rm LH}_\infty$. If $q(\cdot) \geq p(\cdot)$, then $A^{\rm loc}_{q(\cdot)} \supset A^{\rm loc}_{p(\cdot)}$.*

A clarifying remark may be in order.

Remark 2.10. Let $p(\cdot), q(\cdot) \in \mathcal{P} \cap {\rm LH}_0 \cap {\rm LH}_\infty$. Combining the result by Diening and Hästo [14] and the one by Cruz-Uribe, Fiorenza and Neugebauer [10], we learn that $A_{q(\cdot)} \supset A_{p(\cdot)}$ whenever $q(\cdot) \geq p(\cdot)$. In this paper, we will follow the idea of [14] to prove Proposition 2.9.

We move on to the vector-valued inequality, which is an extension of [7] to the setting of the $A^{\rm loc}_\infty$-class.

Lemma 2.11. [45, Theorem 1.11] *Let $p(\cdot) \in \mathcal{P} \cap {\rm LH}_0 \cap {\rm LH}_\infty$. Let $1 < q \leq \infty$ and $w \in A^{\rm loc}_{p(\cdot)}$. Then*

$$\left\| \left(\sum_{j=1}^{\infty} (M^{\rm loc} f_j)^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}(w)} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}(w)}$$

for all $\{f_j\}_{j=1}^\infty \subset L^0(\mathbb{R}^n)$.

Proposition 2.12. [30, Proposition 2.11] *Let $1<q_1,q_2<\infty$. Assume that $p(\cdot)\in\mathcal{P}\cap \mathrm{LH}_0\cap \mathrm{LH}_\infty$ and $w\in A^{\rm loc}_{p(\cdot)}$. Then*

$$\left\|\left(\sum_{j_2=1}^\infty\left(\sum_{j_1=1}^\infty (M^{\rm loc}f_{j_1,j_2})^{q_1}\right)^{\frac{q_2}{q_1}}\right)^{\frac{1}{q_2}}\right\|_{L^{p(\cdot)}(w)} \lesssim \left\|\left(\sum_{j_2=1}^\infty\left(\sum_{j_1=1}^\infty |f_{j_1,j_2}|^{q_1}\right)^{\frac{q_2}{q_1}}\right)^{\frac{1}{q_2}}\right\|_{L^{p(\cdot)}(w)} \tag{2.12}$$

for all $\{f_{j_1,j_2}\}_{j_1,j_2=1}^\infty\subset L^0(\mathbb{R}^n)$.

2.4. **Openness property–A variant of Lemma 2.7.** Let $R\in\mathfrak{D}$. We set

$$[w]^{\mathfrak{D}}_{A_{\infty,R^\times}}=\sup_{Q\in\mathfrak{D},|Q\setminus R|>0} m_{Q\setminus R}(w)\exp\left(-m_{Q\setminus R}(\log w)\right)$$

and define the class $A^{\mathfrak{D}}_{\infty,R^\times}$ is the set of all weights such that $[w]^{\mathfrak{D}}_{A_{\infty,R^\times}}<\infty$. Next, we define maximal operators $M^{\mathfrak{D}}_{R^\times}$ and $M^{0,\mathfrak{D}}_{R^\times}$ as follows:

$$M^{\mathfrak{D}}_{R^\times}f(x)\equiv \sup_{S\in\mathfrak{D},\,|S\setminus R|>0}\frac{\chi_{S\setminus R}(x)}{|S\setminus R|}\int_{S\setminus R}|f(y)|{\rm d}y \quad (x\in\mathbb{R}^n),$$

and

$$M^{0,\mathfrak{D}}_{R^\times}f(x)\equiv \sup_{S\in\mathfrak{D}}\chi_{S\setminus R}(x)\exp(m_{S\setminus R}(-\log|f|)) \quad (x\in\mathbb{R}^n).$$

Here roughly speaking, “0” stands for the maximal operator based on the $L^{0+}(\mathbb{R}^n)$-average. Following the idea in [27, Lemma 2.1], we investigate the maximal operators $M^{\mathfrak{D}}_{R^\times}$ and $M^{0,\mathfrak{D}}_{R^\times}$. Note that by the stndard argument for the weak type estimate, we have

$$|\{x\in\mathbb{R}^n\setminus R\,:\,M^{\mathfrak{D}}_{R^\times}f(x)>\lambda\}|\le\frac{1}{\lambda}\|\chi_{\{x\in\mathbb{R}^n\setminus R\,:\,M^{\mathfrak{D}}_{R^\times}f(x)>\lambda\}}f\|_{L^1(\mathbb{R}^n\setminus R)}. \tag{2.13}$$

By Jensen's inequality, the layer cake formula and (2.13),

$$\|M^{0,\mathfrak{D}}_{R^\times}f\|_{L^p}\le\|M^{\mathfrak{D}}_{R^\times}f\|_{L^p}\le\frac{p}{p-1}\|f\|_{L^p}\quad(1<p<\infty)$$

for all measurable functions f.

Since

$$M^{0,\mathfrak{D}}_{R^\times}[|f|^u]=(M^{0,\mathfrak{D}}_{R^\times}f)^u \tag{2.14}$$

for all $u>0$, we have

$$\|M^{0,\mathfrak{D}}_{R^\times}f\|_{L^1}\le\left(\frac{p}{p-1}\right)^p\|f\|_{L^1}$$

Letting $p\to\infty$ gives

$$\|M^{0,\mathfrak{D}}_{R^\times}f\|_{L^1}\le e\|f\|_{L^1} \tag{2.15}$$

for all measurable functions f.

Lemma 2.13. *Let $R\in\mathfrak{D}$, $w\in A^{\mathfrak{D}}_{\infty,R^\times}$, and*

$$q\equiv 1+\frac{1}{4^{n+6}[w]_{A^{\mathfrak{D}}_{\infty,R^\times}}}. \tag{2.16}$$

Then for all cubes $Q\in\mathfrak{D}$ satisfying $|Q\setminus R|>0$,

$$m_{Q\setminus R}(w^q)^{\frac{1}{q}}\le 2m_{Q\setminus R}(w). \tag{2.17}$$

Proof. Fix $Q \in \mathfrak{D}$. We can assume that w is bounded by approximating w with a function in the form

$$\sum_{S \in \mathcal{D}_j(Q)} m_{S\setminus R}(w)\chi_{S\setminus R},$$

where $\mathcal{D}_j(Q)$ denotes the set of all cubes obtained by bisecting Q j times. Let $\varepsilon \equiv q-1$ and $\mathcal{D}(Q) \equiv \bigcup_{j=0}^{\infty} \mathcal{D}_j(Q)$. When denoted by $\tilde{M}^{\mathcal{D}(Q)}$, the dyadic maximal operator given by

$$\tilde{M}^{\mathcal{D}(Q)} f(x) := \sup_{S\in\mathcal{D}(Q),\, \frac{1}{2n}\log_2\frac{|Q|}{|S|}\in\mathbb{Z}} \chi_S(x) m_S(|f|) = \sup_{S\in\mathcal{D}(Q)\cap\mathfrak{D}} \chi_S(x) m_S(|f|).$$

We use the layer cake formula to obtain

$$\int_{Q\setminus R} \tilde{M}^{\mathcal{D}(Q)}[\chi_{Q\setminus R} w](x)^{\varepsilon} w(x)\mathrm{d}x = \varepsilon \int_0^{\infty} \lambda^{\varepsilon-1} w((Q\setminus R)\cap\{\tilde{M}^{\mathcal{D}(Q)}[\chi_{Q\setminus R}w] > \lambda\})\mathrm{d}\lambda.$$

We suppose

$$\lambda > m_{Q\setminus R}(w).$$

Consider the set of all maximal dyadic cubes $\{U_j\}_{j\in J(\lambda)}$ in $\mathcal{D}(Q)\cap\mathfrak{D}$ in the set $(Q\setminus R)\cap \{\tilde{M}^{\mathcal{D}(Q)}[\chi_{Q\setminus R}w] > \lambda\}$ whose average of $\chi_{Q\setminus R}w$ exceeds λ. Then due to the maximality of each U_j, the grand parent $\tilde{\tilde{U}}_j$ satisfies $m_{\tilde{\tilde{U}}_j}(w) \le \lambda$. Thus,

$$\begin{aligned}
\frac{w((Q\setminus R)\cap\{\tilde{M}^{\mathcal{D}(Q)}[\chi_{Q\setminus R}w] > \lambda\})}{4^n\lambda} &= \sum_{j\in J(\lambda)} \frac{w(U_j)}{4^n\lambda} \\
&\lesssim \sum_{j\in J(\lambda)} |U_j| \\
&= |(Q\setminus R)\cap\{\tilde{M}^{\mathcal{D}(Q)}[\chi_{Q\setminus R}w] > \lambda\}|.
\end{aligned}$$

We also note that $(Q\setminus R)\cap\{\tilde{M}^{\mathcal{D}(Q)}[\chi_{Q\setminus R}w] > \lambda\} \subset (Q\setminus R)$ for any $\lambda > 0$. Thus,

$$\begin{aligned}
&\int_{Q\setminus R} \tilde{M}^{\mathcal{D}(Q)} w(x)^{\varepsilon} w(x)\mathrm{d}x \\
&\quad \le \varepsilon \int_0^{m_{Q\setminus R}(w)} \lambda^{\varepsilon-1} w((Q\setminus R)\cap\{\tilde{M}^{\mathcal{D}(Q)}[\chi_{Q\setminus R}w] > \lambda\})\mathrm{d}\lambda \\
&\qquad + 4^n\varepsilon \int_{m_{Q\setminus R}(w)}^{\infty} \lambda^{\varepsilon} |(Q\setminus R)\cap\{\tilde{M}^{\mathcal{D}(Q)}[\chi_{Q\setminus R}w] > \lambda\}|\mathrm{d}\lambda \\
&\quad \le |Q\setminus R| \left(m_{Q\setminus R}(w)\right)^{1+\varepsilon} + \frac{4^n\varepsilon}{1+\varepsilon} \int_{Q\setminus R} \tilde{M}^{\mathcal{D}(Q)}[\chi_{Q\setminus R}w](x)^{1+\varepsilon}\mathrm{d}x.
\end{aligned} \tag{2.18}$$

Since $w \in A_{\infty,R^\times}^{\mathfrak{D}}$, we have

$$m_S(w) \le \frac{2^n}{2^n-1} m_{S\setminus R}(\chi_{\mathbb{R}^n\setminus R} w) \le [w]_{A_{\infty,R^\times}^{\mathfrak{D}}} \exp(m_{S\setminus R}(-\log w)) \tag{2.19}$$

for all $S \in \mathcal{D}(Q)\cap\mathfrak{D}$ such that $|S\setminus R| > 0$.

A geometric observation shows

$$\tilde{M}^{\mathcal{D}(Q)}[\chi_{Q\setminus R}w](x) \le \frac{2^n}{2^n-1}[w]_{A_{\infty,R^\times}^{\mathfrak{D}}} M_{R^\times}^{0,\mathfrak{D}} w. \tag{2.20}$$

Inserting (2.20) into (2.18), we obtain

$$\begin{aligned}
&\int_{Q\setminus R} \tilde{M}^{\mathcal{D}(Q)} w(x)^{\varepsilon} w(x)\mathrm{d}x \\
&\le |Q\setminus R| \left(m_{Q\setminus R}(w)\right)^{1+\varepsilon} + \frac{4^n\varepsilon}{1+\varepsilon}\left(\frac{2^n}{2^n-1}[w]_{A^{\mathfrak{D}}_{\infty,R^\times}}\right)^{1+\varepsilon} \int_{Q\setminus R} M^{0,\mathfrak{D}}_{R^\times}[\chi_{Q\setminus R} w](x)^{1+\varepsilon}\mathrm{d}x.
\end{aligned}$$

By the Lebesgue differentiation theorem and (2.14), we have

$$\begin{aligned}
&\int_{Q\setminus R} w(x)^{1+\varepsilon}\mathrm{d}x \\
&\le |Q\setminus R| \left(m_{Q\setminus R}(w)\right)^{1+\varepsilon} + \frac{4^n\varepsilon}{1+\varepsilon}\left(\frac{2^n}{2^n-1}[w]_{A^{\mathfrak{D}}_{\infty,R^\times}}\right)^{1+\varepsilon} \int_{Q\setminus R} M^{0,\mathfrak{D}}_{R^\times}[\chi_{Q\setminus R} w^{1+\varepsilon}](x)\mathrm{d}x.
\end{aligned}$$

From (2.15)

$$m^{(1+\varepsilon)}_{Q\setminus R}(w)^{1+\varepsilon} \le (m_{Q\setminus R}(w))^{1+\varepsilon} + \frac{4^n\varepsilon e}{1+\varepsilon}\left(\frac{2^n}{2^n-1}[w]_{A^{\mathfrak{D}}_{\infty,R^\times}}\right)^{1+\varepsilon} m^{(1+\varepsilon)}_{Q\setminus R}(w)^{1+\varepsilon}. \tag{2.21}$$

Arithmetic shows

$$\left(\frac{2^n}{2^n-1}\right)^{\frac{1}{4^n}} = \left(1+\frac{1}{2^n-1}\right)^{\frac{1}{4^n}} \le \left(1+\frac{1}{2^n-1}\right)^{\frac{1}{2^n-1}} \le e.$$

Since $[w]_{A^{\mathfrak{D}}_{\infty,R^\times}} \ge 1$ and $1+\varepsilon = q$, we have

$$\begin{aligned}
\frac{4^n\varepsilon}{1+\varepsilon}\left(\frac{2^n}{2^n-1}[w]_{A^{\mathfrak{D}}_{\infty,R^\times}}\right)^{1+\varepsilon} &\le 2\left(\frac{2^n}{2^n-1}\right)^{\frac{1}{4^n}} \frac{4^n[w]_{A^{\mathfrak{D}}_{\infty,R^\times}}\left([w]_{A^{\mathfrak{D}}_{\infty,R^\times}}\right)^{\frac{1}{4^{n+6}[w]_{A^{\mathfrak{D}}_{\infty,R^\times}}}}}{4^{n+6}[w]_{A^{\mathfrak{D}}_{\infty,R^\times}}+1} \\
&\le \frac{1}{512}\exp\left(\frac{1}{4^{n+6}e}\right) \\
&\le \frac{1}{7}.
\end{aligned}$$

It follows that $m_{Q\setminus R}(w^{1+\varepsilon}) \le \left(m_{Q\setminus R}(w)\right)^{1+\varepsilon} + \frac{e}{7} m_{Q\setminus R}(w^{1+\varepsilon})$. Since w is assumed to be bounded, $2e \le 7$ and $1+\varepsilon = q$, if we absorb the second term of the right-hand side of (2.21) into the left-hand side, we obtain

$$m_{Q\setminus R}(w^q)^{\frac{1}{q}} \le 2 m_{Q\setminus R}(w),$$

which proves (2.17).

□

2.5. **The operator K_B.** For $B>0$, define a convolution operator K_B by

$$K_B f(x) \equiv \int_{\mathbb{R}^n} e^{-B|x-y|} f(y)\mathrm{d}y$$

for $f \in L^0(\mathbb{R}^n)$ as long as the definition makes sense. We invoke an estimate from [30, Corollary 2.6].

Lemma 2.14. *Let $p(\cdot) \in \mathcal{P}\cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Let D be the constant satisfying* (2.11). *If $B > 8n + 6\log D$ and $w \in A^{\mathrm{loc}}_{p(\cdot)}$, then K_B is bounded on $L^{p(\cdot)}(w)$.*

Recall that we used a pointwise estimate in the proof of Lemma 2.14. Thus, if we examine its proof, then we can obtain a vector-valued inequality from Lemmas 2.11 and 2.14. Actually, the following inequality can be proven using the local maximal operator, whose proof we omit.

Corollary 2.15. *Let $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Let $w \in A_{p(\cdot)}^{\mathrm{loc}}$ and $1 < q \le \infty$. Let D be the constant satisfying* (2.11)*. If $B > 8n + 6 \log D$, then*

$$\left\| \left(\sum_{j=1}^{\infty} |K_B f_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}(w)} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}(w)}$$

for all $\{f_j\}_{j=1}^{\infty} \subset L^{p(\cdot)}(w)$.

We transform Corollary 2.15 into a form for later considerations by writing

$$m_{j,A,B}(x) \equiv (1 + 2^j|x|)^A e^{|x|B} \quad (x \in \mathbb{R}^n) \tag{2.22}$$

for $j = 0, 1, \dots$ and $A, B > 0$.

Corollary 2.16. *Let $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and $w \in A_{p(\cdot)}^{\mathrm{loc}}$. Let D be the constant satisfying* (2.11)*. If $A > 0$, $B > 8n + 6 \log D$ and $1 < r < \infty$, then*

$$\left\| \left(\sum_{j=1}^{\infty} \left| 2^{jn} \int_{\mathbb{R}^n} \frac{f_j(\cdot - y)}{m_{j,A,B}(y)} \mathrm{d}y \right|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}(w)} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}(w)}$$

for all $\{f_j\}_{j=1}^{\infty} \subset L^{p(\cdot)}(w)$.

Proof. Simply observe that

$$\int_{\mathbb{R}^n} \frac{|f_j(x-y)|}{m_{j,A,B}(y)} \mathrm{d}y \lesssim K_B|f_j|(x) + M^{\mathrm{loc}} f_j(x)$$

for all $x \in \mathbb{R}^n$, $j = 1, 2, \dots$ and $A, B > 0$ (cf. [47, Lemma 2.10]). Thus, we are in the position to use Lemma 2.11 and Corollary 2.15. $\square$

2.6. **Powered local weighted maximal operator.** For $0 < u < \infty$ and a weight w, define the powered local weighted maximal operator $M_w^{(u),\mathrm{loc}}$ by

$$M_w^{(u),\mathrm{loc}} f(x) \equiv \sup_{Q \in \mathcal{Q}, |Q| \le 1} \left(\frac{\chi_Q(x)}{w(Q)} \int_Q |f(y)|^u w(y) \mathrm{d}y \right)^{\frac{1}{u}} \qquad (f \in L^0(\mathbb{R}^n)).$$

We write $M_w^{\mathrm{loc}} \equiv M_w^{(1),\mathrm{loc}}$.

We work in the Euclidean space with the weighted measure $w\mathrm{d}x$.

Proposition 2.17. *Let $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Let $0 < u < p_-$ and $w \in A_\infty^{\mathrm{loc}}$. Then*

$$\left\| M_w^{(u),\mathrm{loc}} f \right\|_{L^{p(\cdot)}(w)} \lesssim \|f\|_{L^{p(\cdot)}(w)}$$

for $f \in L^{p(\cdot)}(w)$.

Some lemmas and intricate arguments are needed to prove this lemma. First, we prove Proposition 2.17 if the exponent is constant. Since $0 < u < p_-$, we can assume $u = 1$ by a scaling argument. Since $w \in A_\infty^{\mathrm{loc}}$, w is a locally doubling weight. That is, w satisfies $w(5Q) \lesssim w(2Q) \lesssim w(Q)$ for all cubes Q with $|Q| \le 1$. In the case where $p(\cdot)$ is a constant we can use the theory of general Radon measures in [56, Section 3]. In fact, we can replace $M_w^{(u),\mathrm{loc}}$ by the maximal operator given by

$$\tilde{M}_w^{(u)} f(x) \equiv \sup_{Q \in \mathcal{Q}} \left(\frac{\chi_Q(x)}{w(5Q)} \int_Q |f(y)|^u w(y) \mathrm{d}y \right)^{\frac{1}{u}} \qquad (f \in L^0(\mathbb{R}^n)).$$

Thus, the proof of Proposition 2.17 is complete if $p(\cdot)$ is a constant exponent.

Now we consider the case where $p(\cdot)$ is a variable exponent. We define

$$[w]_{A^{\mathfrak{D}}_{p(\cdot)}} \equiv \sup_{Q\in\mathfrak{D}} \frac{1}{|Q|}\|\chi_Q\|_{L^{p(\cdot)}(w)}\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)},$$

where $\sigma \equiv w^{-\frac{1}{p(\cdot)-1}}$ is the dual weight. The class $A^{\mathfrak{D}}_{p(\cdot)}$ collects all weights w for which $[w]_{A^{\mathfrak{D}}_{p(\cdot)}} < \infty$. Hence, we have only to deal with $M^{\mathfrak{D}}_w$ instead of $M^{\rm loc}_w$ assuming that $w \in A^{\mathfrak{D}}_\infty$ instead of $w \in A^{\rm loc}_\infty$ by the use of a technique similar to that developed in [45]. Here, $M^{\mathfrak{D}}_w$ stands for

$$M^{\mathfrak{D}}_w f(x) \equiv \sup_{Q\in\mathfrak{D}} \frac{\chi_Q(x)}{w(Q)}\int_Q |f(y)|w(y){\rm d}y \qquad (f \in L^0(\mathbb{R}^n)).$$

First, we consider the case where f is unbounded to find a pointwise estimate of $M^{\mathfrak{D}}_w f$.

Lemma 2.18. *Let $p(\cdot) \in \mathcal{P}_0$, $w \in A^{\mathfrak{D}}_\infty$ and $Q \in \mathfrak{D}$. Let $f \in L^{p(\cdot)}(w)$ be a non-negative real-valued function with $\|f\|_{L^{p(\cdot)}(w)} \le 1$. Assume $f \le f^2$. Then*

$$\left(\frac{1}{w(Q)}\int_Q f(y)w(y){\rm d}y\right)^{p(x)} \lesssim \left(\frac{1}{w(Q)}\int_Q f(y)^{\frac{p(y)}{p_-}}w(y){\rm d}y\right)^{p_-}$$

for all $x \in Q$.

Proof. Fix $x \in Q$. If

$$k \equiv \frac{1}{w(Q)}\int_Q f(y)^{\frac{p(y)}{p_-}}w(y){\rm d}y \le 1,$$

then the desired estimate is clear since

$$\frac{1}{w(Q)}\int_Q f(y)w(y){\rm d}y \le \frac{1}{w(Q)}\int_Q f(y)^{\frac{p(y)}{p_-}}w(y){\rm d}y \le \left(\frac{1}{w(Q)}\int_Q f(y)^{\frac{p(y)}{p_-}}w(y){\rm d}y\right)^{\frac{p_-}{p(x)}}.$$

Otherwise, assume $k \ge 1$. Then f can be decomposed according to $\{y \in \mathbb{R}^n : f(y) \ge k^{\frac{p_-}{p(x)}}\}$ to give

$$\begin{aligned}\frac{1}{w(Q)}\int_Q f(y)w(y){\rm d}y &\le k^{\frac{p_-}{p(x)}} + \frac{1}{w(Q)}\int_Q f(y)\chi_{[k,\infty]}(f(y)^{\frac{p(x)}{p_-}})w(y){\rm d}y\\ &\le k^{\frac{p_-}{p(x)}} + \frac{1}{w(Q)}\int_Q f(y)^{\frac{p(y)}{p_-}}k^{\left(-\frac{p(y)}{p_-}+1\right)\frac{p_-}{p(x)}}w(y){\rm d}y. \end{aligned}\tag{2.23}$$

Since $w \in A^{\mathfrak{D}}_\infty$ and

$$1 \le k = \frac{1}{w(Q)}\int_Q f(y)^{\frac{p(y)}{p_-}}w(y){\rm d}y \le \frac{1}{w(Q)},$$

we have

$$k^{1-\frac{p(y)}{p(x)}} \sim \left(\frac{1}{w(Q)}\right)^{1-\frac{p(y)}{p(x)}} \sim 1 \tag{2.24}$$

thanks to Remark 2.4 and the global counterpart to [45, Lemma 2.13], whose proof is similar to the original proposition [45, Lemma 2.13]. If we insert (2.24) into (2.23), then

$$\begin{aligned}\frac{1}{w(Q)}\int_Q f(y)w(y){\rm d}y &\lesssim k^{\frac{p_-}{p(x)}} + \frac{1}{w(Q)}\int_Q f(y)^{\frac{p(y)}{p_-}}k^{\left(-\frac{p(x)}{p_-}+1\right)\frac{p_-}{p(x)}}w(y){\rm d}y\\ &= k^{\frac{p_-}{p(x)}} + \frac{k^{\frac{p_-}{p(x)}-1}}{w(Q)}\int_Q f(y)^{\frac{p(y)}{p_-}}w(y){\rm d}y\\ &= 2k^{\frac{p_-}{p(x)}}.\end{aligned}$$

Thus, from the definition of k, we conclude

$$\frac{1}{w(Q)}\int_Q f(y)w(y)\mathrm{d}y \lesssim \left(\frac{1}{w(Q)}\int_Q f(y)^{\frac{p(y)}{p_-}}w(y)\mathrm{d}y\right)^{\frac{p_-}{p(x)}}.$$

Recall that $p_+ < \infty$. Hence if we take the $p(x)$-th power of the above inequality, then we obtain the desired result. □

We define a variable exponent $s(\cdot)$ by

$$\frac{1}{s(x)} \equiv \left|\frac{1}{p_\infty} - \frac{1}{p(x)}\right| \left(\lesssim \frac{1}{\log(e+|x|)}\right) \tag{2.25}$$

for $x \in \mathbb{R}^n$. Roughly speaking, the function $s(\cdot)$ measures how differs $p(\cdot)$ from p_∞. It turns out that the log-Hölder condition at infinity is transformed into the integrability of $\gamma^{s(\cdot)}$ for small $\gamma > 0$.

Lemma 2.19. *Let $p(\cdot) \in \mathcal{P}_0$, $w \in A_\infty^{\mathfrak{D}}$ and $Q \in \mathfrak{D}$. Let $f \in L^{p(\cdot)}(w)$ be a real-valued function with $\|f\|_{L^{p(\cdot)}(w)} \le 1$. Assume $0 \le f \le 1$. Then for all $\gamma \in (0,1)$,*

$$\left(\frac{1}{w(Q)}\int_Q f(y)w(y)\mathrm{d}y\right)^{p(x)} \lesssim \left(\frac{1}{w(Q)}\int_Q f(y)^{\frac{p(y)}{p_-}}w(y)\mathrm{d}y\right)^{p_-} + M_w^{\mathfrak{D}}[\gamma^{\frac{s(\cdot)}{p_-}}](x)^{p_-}$$

for all $x \in Q$.

Proof. Fix $x \in Q$. We set

$$f_1(y) \equiv \chi_{[p(y),\infty)}(p(x))f(y), \quad f_2(y) \equiv f(y) - f_1(y)$$

for $y \in \mathbb{R}^n$. Then $f = f_1 + f_2$. Hence, we have

$$\left(\frac{\gamma^2}{2w(Q)}\int_Q f(y)w(y)\mathrm{d}y\right)^{p(x)} \le \frac{1}{2}\sum_{j=1}^{2}\left(\frac{\gamma^2}{w(Q)}\int_Q f_j(y)w(y)\mathrm{d}y\right)^{p(x)}.$$

As for f_1, we have

$$\begin{aligned}\frac{\gamma^2}{w(Q)}\int_Q f_1(y)w(y)\mathrm{d}y &\le \frac{1}{w(Q)}\int_Q f_1(y)w(y)\mathrm{d}y \\ &\le \left(\frac{1}{w(Q)}\int_Q f_1(y)^{\frac{p(x)}{p_-}}w(y)\mathrm{d}y\right)^{\frac{p_-}{p(x)}} \\ &\le \left(\frac{1}{w(Q)}\int_Q f(y)^{\frac{p(y)}{p_-}}w(y)\mathrm{d}y\right)^{\frac{p_-}{p(x)}}\end{aligned} \tag{2.26}$$

by Hölder's inequality and the fact that $f_1(y) \in [0,1]$ and $p(x) \ge p(y)$ for all $y \in Q$ such that $f_1(y) \ne 0$.

As for f_2, we define a variable exponent $q(x,y)$ by

$$\frac{1}{q(x,y)} = \frac{1}{p(x)} - \frac{1}{p(y)} > 0$$

for all $y \in Q$ with $p(x) < p(y)$. Then $2q(x,y) \ge \min(s(x), s(y))$, since

$$\frac{1}{q(x,y)} \le \frac{1}{s(x)} + \frac{1}{s(y)} \le 2\max\left(\frac{1}{s(x)}, \frac{1}{s(y)}\right).$$

Thus, using the Hölder inequality and then the Young inequality, we obtain

$$\begin{aligned}
(2.27)\qquad \left(\frac{\gamma^2}{w(Q)}\int_Q f_2(y)w(y)\mathrm{d}y\right)^{\frac{p(x)}{p_-}} &\le \frac{1}{w(Q)}\int_Q \gamma^{\frac{2p(x)}{p_-}} f_2(y)^{\frac{p(x)}{p_-}} w(y)\mathrm{d}y\\
&\le \frac{1}{w(Q)}\int_Q \left(\gamma^{\frac{2q(x,y)}{p_-}} + f(y)^{\frac{p(y)}{p_-}}\right) w(y)\mathrm{d}y\\
&\le \frac{1}{w(Q)}\int_Q \left(\gamma^{\frac{s(x)}{p_-}} + \gamma^{\frac{s(y)}{p_-}} + f(y)^{\frac{p(y)}{p_-}}\right) w(y)\mathrm{d}y.
\end{aligned}$$

If we use the Lebesgue differentiation theorem, then

$$(2.28)\qquad \gamma^{\frac{s(x)}{p_-}} \le M_w^{\mathfrak{D}}[\gamma^{\frac{s(\cdot)}{p_-}}](x).$$

If we insert (2.28) into (2.27), then

$$(2.29)\qquad \left(\frac{\gamma^2}{2w(Q)}\int_Q f_2(y)w(y)\mathrm{d}y\right)^{p(x)} \lesssim \left(\frac{1}{w(Q)}\int_Q f(y)^{\frac{p(y)}{p_-}} w(y)\mathrm{d}y\right)^{p_-} + M_w^{\mathfrak{D}}[\gamma^{\frac{s(\cdot)}{p_-}}](x)^{p_-}.$$

Combining (2.26) and (2.29), we obtain the desired result. □

We use the local log-Hölder continuity at infinity to show that $\gamma^{s(\cdot)}$ is integrable as long as $\gamma \ll 1$. We solidify this idea in the context of weights as follows:

Lemma 2.20. *Let $p(\cdot) \in \mathcal{P}\cap\mathrm{LH}_0\cap\mathrm{LH}_\infty$. If $0 < \gamma \ll 1$ and $w \in A_\infty^{\mathfrak{D}}$, then $M_w^{\mathfrak{D}}[\gamma^{\frac{s(\cdot)}{p_-}}] \in L^{p_-}(w)$.*

Proof. It suffices to show that

$$\int_{\mathbb{R}^n} M_w^{\mathfrak{D}}[\gamma^{\frac{s(\cdot)}{p_-}}](x)^{p_-} w(x)\mathrm{d}x < \infty.$$

A geometric observation shows that $M_w^{\mathfrak{D}}$ is weak $L^1(w)$-bounded. As we mentioned in the beginning of Proposition 2.17, $M_w^{\mathfrak{D}}$ is bounded on $L^{p_-}(w)$, assuming that $p(\cdot)$ is a constant exponent. Thus, thanks to the log-Hölder continuity, this is equivalent to

$$(2.30)\qquad \int_{\mathbb{R}^n} \gamma^{C\log(e+|x|)} w(x)\mathrm{d}x < \infty$$

for some $C > 0$. Note that (2.30) paraphrases [45, Corollary 2.14]. □

We conclude the proof of Proposition 2.17. Let $f \in L^{p(\cdot)}(w)$ with $\|f\|_{L^{p(\cdot)}(w)} \le 1$. Combining Lemmas 2.18 and 2.19, we have

$$M_w^{\mathfrak{D}} f(x)^{p(x)} \lesssim \left(M_w^{\mathfrak{D}}[|f(\cdot)|^{\frac{p(\cdot)}{p_-}}](x)\right)^{p_-} + M_w^{\mathfrak{D}}[\gamma^{\frac{s(\cdot)}{p_-}}](x)^{p_-}$$

for all $x \in \mathbb{R}^n$. Due to Lemma 2.20, the right-hand side is integrable with respect to the weighted measure $w(x)\mathrm{d}x$. Thus, we have the desired result.

2.7. **Diening's comparison principle.** We invoke the following variant of the norm equivalence due to Diening for weights that have at most polynomial growth. We remark that a weight has at most polynomial growth, if $w(\{y \in \mathbb{R}^n : |y| \le |x|\}) \lesssim (1+|x|)^N$ for some large N.

Lemma 2.21 (cf. [14, Lemma 2.7])**.** *Let $p(\cdot) \in \mathrm{LH}_0\cap\mathrm{LH}_\infty\cap\mathcal{P}_0$. Also, let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfy $|f(x)| \le (1+|x|)^N$ for some large N. Suppose that a weight w has at most polynomial growth. Then,*

(i) *If $\|f\|_{L^{p(\cdot)}(w)} \le A$, then $\|f\|_{L^{p_\infty}(w)} \le C_A$.*

(ii) *If* $\|f\|_{L^{p_\infty}(w)} \le B$, *then* $\|f\|_{L^{p(\cdot)}(w)} \le C_B$.

Here, C_A *and* C_B *are constants, which depend on* A *and* B*, respectively.*

Proof. We let $\tilde{p}(\cdot) \equiv \min(p_\infty, p(\cdot))$ and $p^\dagger(\cdot) \equiv \max(p_\infty, p(\cdot))$. Denote by X the set of all measurable functions satisfying $|f(x)| \le (1+|x|)^N$ for some large N. We claim that there exist constants $K_1, K_2, K_3, K_4 > 1$ with the following properties:

(1) There exists a constant $K_1 \ge 1$ such that $\|f\|_{L^{\tilde{p}(\cdot)}(w)} \le K_1 \|f\|_{L^{p(\cdot)}(w)}$ for $f \in L^{p(\cdot)}(w)$.
(2) If $f \in L^{\tilde{p}(\cdot)}(w) \cap X$ satisfies $\|f\|_{L^{\tilde{p}(\cdot)}(w)} \le 1$, then $\|f\|_{L^{p(\cdot)}(w)} \le K_2$.
(3) There exists a constant $K_3 \ge 1$ such that $\|f\|_{L^{p(\cdot)}(w)} \le K_3 \|f\|_{L^{p^\dagger(\cdot)}(w)}$ for $f \in L^{p^\dagger(\cdot)}(w)$.
(4) If $f \in L^{p(\cdot)}(w) \cap X$ satisfies $\|f\|_{L^{p(\cdot)}(w)} \le 1$, then $\|f\|_{L^{p_\dagger(\cdot)}(w)} \le K_4$.

Once we prove (1)–(4), we obtain the equivalence as follows:

(i) Taking A as $\max(1, A)$, we can assume that $A \ge 1$. Suppose that $f \in L^{p(\cdot)}(w) \cap X$ with $\|f\|_{L^{p(\cdot)}(w)} \le A$. Then by (1) we have $\|f\|_{L^{\tilde{p}(\cdot)}(w)} \le K_1 A$. Since $K_1 A \ge 1$, we have $(K_1 A)^{-1} f \in X$. By using (4) for the exponent $\tilde{p}(\cdot)$, $\|(K_1 A)^{-1} f\|_{L^{\max(p_\infty, \tilde{p}(\cdot))}(w)} \le K_4$. This implies that $\|f\|_{L^{p_\infty}(w)} \le K_1 K_4 A$.
(ii) For the same reason as above, we can assume that $B \ge 1$. Suppose instead that $f \in L^{p_\infty} \cap X$ with $B \ge \|f\|_{L^{p_\infty}(w)} = \|f\|_{L^{\min(p_\infty, p^\dagger(\cdot))}(w)}$. Since $B \ge 1$, we have $\|B^{-1} f\|_{L^{\max(p_\infty, p(\cdot))}(w)} \le K_2$ by (2). Hence $\|(K_2 B)^{-1} f\|_{L^{\max(p_\infty, p(\cdot))}(w)} \le 1$. This implies that $\|f\|_{L^{p(\cdot)}(w)} \le K_2 K_3 B$ by (3).

So, let us prove (1)–(4).

(1) We prove $L^{p(\cdot)}(w) \hookrightarrow L^{\tilde{p}(\cdot)}(w)$. Let $1/\tilde{p}(\cdot) = 1/\tilde{r}(\cdot) + 1/p(\cdot)$. To this end, we claim
$$\int_{\mathbb{R}^n} \lambda^{\tilde{r}(x)} w(x) \mathrm{d}x < \infty.$$
Once this is proved, we have $L^{p(\cdot)}(w) \hookrightarrow L^{\tilde{p}(\cdot)}(w) = L^{\min(p_\infty, p(\cdot))}(w)$ by the Hölder inequality and the fact $w \in L^{\tilde{r}(\cdot)}(\mathbb{R}^n)$.
Note that
$$\frac{1}{\tilde{r}(\cdot)} = \max\left\{\frac{1}{p_\infty} - \frac{1}{p(\cdot)}, 0\right\} \lesssim \frac{1}{\log(e + |\cdot|)}.$$
Thus, $\tilde{r}(\cdot) \gtrsim \log(e + |\cdot|)$. Consequently, assuming that $\tilde{r}(\cdot) < \infty$ everywhere (since it is trivial that $1 \in L^\infty(w)$), for small $\lambda \in (0,1)$, we have
$$\int_{\mathbb{R}^n} \lambda^{\tilde{r}(x)} w(x) \mathrm{d}x \lesssim \sum_{j=1}^\infty (e+j)^{c \log \lambda} w(B(j)) < \infty.$$
(2) Suppose that $f \in L^{\tilde{p}(\cdot)}(w) \cap X$ satisfies $\|f\|_{L^{\tilde{p}(\cdot)}(w)} \le 1$. Since $|f(x)| \le (1+|x|)^N$,
$$|f(x)|^{p(x) - \tilde{p}(x)} \le (1+|x|)^{N(p(x) - \tilde{p}(x))} \le \max(1, e^{Nc^*}).$$
Here, we use the estimate
$$(1+|x|)^{(p(x)-\tilde{p}(x))} = \max\left(1, (1+|x|)^{p_\infty - \tilde{p}(x)}\right) \le \max\left(1, (1+|x|)^{\frac{c^*}{\log(e+|x|)}}\right) \le e^{c^*}$$
by the log-Hölder-type decay condition. This proves that
$$\int_{\mathbb{R}^n} |f(x)|^{p(x)} w(x) \mathrm{d}x \le \max(1, e^{Nc^*}) \int_{\mathbb{R}^n} |f(x)|^{\tilde{p}(x)} w(x) \mathrm{d}x \le \max(1, e^{Nc^*}).$$
Thus, $\|f\|_{L^{p(\cdot)}(w)} \le C_{c^*}$.

(3) Define an exponent $r_\dagger(\cdot)$ by

$$\frac{1}{\tilde{p}(\cdot)} = \frac{1}{p(\cdot)} - \frac{1}{r_\dagger(\cdot)}.$$

We claim that $1 \in L^{r_\dagger(\cdot)}(w)$ if weight w has at most polynomial growth.

Note that

$$\frac{1}{r_\dagger(\cdot)} = \max\left\{\frac{1}{p(\cdot)} - \frac{1}{p_\infty}, 0\right\} \lesssim \frac{1}{\log(e+|\cdot|)}.$$

Thus, $r_\dagger(\cdot) \gtrsim \log(e+|\cdot|)$.

Assuming that $r_\dagger(\cdot) < \infty$ everywhere (since it is trivial thati $1 \in L^\infty(w)$), for small $\lambda \in (0,1)$, we have

$$\int_{\mathbb{R}^n} \lambda^{r_\dagger(x)} w(x) \mathrm{d}x \lesssim \sum_{j=1}^\infty (e+j)^{c\log\lambda} w(B(j)) < \infty.$$

Consequently, since $w \in L^{r_\dagger(\cdot)}(\mathbb{R}^n)$, we have $L^{p(\cdot)}(w) \hookleftarrow L^{p_\dagger(\cdot)}(w) = L^{\max(p_\infty, p(\cdot))}(w)$ by the Hölder inequality.

(4) Since $|f(x)| \le (1+|x|)^N$,

$$|f(x)|^{p_\dagger(x)-p(x)} \le (1+|x|)^{N(p_\dagger(x)-p(x))} \le \max(1, e^{Nc_*}).$$

This proves that

$$\int_{\mathbb{R}^n} |f(x)|^{p_\dagger(x)} w(x) dx \le \max(1, e^{Nc_*}) \int_{\mathbb{R}^n} |f(x)|^{p(x)} w(x) dx \le \max(1, e^{Nc_*}).$$

Hence, we have $\|f\|_{L^{p_\dagger(\cdot)}(w)} \le C_{c_*}$.

□

2.8. **Inequality in $\mathcal{D}(\mathbb{R}^n)$.** In this subsection, we prepare some lemmas by Rychkov [47]. Especially, these lemmas play an important role to consider the Littlewood–Paley and wavelet characterization. (See Section 7 and Section 8, respectively.)

First, we recall the following lemma on the moment condition of functions:

Lemma 2.22 (Grafakos [21, p.466] or [22, p.595])**.** *Let $\mu, \nu \in \mathbb{R}$, $M, N > 0$, and $L \in \mathbb{N}_0$ satisfy $\nu \ge \mu$ and $N > M + L + n$. Suppose that $\phi_{(\mu)} \in C^L(\mathbb{R}^n)$ satisfies*

$$|\partial^\alpha \phi_{(\mu)}(x)| \le A_\alpha \frac{2^{\mu(n+L)}}{(1+2^\mu|x-x_\mu|)^M} \quad \textit{for all } |\alpha| = L.$$

Furthermore, suppose that $\phi_{(\nu)}$ is a measurable function satisfying

$$\int_{\mathbb{R}^n} \phi_{(\nu)}(x)(x-x_\nu)^\beta \, dx = 0 \quad \textit{for all } |\beta| \le L-1, \ \textit{and } |\phi_{(\nu)}(x)| \le B \frac{2^{\nu n}}{(1+2^\nu|x-x_\nu|)^N},$$

where the former condition is supposed to be vacuous when $L = 0$. Then it holds

$$\left|\int_{\mathbb{R}^n} \phi_{(\mu)}(x)\phi_{(\nu)}(x) dx\right| \le C_{A_\alpha,B,L,M,N}\, 2^{\mu n-(\nu-\mu)L}(1+2^\mu|x_\mu-x_\nu|)^{-M}$$

with a constant $C_{A_\alpha,B,L,M,N}$ taken as

$$C_{A_\alpha,B,L,M,N} = B\left(\sum_{|\alpha|=L} \frac{A_\alpha}{\alpha!}\right) \frac{\omega_n(N-M-L)}{N-M-L-n},$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

We also recall the decomposition formula of Dirac's delta and invoke [47, Theorem 1.6]. Note that for $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $t > 0$, we write $\varphi_t \equiv t^{-n}\varphi(t^{-1}\cdot)$.

Lemma 2.23. *Let $L \in \mathbb{N} \cup \{-1, 0\}$ and $\phi \in \mathcal{D}(\mathbb{R}^n) \setminus \mathcal{P}_0^{\perp}(\mathbb{R}^n)$. Then there exist $\phi^*, \psi, \psi^* \in \mathcal{D}(\mathbb{R}^n)$ such that*

$$\phi^* = \phi - 2^{-n}\phi\left(\frac{\cdot}{2}\right), \quad \phi^*, \psi^* \in \mathcal{P}_L(\mathbb{R}^n), \quad \phi * \psi + \sum_{j=1}^{\infty} \phi^*_{2^{-j}} * \psi^*_{2^{-j}} = \delta$$

in the topology of $\mathcal{D}'(\mathbb{R}^n)$.

Remark that if ϕ is even (resp. radial) then the actual construction in [47, Theorem 1.6] shows that ϕ^*, ψ, ψ^* are even (resp. radial).

Furthermore, in [47, Lemma 2.9], Rychkov proved the following estimate for the functions which are constructed in Lemma 2.23:

Lemma 2.24. *Let $A, B, r > 0$ and $L \in \mathbb{N} \cap [A, \infty)$. Then in Lemma 2.23, for all $j \in \mathbb{N}_0$, $t > 0$ and $f \in \mathcal{D}'(\mathbb{R}^n)$,*

$$\begin{aligned}&|\phi_{2^{-j}t} * f(x)|^r \\ &\lesssim 2^{jn} \int_{\mathbb{R}^n} \frac{|\phi_{2^{-j}t} * f(x-y)|^r}{m_{j,Ar,Br}(y)} \mathrm{d}y + \sum_{k=j+1}^{\infty} 2^{kn+(j-k)(L+1)r} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-k}t} * f(x-y)|^r}{m_{j,Ar,Br}(y)} \mathrm{d}y\end{aligned}$$

and

$$|\phi^*_{2^{-j}t} * f(x)|^r \lesssim \sum_{k=j}^{\infty} 2^{kn+(j-k)(L+1)r} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-k}t} * f(x-y)|^r}{m_{j,Ar,Br}(y)} \mathrm{d}y.$$

In particular,

$$\begin{aligned}&\sup_{y \in \mathbb{R}^n} \frac{|\phi_{2^{-j}t} * f(x-y)|^r}{m_{j,Ar,Br}(y)} \\ &\lesssim 2^{jn} \int_{\mathbb{R}^n} \frac{|\phi_{2^{-j}t} * f(x-y)|^r}{m_{j,Ar,Br}(y)} \mathrm{d}y + \sum_{k=j+1}^{\infty} 2^{kn+(j-k)(L+1)r} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-k}t} * f(x-y)|^r}{m_{j,Ar,Br}(y)} \mathrm{d}y\end{aligned}$$

and

$$\sup_{y \in \mathbb{R}^n} \frac{|\phi^*_{2^{-j}t} * f(x-y)|^r}{m_{j,Ar,Br}(y)} \lesssim \sum_{k=j}^{\infty} 2^{kn+(j-k)(L+1)r} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-k}t} * f(x-y)|^r}{m_{j,Ar,Br}(y)} \mathrm{d}y.$$

In fact, these estimates hold if the function ϕ in the right-hand side is replaced by a function having similar properties.

Lemma 2.25 ([47, Theorem 2.5])**.** *In addition to the assumptions in Lemmas 2.23 and 2.24, let $\zeta \in \mathcal{S}(\mathbb{R}^n)$. Then $j \in \mathbb{N}_0$, $t > 0$ and $f \in \mathcal{D}'(\mathbb{R}^n)$,*

$$\begin{aligned}&|\zeta_{2^{-j}t} * f(x)|^r \\ &\lesssim 2^{jn} \int_{\mathbb{R}^n} \frac{|\phi_{2^{-j}t} * f(x-y)|^r}{m_{j,Ar,Br}(y)} \mathrm{d}y + \sum_{k=j+1}^{\infty} 2^{kn+(j-k)(L+1)r} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-k}t} * f(x-y)|^r}{m_{j,Ar,Br}(y)} \mathrm{d}y.\end{aligned}$$

3. Fundamental properties of $h^{p(\cdot)}(w)$ (including the proof of Theorem 1.6)

Here, we investigate the structure of $h^{p(\cdot)}(w)$. We first verify that $\mathcal{D}'(\mathbb{R}^n)$ is a suitable space to consider $h^{p(\cdot)}(w)$. Note that Propositions 3.1 and 3.2 are proved by Tang [55, Propositions 3.1 and 3.2] when $p(\cdot)$ is a constant exponent in $(0, 1]$.

Proposition 3.1. *Let $w \in A_\infty^{\rm loc}$. If $N \ge N_{p(\cdot),w}$, then the inclusion $h^{p(\cdot)}(w) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$ is continuous.*

Proof. The proof is the same as that in [55, Proposition 3.1], where we take the $L^{p(\cdot)}(w)$-norm instead of $L^p(w)$-norm. □

Proposition 3.2. *Let $w \in A_\infty^{\rm loc}$. If $N \ge N_{p(\cdot),w}$, then $h^{p(\cdot)}(w)$ is complete.*

Proof. This is a direct consequence of Proposition 3.1. The proof is similar to [47, Lemma 2.15]. Here, we omit the details. □

We prove Theorem 1.6.

Proof of Theorem 1.6. Let $f \in h^{p(\cdot)}(w)$. Take $\psi \in \mathcal{D}(\mathbb{R}^n) \setminus \mathcal{P}_0(\mathbb{R}^n)^\perp$. Write $\psi_t \equiv t^{-n}\psi(t^{-1}\cdot)$ as before. Then $\{\psi_t * f\}_{t>0}$ is a bounded set of $L^{p(\cdot)}(w) = (L^{p'(\cdot)}(\sigma))^*$. By the Banach–Alaoglu theorem there exists a sequence $\{t_j\}_{j=1}^\infty$ decreasing to 0 such that $\{\psi_{t_j} * f\}_{j=1}^\infty$ converges to a function g in the weak-* topology of $L^{p(\cdot)}(w)$. Meanwhile, it can be shown that $\lim\limits_{t\downarrow 0}\psi_t * f = f$ in the topology of $\mathcal{D}'(\mathbb{R}^n)$. Since the weak-* topology of $L^{p(\cdot)}(w)$ is stronger than the topology of $\mathcal{D}'(\mathbb{R}^n)$, it follows that $f = g \in L^{p(\cdot)}(w)$. □

4. Proof of Theorem 1.1

From the definition of the three local grand maximal operators, it suffices to handle the most right-hand inequality. That is,

$$\|\mathcal{M}_N f\|_{L^{p(\cdot)}(w)} \lesssim \|\mathcal{M}_N^0 f\|_{L^{p(\cdot)}(w)}.$$

Let $L \in \mathbb{N}$ be sufficiently large. Fix $x \in \mathbb{R}^n$ for now. Let $(z,t) \in \mathbb{R}^n_+$ satisfy $|x-z| < t \le 1$. Fix $\phi \in \mathcal{D}(\mathbb{R}^n) \setminus \mathcal{P}_0^\perp$ and take $\phi^*, \psi, \psi^* \in \mathcal{D}(\mathbb{R}^n)$ as in Lemma 2.23.

Then

$$\varphi_t * f(z) = [\varphi_t * \psi_t(\cdot - x + z)] * \phi_t * f(x) + \sum_{j=1}^\infty [\varphi_t * \psi^*_{2^{-j}t}(\cdot - x + z)] * \phi^*_{2^{-j}t} * f(x).$$

Let $A > \frac{n}{r}$, $B > \frac{8n + 6\log D}{r}$ and assume $L \in \mathbb{N} \cap (A, \infty)$. By the assumption on N, we can assume that

$$\frac{p_-}{r} > q_w.$$

By the moment condition on ϕ^* and ψ^* and the equality $\phi^* = \phi - 2^{-n}\phi(2^{-1}\cdot)$,

$$\begin{aligned}
\mathcal{M}_N f(x) &= \sup_{(z,t)\in\mathbb{R}^n, |z-x|<t\le 1} |\varphi_t * f(z)| \\
&\lesssim \left\{ \sum_{j=0}^\infty 2^{-j(L-A)r} M^{\rm loc}\left[\int_{\mathbb{R}^n} \sup_{t\in(0,1]} \frac{|\phi_{2^{-j}t} * f(\cdot - y)|^r}{(1+2^j|y|)^{Ar} 2^{|y|Br}} dy\right](x)\right\}^{\frac{1}{r}} \\
&\lesssim \left\{ \sum_{j=0}^\infty 2^{-j(L-n-A)r} M^{\rm loc} \circ K_{Br}\left[\sup_{t\in(0,1]} |\phi_{2^{-j}t} * f|^r\right](x)\right\}^{\frac{1}{r}} \\
&\lesssim \left\{ M^{\rm loc} \circ K_{Br}\left[(\mathcal{M}_N^0 f)^r\right](x)\right\}^{\frac{1}{r}}.
\end{aligned}$$

If we use Lemmas 2.6 and 2.14, we obtain the desired result.

5. Proof of Theorem 1.5, including the implication (II)/(III) $\Longrightarrow$ (I) in Theorem 1.7

The proof of Theorem 1.5 has two parts. One controls the sum of atoms by the norm of $h^{p(\cdot)}(w)$ (Sections 5.1 and 5.2). The other decomposes the distributions $h^{p(\cdot)}(w)$ into the sum of atoms (Section 5.3).

5.1. **Some norm estimates.**

Here and below we write

$$m^{(u)}_{Q,w}(f) \equiv \frac{\|\chi_Q f\|_{L^u(w)}}{w(Q)^{\frac{1}{u}}}$$

for a cube Q, $0 < u < \infty$ and $f \in L^0(\mathbb{R}^n)$. We establish the following key estimate for the proof of Theorem 1.5 and the proof of the implication (II) $\Longrightarrow$ (I) in Theorem 1.7:

Theorem 5.1. *Let $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and $w \in A^{\rm loc}_\infty$. Assume that $(u,v) \in (0,\infty) \times ((0,p_-) \cap [0,1])$ satisfies $uv > p_+$. Suppose $f_j \in L^{uv}(\mathbb{R}^n)$ which is supported on a cube Q_j with $|Q_j| \le 1$ for each j. Then*

$$\left\| \left(\sum_{j=1}^\infty |f_j|^v \right)^{\frac{1}{v}} \right\|_{L^{p(\cdot)}(w)} \lesssim \mathcal{A}_{p(\cdot),w,v}(\{m^{(uv)}_{Q_j,w}(f_j)\}_{j=1}^\infty; \{Q_j\}_{j=1}^\infty).$$

It is noteworthy that the case where $v = 1 < p_-$ proves the implication (II) $\Longrightarrow$ (I) in Theorem 1.7.

Proof. Write $P(\cdot) \equiv \frac{p(\cdot)}{v}$ and $\Sigma \equiv w^{-\frac{1}{P(\cdot)-1}}$. Since $p_- > v$, $P_- > 1$. By duality and the definition of $P(\cdot)$, the matters are reduced to the estimate

$$\left(\int_{\mathbb{R}^n} \sum_{j=1}^\infty |f_j(x)|^v |g(x)| \mathrm{d}x \right)^{\frac{1}{v}} \lesssim \mathcal{A}_{p(\cdot),w,v}(\{m^{(uv)}_{Q_j,w}(f_j)\}_{j=1}^\infty; \{Q_j\}_{j=1}^\infty) \left(\|g\|_{L^{P'(\cdot)}(\Sigma)} \right)^{\frac{1}{v}} \tag{5.1}$$

for all $g \in L^0(\mathbb{R}^n)$. Since the functions in the summand of the left-hand side are non-negative,

$$\begin{aligned} \left(\int_{\mathbb{R}^n} \sum_{j=1}^\infty |f_j(x)|^v |g(x)| \mathrm{d}x \right)^{\frac{1}{v}} &= \left(\sum_{j=1}^\infty \int_{\mathbb{R}^n} |f_j(x)|^v |g(x)| \mathrm{d}x \right)^{\frac{1}{v}} \\ &= \left(\sum_{j=1}^\infty \int_{\mathbb{R}^n} |f_j(x)|^v |g(x) w(x)^{-1}| w(x) \mathrm{d}x \right)^{\frac{1}{v}}. \end{aligned}$$

By the Hölder inequality, we have

$$\left(\int_{\mathbb{R}^n} \sum_{j=1}^\infty |f_j(x)|^v |g(x)| \mathrm{d}x \right)^{\frac{1}{v}} \le \left\{ \sum_{j=1}^\infty w(Q_j) m^{(u)}_{Q_j,w}(|f_j|^v) m^{(u')}_{Q_j,w}(g w^{-1}) \right\}^{\frac{1}{v}}.$$

Using the powered weighted local maximal operator $M^{(u'),\mathrm{loc}}_w$, we have

$$\left(\int_{\mathbb{R}^n} \sum_{j=1}^\infty |f_j(x)|^v |g(x)| \mathrm{d}x \right)^{\frac{1}{v}} \le \left\{ \int_{\mathbb{R}^n} \sum_{j=1}^\infty m^{(u)}_{Q_j,w}(|f_j|^v) \chi_{Q_j}(x) M^{(u'),\mathrm{loc}}_w(g w^{-1})(x) w(x) \mathrm{d}x \right\}^{\frac{1}{v}}.$$

By the Hölder inequality for Lebesgue spaces with variable exponents (see Lemma 2.1), we have

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^n}\sum_{j=1}^{\infty}|f_j(x)|^v|g(x)|\mathrm{d}x\right)^{\frac{1}{v}}\\
&\le 2^{\frac{1}{v}}\left\{\mathcal{A}_{P(\cdot),w,1}(\{m^{(u)}_{Q_j,w}(|f_j|^v)\}_{j=1}^{\infty};\{Q_j\}_{j=1}^{\infty})\left\|M_w^{(u'),\mathrm{loc}}(gw^{-1})w\right\|_{L^{P'(\cdot)}(\Sigma)}\right\}^{\frac{1}{v}}\\
&= 2^{\frac{1}{v}}\mathcal{A}_{p(\cdot),w,v}(\{m^{(uv)}_{Q_j,w}(f_j)\}_{j=1}^{\infty};\{Q_j\}_{j=1}^{\infty})\left(\left\|M_w^{(u'),\mathrm{loc}}(gw^{-1})\right\|_{L^{P'(\cdot)}(w)}\right)^{\frac{1}{v}}. \qquad (5.2)
\end{aligned}
$$

Recall that we assume $uv > p_+$. That is, $P_+ < u$. Hence $(P')_- > u'$. Since $w \in A_\infty^{\mathrm{loc}}$, by Proposition 2.17, we have

$$
\begin{aligned}
\|M_w^{(u'),\mathrm{loc}}(gw^{-1})\|_{L^{P'(\cdot)}(w)} &= (\|M_w^{\mathrm{loc}}[|gw^{-1}|^{u'}]\|_{L^{P'(\cdot)/u'}(w)})^{\frac{1}{u'}}\\
&\lesssim \|gw^{-1}\|_{L^{P'(\cdot)}(w)}\\
&= \|g\|_{L^{P'(\cdot)}(\Sigma)}. \qquad (5.3)
\end{aligned}
$$

Combining (5.2) and (5.3), gives (5.1). Thus, the proof is complete. $\square$

As mentioned, the implication (I) $\Longrightarrow$ (II)/(III) is included in Theorems 1.5 and 1.6. Now let us prove the implication (II)/(III) $\Longrightarrow$ (I).

We rephrase and prove the implication (III) $\Longrightarrow$ (I) of in Theorem 1.7 as follows:

Theorem 5.2. *Let $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Also let $w \in A_{p(\cdot)}^{\mathrm{loc}}$ and $q_0 > p_+$. Assume that $\sigma = w^{-\frac{1}{p(\cdot)-1}} \in A_{p'(\cdot)/q_0'}^{\mathrm{loc}}$ If we have a collection $\{\lambda_j\}_{j=1}^{\infty}$ of complex constants and a collection $\{a_j\}_{j=1}^{\infty} \subset L^{q_0}(\mathbb{R}^n)$ such that each $a_j, j \in \mathbb{N}$ is a $(p(\cdot), q_0, L)$-atom supported on a cube Q_j with $|Q_j| \le 1$ and that*

$$\mathcal{A}_{p(\cdot),w,1}(\{\lambda_j\}_{j=1}^{\infty};\{Q_j\}_{j=1}^{\infty}) = \left\|\sum_{j=1}^{\infty}|\lambda_j|\chi_{Q_j}\right\|_{L^{p(\cdot)}(w)} < \infty,$$

then

$$f = \sum_{j=1}^{\infty}\lambda_j a_j$$

converges in $L^{p(\cdot)}(w)$ and satisfies

$$\|f\|_{L^{p(\cdot)}(w)} \lesssim \mathcal{A}_{p(\cdot),w,1}(\{\lambda_j\}_{j=1}^{\infty};\{Q_j\}_{j=1}^{\infty}). \qquad (5.4)$$

Unlike Theorem 1.7, we do not have to distinguish two cases. In fact, if $q_0 > p_+$ and $w \in L^1(\mathbb{R}^n)$, then $L^{q_0}(w) \subset L^{p(\cdot)}(w)$.

Proof. We can assume that each a_j is a non-negative function and each λ_j is a non-negative real number. We dualize the conclusion

$$\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x \lesssim \mathcal{A}_{p(\cdot),w,1}(\{\lambda_j\}_{j=1}^{\infty};\{Q_j\}_{j=1}^{\infty})\|g\|_{L^{p'(\cdot)}(\sigma)}, \qquad (5.5)$$

where $g \in L^0(\mathbb{R}^n)$ is a non-negative function.

Using Hölder's inequality twice, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\sum_{j=1}^{\infty}\lambda_j a_j(x)g(x)\mathrm{d}x &\le \int_{\mathbb{R}^n}\sum_{j=1}^{\infty}\lambda_j m_{Q_j}^{(q_0)}(a_j)m_{Q_j}^{(q_0')}(g)\chi_{Q_j}(x)\mathrm{d}x\\ &\le \int_{\mathbb{R}^n}\sum_{j=1}^{\infty}\lambda_j m_{Q_j}^{(q_0)}(a_j)M^{\rm loc}[g^{q_0'}](x)^{\frac{1}{q_0'}}\chi_{Q_j}(x)\mathrm{d}x\\ &\lesssim \mathcal{A}_{p(\cdot),w,1}(\{\lambda_j\}_{j=1}^{\infty};\{Q_j\}_{j=1}^{\infty})\|(M^{\rm loc}[g^{q_0'}])^{\frac{1}{q_0'}}\|_{L^{p'(\cdot)}(\sigma)}.\end{aligned}$$

Since $q_0 > p_+(>1)$, we have $q_0' < p_-' = (p_+)'$. Since we assume $\sigma \in A^{\rm loc}_{p'(\cdot)/q_0'}$,

$$\|(M^{\rm loc}[g^{q_0'}])^{\frac{1}{q_0'}}\|_{L^{p'(\cdot)}(\sigma)} \lesssim \|g\|_{L^{p'(\cdot)}(\sigma)}. \tag{5.6}$$

If we insert (5.6) into the above expression, we obtain (5.5). □

5.2. **Proof of $h^{p(\cdot),q,L;v}(w) \hookrightarrow h^{p(\cdot)}(w)$.** The following estimate is a passage of [55, (3.2)] to the setting of variable exponents. Recall that

$$N_{p(\cdot),w} \equiv 2 + \left[n\left(\frac{q_w}{\min(1,p_-)}-1\right)\right].$$

Lemma 5.3. *Let $w \in A_\infty^{\rm loc} \cap L^1(\mathbb{R}^n)$ and $p(\cdot) \in \mathcal{P} \cap {\rm LH}_0 \cap {\rm LH}_\infty$. We assume that $q > \max(q_w, p_+)$. Let a be a single $(p(\cdot),q)_w$-atom. Then $\|\mathcal{M}^0_{N_{p(\cdot),w}}a\|_{L^{p(\cdot)}(w)} \lesssim 1$.*

Proof. Since $q > q_w$, $w \in A_q^{\rm loc}$. Thus, $\mathcal{M}^0_{N_{p(\cdot),w}}$ is bounded on $L^q(w)$ ([55, Proposition 2.2]). Define an exponent $r(\cdot)$ by $1/p(\cdot) = 1/q + 1/r(\cdot)$ while recalling that $q > p_+$. Using the Hölder inequality (see Lemma 2.1), we have

$$\begin{aligned}\|\mathcal{M}^0_{N_{p(\cdot),w}}a\|_{L^{p(\cdot)}(w)} &\lesssim \|\mathcal{M}^0_{N_{p(\cdot),w}}a\|_{L^q(w)}\|\chi_{\mathbb{R}^n}\|_{L^{r(\cdot)}(w)}\\ &\lesssim \|a\|_{L^q(w)}\|\chi_{\mathbb{R}^n}\|_{L^{r(\cdot)}(w)} \le w(\mathbb{R}^n)^{\frac{1}{q}}\|\chi_{\mathbb{R}^n}\|_{L^{r(\cdot)}(w)} \lesssim 1.\end{aligned}$$

This is the desired result. □

Tang pointed out an important feature of $\mathcal{M}^0_{N_{p(\cdot),w}}a$ for $(p(\cdot),q,L)_w$-atoms a.

Lemma 5.4. [55, (3.3)] *Let $w \in A_\infty^{\rm loc}$ and $p(\cdot) \in \mathcal{P}\cap{\rm LH}_0\cap{\rm LH}_\infty$. Suppose $L \in \mathbb{Z}\cap[-1, N_{p(\cdot),w}]$ and $q > q_w$. Let a be a $(p(\cdot),q,L)_w$-atom supported on a cube $Q = Q(x_0,r)$ with $|Q| < 1$. Then*

$$\mathcal{M}^0_{N_{p(\cdot),w}}a(x) \lesssim (M^{\rm loc}\chi_Q(x))^{\frac{n+L+1}{n}} \tag{5.7}$$

for $x \in \mathbb{R}^n \setminus 2Q$.

Here we recall the proof of Lemma 5.4 since we must rephrase in terms of the local maximal operator.

Proof. If $x \in \mathbb{R}^n \setminus Q(x_0,4n)$, then $\mathcal{M}^0_{N_{p(\cdot),w}}a(x) = 0$. So, we assume that $x \in Q(x_0,4n)\setminus 2Q$. Let $\varphi \in \mathcal{D}_N^0$. By the support condition of a, if ${\rm supp}\,a \cap {\rm supp}\,\varphi_t(x-\cdot) \ne \phi$, then $r < t$ and $|x-x_0| < 2t$. Let P be the Taylor expansion of φ at the point $(x-x_0)/t$ of order $L \le N_{p(\cdot),w}$. Since a is a $(p(\cdot),q,L)_w$-atom, we have

$$|a*\varphi_t(x)| = \left|t^{-n}\int_{\mathbb{R}^n}a(y)\left(\varphi\left(\frac{x-y}{t}\right)-P\left(\frac{x-y}{t}\right)\right)\mathrm{d}y\right|.$$

Thus, by the Taylor remainder theorem,

$$|a*\varphi_t(x)| \lesssim t^{-n}\int_Q |a(y)|\left|\frac{x_0-y}{t}\right|^{L+1} {\rm d}y. \tag{5.8}$$

By the triangle inequality and Hölder's inequality,

$$\begin{aligned}
|a*\varphi_t(x)| &\lesssim |x-x_0|^{-n-L-1}\ell(Q)^{L+1}\int_Q |a(y)|{\rm d}y\\
&\le |x-x_0|^{-n-L-1}\ell(Q)^{L+1}\|a\|_{L^q(w)}\left(\int_Q w(y)^{-\frac{1}{q-1}}{\rm d}y\right)^{1-\frac{1}{q}}\\
&\le |x-x_0|^{-n-L-1}\ell(Q)^{L+1}w(Q)^{\frac{1}{q}}\left(\int_Q w(y)^{-\frac{1}{q-1}}{\rm d}y\right)^{1-\frac{1}{q}}\\
&\lesssim (M^{\rm loc}\chi_Q(x))^{\frac{n+L+1}{n}}|Q|^{-1}w(Q)^{\frac{1}{q}}\left(\int_Q w(y)^{-\frac{1}{q-1}}{\rm d}y\right)^{1-\frac{1}{q}}.
\end{aligned}$$

Since $w \in A_q^{\rm loc}$, we have

$$|a*\varphi_t(x)| \lesssim (M^{\rm loc}\chi_Q(x))^{\frac{n+L+1}{n}}.$$

If we take the supremum over $t \in (0,1)$, then the desired inequality is obtained. □

Keeping Lemma 5.4 in mind, let us prove the one inclusion of Theorem 1.5. That is, we will prove $h^{p(\cdot),q,L;v}(w) \hookrightarrow h^{p(\cdot)}(w)$.

We start with the setup. Let $u>0$ satisfy

$$\max(q_w,p_+) < uv < q.$$

Since $w \in A_{p(\cdot)}^{\rm loc}$ and $q>q_w$, $w \in A_q^{\rm loc}$ thanks to Proposition 2.9.

Let $f \in h^{p(\cdot),q,L;v}(w)$. We suppose that $w(\mathbb{R}^n)<\infty$. Otherwise we can modify the proof below. There is a decomposition $f=\sum\limits_{j=0}^{\infty}\lambda_j a_j+\sum\limits_{j=1}^{\infty}\lambda'_j b_j$, where a_0 is a simple $(p(\cdot),q)_w$-atom, each a_j, $j \in \mathbb{N}$ is a $(p(\cdot),q,L)_w$-atom supported on $Q_j=Q(x_j,r_j)$ with $r_j<1/2$, each b_j, $j \in \mathbb{N}$ is a $(p(\cdot),q,L)_w$-atom supported on $R_j=Q(x_j,1/2)$ and coefficients $\{\lambda_j\}_{j=0}^\infty$ and $\{\lambda'_j\}_{j=1}^\infty$ satisfy

$$|\lambda_0|+\mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^\infty;\{Q_j\}_{j=1}^\infty)+\mathcal{A}_{p(\cdot),w,v}(\{\lambda'_j\}_{j=1}^\infty;\{R_j\}_{j=1}^\infty)<\infty.$$

By the triangle inequality, the sublinearlity of $\mathcal{M}^0_{N_{p(\cdot),w}}$ and Lemma 5.4, we have

$$\begin{aligned}
&\mathcal{M}^0_{N_{p(\cdot),w}}f\\
&\le |\lambda_0|\mathcal{M}^0_{N_{p(\cdot),w}}a_0+\sum_{j=1}^\infty|\lambda_j|\mathcal{M}^0_{N_{p(\cdot),w}}a_j+\sum_{j=1}^\infty|\lambda'_j|\mathcal{M}^0_{N_{p(\cdot),w}}b_j\\
&\lesssim |\lambda_0|\mathcal{M}^0_{N_{p(\cdot),w}}a_0+\sum_{j=1}^\infty|\lambda_j|(M^{\rm loc}\chi_{Q(x_j,r_j)})^{\frac{n+L+1}{n}}+\sum_{j=1}^\infty|\lambda_j|\chi_{2Q_j}M^{\rm loc}a_j+\sum_{j=1}^\infty|\lambda'_j|\mathcal{M}^0_{N_{p(\cdot),w}}b_j.
\end{aligned}$$

The first term is controlled by Lemma 5.3 as

$$\|\mathcal{M}^0_{N_{p(\cdot),w}}a_0\|_{L^{p(\cdot)}(w)} \lesssim 1. \tag{5.9}$$

To handle the second term we use

$$\sum_{j=1}^\infty|\lambda_j|(M^{\rm loc}\chi_{Q(x_j,r_j)})^{\frac{n+L+1}{n}} \le \left(\sum_{j=1}^\infty|\lambda_j|^v(M^{\rm loc}\chi_{Q(x_j,r_j)})^{v\frac{n+L+1}{n}}\right)^{\frac{1}{v}}.$$

We recall that $v \le \min(1,p_-)$ and that (1.5) holds. Arithmetic shows that

$$v\frac{n+L+1}{n} \ge \frac{v}{n}\left(n+1+\left[n\left(\frac{q_w}{v}-1\right)\right]\right) > q_w \ge 1,$$

and that

$$\frac{n+L+1}{n}p(\cdot) \ge \frac{p_-}{n}\left(n+1+\left[n\left(\frac{q_w}{v}-1\right)\right]\right) > q_w\frac{p_-}{v} > q_w \ge 1.$$

So we can use the vector-valued inequality of $M^{\rm loc}$ for weighted Lebesgue spaces with variable exponents (see Lemma 2.11) to give

$$\left\|\sum_{j=1}^{\infty}|\lambda_j|(M^{\rm loc}\chi_{Q(x_j,r_j)})^{\frac{n+L+1}{n}}\right\|_{L^{p(\cdot)}(w)} \lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^{\infty};\{Q_j\}_{j=1}^{\infty}). \tag{5.10}$$

For the third and fourth terms, while recalling that $\mathcal{M}^0_{N_{p(\cdot),w}}b_j$ is supported on $20nR_j$, we use Theorem 5.1 to deduce

$$\left\|\sum_{j=1}^{\infty}|\lambda_j|\chi_{2Q_j}M^{\rm loc}a_j\right\|_{L^{p(\cdot)}(w)} \lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j m^{(uv)}_{2Q_j,w}(M^{\rm loc}a_j)\}_{j=1}^{\infty};\{2Q_j\}_{j=1}^{\infty}) \tag{5.11}$$

and

$$\left\|\sum_{j=1}^{\infty}|\lambda_j'|\mathcal{M}^0_{N_{p(\cdot),w}}b_j\right\|_{L^{p(\cdot)}(w)} \lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j m^{(uv)}_{20nR_j,w}(M^{\rm loc}b_j)\}_{j=1}^{\infty};\{20nR_j\}_{j=1}^{\infty}). \tag{5.12}$$

We estimate $m^{(uv)}_{Q_j,w}(M^{\rm loc}a_j)$. Recall that $q>q_w$. Hence, $M^{\rm loc}$ is bounded on $L^q(w)$. Since we assume that $1<uv<q$, we choose $r\in(0,\infty)$ such that $\frac{1}{uv}=\frac{1}{q}+\frac{1}{r}$. Then, using the Hölder inequality and the condition of the $(p(\cdot),q,L)_w$-atom, we have

$$\begin{aligned}
m^{(uv)}_{2Q_j,w}(M^{\rm loc}a_j) &= \frac{1}{w(Q_j)^{\frac{1}{uv}}}\left\|\chi_{2Q_j}M^{\rm loc}a_j\right\|_{L^{uv}(w)}\\
&\le \frac{1}{w(Q_j)^{\frac{1}{uv}}}\left\|M^{\rm loc}a_j\right\|_{L^q(w)}\left\|\chi_{2Q_j}\right\|_{L^r(w)}\\
&\lesssim \frac{1}{w(Q_j)^{\frac{1}{uv}}}\left\|a_j\right\|_{L^q(w)}\left\|\chi_{2Q_j}\right\|_{L^r(w)} \le \frac{1}{w(Q_j)^{\frac{1}{uv}}}w(Q_j)^{\frac{1}{q}}w(Q_j)^{\frac{1}{r}}=1.
\end{aligned}$$

A geometric observation shows that $\chi_{2Q_j}\lesssim M^{\rm loc}\chi_{Q_j}$. By virtue of Lemma 2.11 and Theorem 5.1 along with the above estimate, we obtain

$$\begin{aligned}
\left\|\sum_{j=1}^{\infty}|\lambda_j|\chi_{2Q_j}M^{\rm loc}a_j\right\|_{L^{p(\cdot)}(w)} &\lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j m^{(u)}_{2Q_j,w}(M^{\rm loc}a_j)\}_{j=1}^{\infty};\{2Q_j\}_{j=1}^{\infty})\\
&\lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^{\infty};\{2Q_j\}_{j=1}^{\infty})\\
&\lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^{\infty};\{Q_j\}_{j=1}^{\infty}).
\end{aligned}$$

In total,

$$\left\|\sum_{j=1}^{\infty}|\lambda_j|\chi_{2Q_j}M^{\rm loc}a_j\right\|_{L^{p(\cdot)}(w)} \lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^{\infty};\{Q_j\}_{j=1}^{\infty}). \tag{5.13}$$

In a similar fashion, we estimate the right-hand side of (5.12). The result is

$$\left\| \sum_{j=1}^{\infty} |\lambda'_j| \mathcal{M}^0_{N_{p(\cdot),w}} b_j \right\|_{L^{p(\cdot)}(w)} \lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda'_j\}_{j=1}^{\infty}; \{R_j\}_{j=1}^{\infty}). \tag{5.14}$$

Combining (5.9), (5.10), (5.13) and (5.14), we obtain the desired result.

Let us reexamine the above proof to polish the conditions on atoms.

Remark 5.5. Let $v \in (0, p_-) \cap [0,1]$ and $s_0 \equiv \left[n\left(\frac{q_w}{v} - 1\right)\right]_+$. A close inspection of the proof shows that the condition on the atom a_j may be relaxed. It suffices to assume each a_j satisfies the pointwise estimate

$$|a_j| \le |a_j|\chi_{3Q_j} + (M^{\mathrm{loc}}\chi_{Q_j})^{\frac{n+s_0+1}{n}} \tag{5.15}$$

and the norm estimate $\|a_j\chi_{3Q_j}\|_{L^q(w)} \le w(Q_j)^{\frac{1}{q}}$ for $q < \infty$. In fact, using the same argument as Lemma 6.7 below, we can establish estimate (5.7) under a milder condition (5.15). Thus, it is not necessary to assume that each a_j is compactly supported. Instead, a weaker assumption (5.15) is sufficient.

5.3. **Proof of $h^{p(\cdot),q,L;v}(w) \hookleftarrow h^{p(\cdot)}(w)$.** Let us consider the decomposition following [55, pp. 461–462]. Set $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\lambda > 0$. We set

$$\Omega_\lambda \equiv \{x \in \mathbb{R}^n \,:\, \mathcal{M}_N f(x) > \lambda\}.$$

Here and below, consider a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ such that $w(\Omega_\lambda) < \infty$ for all $\lambda > 0$. From the Whitney decomposition $\{Q_k\}_{k\in K}$ of Ω_λ, a decomposition $\Omega_\lambda = \bigcup_{k\in K} Q_k$ such that

$$\mathrm{diam}(Q_k) \le 2^{-n-6}\mathrm{dist}(Q_k, \mathbb{R}^n \setminus \Omega_\lambda) \le 4\mathrm{diam}(Q_k) \quad (k \in K)$$

and that the overlapping property is satisfied as

$$\sum_{k\in K} \chi_{(1+2^{-n-10})Q_k} \lesssim 1.$$

Fix $k \in K$. Set

$$Q_k \equiv Q(x_k, l_k), \quad Q_k^* \equiv (1+2^{-n-10})Q_k. \tag{5.16}$$

Let $\xi \in C^\infty(\mathbb{R}^n)$ be a bump function such that $\chi_{Q(1+2^{-n-11})} \le \xi \le \chi_{Q(1+2^{-n-10})}$. Define $\xi_k \equiv \xi\left(\frac{\cdot - x_k}{l_k}\right)$ and $\eta_k \equiv \xi_k \div \sum_{l\in K} \xi_l$. Choose a polynomial $P_k \in \mathcal{P}_L(\mathbb{R}^n)$ so that $\langle f, Q\xi_k\rangle = \langle P_k, Q\xi_k\rangle$ for all $Q \in \mathcal{P}_L(\mathbb{R}^n)$. Set $b_k \equiv (f - P_k)\eta_k$ for each k and $g \equiv f - \sum_{k\in K} b_k$. Thus, $f = g + \sum_{k\in K} b_k$. We invoke the following estimate from [55], which uses the maximal operators $\mathcal{M}^0_N$ and $\mathcal{M}_N$.

Lemma 5.6. [55, Lemmas 4.3 and 4.5] *Let $L \in [0, N) \cap \mathbb{Z}$. Then there exists $D > 0$ such that, for all $k \in \mathbb{N}$,*

$$\mathcal{M}^0_N b_k \lesssim \chi_{Q_k^*}\mathcal{M}_N f + \lambda\chi_{(0,D)}(|Q_k|)(M^{\mathrm{loc}}\chi_{Q_k})^{\frac{n+L+1}{n}}.$$

Since we have the vector-valued boundedness of M^{loc} (see Lemma 2.11), taking the norm on both sides gives a norm estimate for the bad parts.

Corollary 5.7. *Let $w \in A^{\mathrm{loc}}_\infty$ and $p(\cdot) \in \mathcal{P}_0 \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Let $v \in (0, p_-)$ and $q > q_w$. Let $L, N \in \mathbb{Z}$ satisfy*

$$N \ge L \ge \left[n\left(\frac{q_w}{v} - 1\right)\right]. \tag{5.17}$$

Then

$$\left\| \left(\sum_{k \in K} (\mathcal{M}_N^0 b_k)^v \right)^{\frac{1}{v}} \right\|_{L^{p(\cdot)}(w)} \lesssim \|\mathcal{M}_N f\|_{L^{p(\cdot)}(w)}. \tag{5.18}$$

In particular, if $v \leq 1$, $\sum\limits_{k \in K} b_k$ converges in $h^{p(\cdot)}(w)$. Hence it converges also in $\mathcal{D}'(\mathbb{R}^n)$.

Proof. Using Lemma 5.6, we estimate

$$\begin{aligned}
&\left\| \left(\sum_{k \in K} (\mathcal{M}_N^0 b_k)^v \right)^{\frac{1}{v}} \right\|_{L^{p(\cdot)}(w)} \\
&\lesssim \left\| \left(\sum_{k \in K} \left(\chi_{Q_k^*} \mathcal{M}_N f + \lambda \chi_{(0,D)}(|Q_k|)(M^{\mathrm{loc}} \chi_{Q_k})^{\frac{n+L+1}{n}} \right)^v \right)^{\frac{1}{v}} \right\|_{L^{p(\cdot)}(w)}.
\end{aligned}$$

We remark that $p_- > v$. Using the constant sequence $\{\lambda\}_{k \in K}$, the triangle inequality, the vector-valued inequality (Lemma 2.11) and the fact that $\{Q_k\}_{k \in K}$ is a Whitney decomposition of $\Omega_\lambda = \{x \in \mathbb{R}^n : \mathcal{M}_N f(x) > \lambda\}$, we have

$$\begin{aligned}
&\left\| \left(\sum_{k \in K} (\mathcal{M}_N^0 b_k)^v \right)^{\frac{1}{v}} \right\|_{L^{p(\cdot)}(w)} \\
&\lesssim \left\| \left(\sum_{k \in K} \left(\chi_{Q_k^*} \mathcal{M}_N f \right)^v \right)^{\frac{1}{v}} \right\|_{L^{p(\cdot)}(w)} + \mathcal{A}_{p(\cdot),w,v}(\{\lambda\}_{k \in K}; \{Q_k\}_{k \in K}) \\
&\lesssim \|\mathcal{M}_N f\|_{L^{p(\cdot)}(w)},
\end{aligned}$$

proving (5.18).

Now assuming that $v \leq 1$, we can easily prove that $\sum\limits_{k \in K} b_k$ converges in $h^{p(\cdot)}(w)$ since

$$\left\| \sum_{k \in K} \mathcal{M}_N^0 b_k \right\|_{L^{p(\cdot)}(w)} \lesssim \left\| \left(\sum_{k \in K} (\mathcal{M}_N^0 b_k)^v \right)^{\frac{1}{v}} \right\|_{L^{p(\cdot)}(w)} \lesssim \|\mathcal{M}_N f\|_{L^{p(\cdot)}(w)}.$$

□

Concerning condition (5.17), a helpful remark is in order.

Remark 5.8. Let $q \geq 1$ and $p_0 > 0$. Due to the right-continuity of the function $v \in (0, \infty) \mapsto \left[n\left(\frac{q}{v} - 1\right)\right] \in \mathbb{R}$, for an integer L, there exists $v \in (0, p_0)$ such that $L \geq \left[n\left(\frac{q}{v} - 1\right)\right]_+$ if and only if $L \geq \left[n\left(\frac{q}{p_0} - 1\right)\right]_+$.

Having guaranteed the convergence of $\sum\limits_{k \in K} b_k$ in $\mathcal{D}'(\mathbb{R}^n)$, we can employ another estimate by Tang, which uses $\mathcal{M}_N^0$ and M^{loc} only.

Lemma 5.9. [55, Lemma 4.8] *Let $L \in [0, N) \cap \mathbb{Z}$. Then*

$$\mathcal{M}_N^0 g \lesssim \chi_{\mathbb{R}^n \setminus \Omega_\lambda} \mathcal{M}_N^0 f + \lambda \sum_{k \in K} (M^{\mathrm{loc}} \chi_{Q_k})^{\frac{n+L+1}{n}}.$$

A direct consequence of Lemma 5.9 is:

Lemma 5.10. *Let $w \in A_\infty^{\rm loc}$ and $p(\cdot) \in \mathcal{P}_0 \cap {\rm LH}_0 \cap {\rm LH}_\infty$. If $L \in [0,N)$, then $g \in L^{p_+ + q_w}(w)$.*

Proof. Using Lemmas 2.11 and 5.9 as well as the fact that $\{Q_k\}_{k\in K}$ is the Whitney decomposition of Ω_λ, we estimate

$$\begin{aligned}\|\mathcal{M}_N^0 g\|_{L^{p_+ + q_w}(w)} &\lesssim \left\| \chi_{\mathbb{R}^n\setminus\Omega_\lambda}\mathcal{M}_N^0 f + \lambda \sum_{k\in K} (M^{\rm loc}\chi_{Q_k})^{\frac{n+L+1}{n}} \right\|_{L^{p_+ + q_w}(w)} \\ &\lesssim \left\| \min\{\mathcal{M}_N^0 f, \lambda\}\right\|_{L^{p_+ + q_w}(w)}.\end{aligned}$$

Note that

$$\min\{\mathcal{M}_N^0 f, \lambda\} \in L^{p(\cdot)}(w) \cap L^\infty(\mathbb{R}^n) \subset L^{p_+ + q_w}(w).$$

In fact, with the implicit constant depending on λ,

$$\int_{\mathbb{R}^n} \min(\lambda, \mathcal{M}_N^0 f(x))^{p_+ + q_w} {\rm d}x \lesssim \int_{\mathbb{R}^n} \mathcal{M}_N^0 f(x)^{p(x)} {\rm d}x < \infty.$$

Since $\min(\lambda, \mathcal{M}_N^0 f) \in L^{p_+ + q_w}(w)$, $\mathcal{M}_N^0 g \in L^{p_+ + q_w}(w)$. Hence $g \in h^{p_+ + q_w}(w) = L^{p_+ + q_w}(w)$ thanks to Theorem 1.6 and the fact that $q_w \ge 1$ and $p_- > 0$. □

Next, let us look for a dense subspace of $h^{p(\cdot)}(w)$, which consists of regular distributions. Specifically, we are interested in distributions in $h^{p(\cdot)}(w)$ which are realized as a function in $L^u(w)$ for some $u \gg 1$. A certain dense subspace which is included in $L^0(\mathbb{R}^n)$ is needed to consider the wavelet characterization in Section 8 below.

Using the same argument as [55, Lemma 4.9], where Tang assumed $f \in L^1(w)$, then we see that g is essentially bounded if $f \in L^{p_+ + q_w}(w)$. We summarize this observation as follows:

Lemma 5.11. [55, Lemma 4.9] *Let $p(\cdot) \in \mathcal{P}_0 \cap {\rm LH}_0 \cap {\rm LH}_\infty$. If $N \ge 2$ and $f \in L^{p_+ + q_w}(w)$, then $g \in L^\infty(\mathbb{R}^n)$ and satisfies $\|g\|_{L^\infty} \lesssim \lambda$.*

Lemma 5.12. *Let $w \in A_\infty^{\rm loc}$ and $p(\cdot) \in \mathcal{P}_0 \cap {\rm LH}_0 \cap {\rm LH}_\infty$. Then the space $h^{p(\cdot)}(w) \cap L^{p_+ + q_w}(w) \cap L^\infty(\mathbb{R}^n)$ is dense in $h^{p(\cdot)}(w)$.*

Proof. Let $L \in \mathbb{N}_0$ satisfy $L \ge \left[n\left(\frac{q_w}{\min(1,p_-)} - 1\right)\right]$. Then

$$p_- \frac{n+L+1}{n} \ge \frac{p_-}{n}\left(n+1+\left[n\left(\frac{q_w}{\min(1,p_-)}-1\right)\right]\right) > q_w.$$

Let $f \in h^{p(\cdot)}(w)$ and $g = f - \sum_{k\in K} b_k$. By the subadditivity of $\mathcal{M}_N^0$,

$$\|f-g\|_{h^{p(\cdot)}(w)} = \|\mathcal{M}_N^0[f-g]\|_{L^{p(\cdot)}(w)} \lesssim \left\| \sum_{k\in K} \mathcal{M}_N^0 b_k \right\|_{L^{p(\cdot)}(w)}.$$

Recall that $\{Q_k\}_{k\in K}$ is the Whitney decomposition. Using Lemmas 2.11 and 5.6, the triangle inequality and the Hölder inequality, we obtain

$$\begin{aligned}\|f-g\|_{h^{p(\cdot)}(w)} &\lesssim \left\| \sum_{k\in K} \chi_{Q_k^*}\mathcal{M}_N f + \sum_{k\in K} \lambda\chi_{(0,D)}(|Q_k|)(M^{\rm loc}\chi_{Q_k})^{\frac{n+L+1}{n}} \right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \|\chi_{\Omega_\lambda}\mathcal{M}_N f + \chi_{\Omega_\lambda}\lambda\|_{L^{p(\cdot)}(w)} \\ &\lesssim \|\chi_{\Omega_\lambda}\mathcal{M}_N f\|_{L^{p(\cdot)}(w)}.\end{aligned}$$

If we let $\lambda \uparrow \infty$, then we obtain $g \to f$ in $h^{p(\cdot)}(w)$. Since $g \in L^{p_+ + q_w}(w) \cap L^\infty(\mathbb{R}^n) \cap h^{p(\cdot)}(w)$ thanks to Lemmas 5.10 and 5.11, we obtain the desired result. □

As Tang noted, if $w \in L^1(\mathbb{R}^n)$, there is a standard method to create single $(p(\cdot),\infty)_w$-atoms.

Lemma 5.13. [55, Lemma 5.4] *Let $p(\cdot) \in \mathcal{P}_0 \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Assume that $w(\mathbb{R}^n) < \infty$ and that $q > \max(q_w, p_+)$. Then there exists a constant $D_0 > 0$ with the following property: Suppose that $\lambda > 0$ satisfies $\lambda \le \inf\limits_{x \in \mathbb{R}^n} \mathcal{M}_N f(x) < 2\lambda$. Then $a_0 \equiv D_0 \lambda^{-1} g$ is a single $(p(\cdot),\infty)_w$-atom.*

With Lemmas 5.6–5.12 in mind, we prove $h^{p(\cdot)}(w) \hookrightarrow h^{p(\cdot),\infty,L;v}(w) (\hookrightarrow h^{p(\cdot),q,L;v}(w))$. We assume $w(\mathbb{R}^n) < \infty$ and that $2^{j_0} \le \inf\limits_{x \in \mathbb{R}^n} \mathcal{M}_N f(x) < 2^{j_0+1}$ for some $j_0 \in \mathbb{Z}$; otherwise we can readily modify the argument below. We use the above observation for $\lambda = 2^j$, where j ranges over $[j_0,\infty) \cap \mathbb{Z}$. We will add a subindex j to what we have obtained to indicate that it comes from Ω_{2^j}. Thus, we obtain cubes $\{Q_{j,k}\}_{k \in K_j}$, smooth functions $\{\eta_{j,k}\}_{k \in K_j}$ and polynomials $\{P_{j,k}\}_{k \in K_j}$. Then we have a decomposition $f = g_j + b_j$, where $b_{j,k} \equiv (f - P_{j,k})\eta_{j,k}$ and $b_j \equiv \sum\limits_{k \in K_j} b_{j,k}$. We write $\tilde{Q}^j_k \equiv (1 + 2^{-n-12}) Q^j_k$. We use the following observation:

Lemma 5.14. *Under the assumption above, we have*

$$\mathcal{A}_{p(\cdot),w,v}(\{\lambda^j_k\}_{j \in [j_0,\infty)\cap\mathbb{Z}, k \in K_j}; \{\tilde{Q}^j_k\}_{j \in [j_0,\infty)\cap\mathbb{Z}, k \in K_j}) \lesssim \|\mathcal{M}_N f\|_{L^{p(\cdot)}(w)}. \tag{5.19}$$

Proof. Due to the bounded overlapping property,

$$\begin{aligned}
&\mathcal{A}_{p(\cdot),w,v}(\{\lambda^j_k\}_{j \in [j_0,\infty)\cap\mathbb{Z}, k \in K_j}; \{\tilde{Q}^j_k\}_{j \in [j_0,\infty)\cap\mathbb{Z}, k \in K_j}) \\
&= \mathcal{A}_{p(\cdot),w,v}(\{2^j\}_{j \in [j_0,\infty)\cap\mathbb{Z}, k \in K_j}; \{\tilde{Q}^j_k\}_{j \in [j_0,\infty)\cap\mathbb{Z}, k \in K_j}) \\
&\lesssim \left\| \left(\sum_{j=j_0}^{\infty} 2^{jv} \chi_{\Omega_{2^j}} \right)^{\frac{1}{v}} \right\|_{L^{p(\cdot)}(w)}.
\end{aligned} \tag{5.20}$$

Let $x \in \mathbb{R}^n$. Then we have

$$\sum_{j=j_0}^{\infty} 2^{jv} \chi_{\Omega_{2^j}}(x) = \sum_{j \in \mathbb{Z} \cap [j_0,\infty) \cap (-\infty, \log_2 \mathcal{M}_N f(x))} 2^{jv} \le \sum_{j \in \mathbb{Z} \cap (-\infty, \log_2 \mathcal{M}_N f(x))} 2^{jv} \lesssim \mathcal{M}_N f(x)^v.$$

Thus, (5.19) follows from (5.20). □

Let us return to the proof of $h^{p(\cdot)}(w) \hookrightarrow h^{p(\cdot),\infty,L,v}(w)$. We follow the idea in [52], which allows us to assume $f \in h^{p(\cdot)}(w) \cap L^{p_+ + q_w}(w) \cap L^\infty(\mathbb{R}^n)$. Also, assume that $f \in h^{p(\cdot)}(w) \cap L^{p_+ + q_w}(w) \cap L^\infty(\mathbb{R}^n)$ keeping Lemma 5.12 in mind.

Let $D_0 > 0$ be a constant from Lemma 5.13. Using the same argument as [55, Lemma 5.4], we have a decomposition $f = g_{j_0} + \sum\limits_{j=j_0}^{\infty} \sum\limits_{k \in K_j} \lambda_{j,k} a_{j,k}$, where $D_0 2^{-j_0} g_{j_0}$ is a single $(p(\cdot),\infty)_w$-atom, each $a_{j,k}$ is a $(p(\cdot),\infty,L)_w$-atom supported on a cube $\tilde{Q}^j_k = (1+2^{-n-12}) Q^j_k$, and $\lambda^j_k = 2^j$. We set

$$K_j^- \equiv \{k \in K_j \,:\, |\tilde{Q}^j_k| < 1\}, \quad K_j^+ \equiv K_j \setminus K_j^-.$$

We write

$$X \equiv \{(j,k,l) \,:\, j \in [j_0,\infty) \cap \mathbb{Z}, k \in K_j^+, \tilde{Q}^j_k \cap (l + [0,1]^n) \ne \emptyset\}.$$

We further decompose

$$
\begin{aligned}
f &= g_{j_0} + \sum_{j=j_0}^{\infty} \sum_{k \in K_j^+} \lambda_{j,k} a_{j,k} + \sum_{j=j_0}^{\infty} \sum_{k \in K_j^-} \lambda_{j,k} a_{j,k} \\
&= g_{j_0} + \sum_{(j,k,l) \in X} \lambda_{j,k} \chi_{l+[0,1]^n} a_{j,k} + \sum_{j=j_0}^{\infty} \sum_{k \in K_j^-} \lambda_{j,k} a_{j,k}.
\end{aligned}
$$

We remark that each $\chi_{l+[0,1]^n} a_{j,k}$ is a $(p(\cdot), \infty, L)_w$-atom supported on a cube $l+[0,1]^n$ as long as $k \in K_j^+$ and $l \in \mathbb{Z}^n$ satisfies $\tilde{Q}_k^j \cap (l + [0,1]^n) \neq \emptyset$.

Let us prove the norm estimate. If $(j,k,l) \in X$, then $l + [0,1]^n \subset 3\tilde{Q}_k^j$. Since $\chi_{3\tilde{Q}_k^j} \lesssim M^{\rm loc}[\chi_{\tilde{Q}_k^j}]^{\frac{1}{v-\varepsilon}}$ for some $\varepsilon \in (0,v)$, we deduce from the vector-valued inequality (Lemma 2.11),

$$
\begin{aligned}
\left\| \left(\sum_{(j,k,l) \in X} (\lambda_{j,k} \chi_{l+[0,1]^n})^v \right)^{\frac{1}{v}} \right\|_{L^{p(\cdot)}(w)} &\lesssim \left\| \left(\sum_{(j,k,l) \in X} (\lambda_{j,k} M^{\rm loc}[\chi_{\tilde{Q}_k^j}]^{\frac{1}{v-\varepsilon}})^v \right)^{\frac{1}{v}} \right\|_{L^{p(\cdot)}(w)} \\
&\lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda_k^j\}_{j \in [j_0,\infty) \cap \mathbb{Z}, k \in K_j^+}; \{\tilde{Q}_k^j\}_{j \in [j_0,\infty) \cap \mathbb{Z}, k \in K_j^+}).
\end{aligned}
$$

Thus by Lemma 5.14, we have

$$
\begin{aligned}
\left\| \left(\sum_{(j,k,l) \in X} (\lambda_{j,k} \chi_{l+[0,1]^n})^v \right)^{\frac{1}{v}} \right\|_{L^{p(\cdot)}(w)} &\lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda_k^j\}_{j \in [j_0,\infty) \cap \mathbb{Z}, k \in K_j}; \{\tilde{Q}_k^j\}_{j \in [j_0,\infty) \cap \mathbb{Z}, k \in K_j}) \\
&\lesssim \|\mathcal{M}_N f\|_{L^{p(\cdot)}(w)}. \qquad (5.21)
\end{aligned}
$$

Meanwhile, from Lemma 5.14, we obtain

$$
\begin{aligned}
&\mathcal{A}_{p(\cdot),w,v}(\{\lambda_k^j\}_{j \in [j_0,\infty) \cap \mathbb{Z}, k \in K_j^-}; \{\tilde{Q}_k^j\}_{j \in [j_0,\infty) \cap \mathbb{Z}, k \in K_j^-}) \qquad (5.22) \\
&\lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda_k^j\}_{j \in [j_0,\infty) \cap \mathbb{Z}, k \in K_j}; \{\tilde{Q}_k^j\}_{j \in [j_0,\infty) \cap \mathbb{Z}, k \in K_j}) \\
&\lesssim \|\mathcal{M}_N f\|_{L^{p(\cdot)}(w)}.
\end{aligned}
$$

Combining (5.21) and (5.22), we obtain the desired norm estimate $\|f\|_{h^{p(\cdot),\infty,L,v}(w)} \lesssim \|f\|_{h^{p(\cdot)}(w)}$.

6. Applications to singular integral operators

Now that the structure of the weighted local Hardy space $h^{p(\cdot)}(w)$ is clarified, we present some applications. In Section 6.1, we establish that generalized local singular integral operators considered in [45] are bounded from $h^{p(\cdot)}(w)$ to $L^{p(\cdot)}(w)$ as long as $p_- > \frac{n}{n+1}$. Section 6.2 is devoted to a special case of Section 6.1. We are interested in removing the condition $p_- > \frac{n}{n+1}$ by considering a narrower but important class of operators. Among the generalized local singular integral operators, we consider the convolution operators generated by compactly supported smooth functions.

We make a brief remark on the method of the proof used in Section 6. There are several ways to prove the boundedness of singular integral operators from Hardy spaces to Hardy spaces. Fefferman and Stein investigated the boundedness of singular integral operators by investigating the distribution function of the image by singular integral operator [16, Lemma 11 and Theorem 12]. We can not employ the method in [16] since we are considering function spaces which is not rearrangement invariant. Our method is to use the atomic decomposition as García-Cuerva and Rubio de Francia [19] and Tang [55] did. García-Cuerva and Rubio de Francia also considered the boundedness property of singular integral operators [19, Theorems

7.8 and 7.9]. They used the atomic decomposition. We can say that our method is akin to theirs. See the proof of Theorem 6.4. We also remark that Tang took the same strategy, where he also analyzed the image of atoms [55, Theorem 7.1]. What is different from [19, 55] is that we must take care of the position of the cubes on which atoms are supported by using Lemma 2.11. This approach is taken in [43]. However, since we need to consider the local grand maximal operators, we can not employ the estimate directly. What we do is to adjust what we did in [43, §5.1].

6.1. **Generalized local singular integral operators.** An L^2-bounded linear operator T is called a generalized local Calderón–Zygmund operator (with the kernel K), if T satisfies the following conditions:

(1) There exists $K \in L^1_{\rm loc}(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) \,:\, x \in \mathbb{R}^n\})$ such that, for all $f \in L^2_{\rm c}(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y){\rm d}y \text{ for almost all } x \notin {\rm supp}(f). \tag{6.1}$$

(2) There exist positive constants γ_0, $D_1 = D_1(T)$ and $D_2 = D_2(T)$ such that the two conditions below hold for all $x, y, z \in \mathbb{R}^n$:
  (*i*) Local size condition:

$$|K(x,y)| \le D_1|x-y|^{-n}\chi_{[-\gamma_0,\gamma_0]^n}(x-y) \tag{6.2}$$

  if $x \neq y$.
  (*ii*) Hörmander's condition:

$$|K(x,z) - K(y,z)| + |K(z,x) - K(z,y)| \le D_2 \frac{|x-y|}{|x-z|^{n+1}} \tag{6.3}$$

  if $0 < 2|x-y| < |z-x|$.

This is analogous to the generalized singular integral operators dealt with in [15], which require

$$|K(x,y)| \le D_1|x-y|^{-n} \tag{6.4}$$

instead of (6.2) if $x \neq y$. [30] shows that all generalized local singular integral operators initially defined on $L^2(\mathbb{R}^n)$ can be extended to a bounded linear operator on $L^{p(\cdot)}(w)$ for any $p(\cdot) \in \mathcal{P} \cap {\rm LH}_0 \cap {\rm LH}_\infty$ and $w \in A^{\rm loc}_{p(\cdot)}$. Recall that such generalized local singular integral operators are bounded on $L^{p(\cdot)}(w)$.

Proposition 6.1. *Suppose that $p(\cdot) \in \mathcal{P} \cap {\rm LH}_0 \cap {\rm LH}_\infty$. Let T be a generalized local singular integral operator and $w \in A^{\rm loc}_{p(\cdot)}$. Then T extends to a bounded linear operator on $L^{p(\cdot)}(w)$ with the norm estimate*

$$\|T\|_{L^{p(\cdot)}(w) \to L^{p(\cdot)}(w)} \lesssim \|T\|_{L^2 \to L^2} + D_1(T) + D_2(T).$$

Proposition 6.1 was proved using the local sharp maximal operator considered in [40]. We extend Proposition 6.1 to weighted local Hardy spaces with variable exponents and investigate how generalized local singular integral operators act on atoms. If we reexamine the proof of [43, (5.2)], then we see that the following pointwise estimate holds.

Lemma 6.2. *Suppose that $p(\cdot) \in \mathcal{P}_0 \cap {\rm LH}_0 \cap {\rm LH}_\infty$. Let T be a generalized local singular integral operator. Then any $(p(\cdot), \infty, 1)_w$-atom a supported on Q satisfies*

$$|Ta(x)| \lesssim |Ta(x)|\chi_{3Q}(x) + D_2(T) M^{\rm loc}\chi_Q(x)^{\frac{n+1}{n}}$$

for all $x \in \mathbb{R}^n$.

Proof. We must consider two cases: $x \in 3Q$, $x \in R\setminus Q$, where R is the cube of volume $(2+2\gamma_0)^n$ concentric to Q. For the first case, there is nothing to prove. We use the Hörmander's condition for the second case. $\square$

A direct consequence of Lemma 6.2 is that generalized local singular integral operators are bounded from $h^{p(\cdot)}(w)$ to $L^{p(\cdot)}(w)$ as long as $p_- > \frac{n}{n+1}$, extending Proposition 6.1 in terms of $h^{p(\cdot)}(w)$ considered in this paper.

Theorem 6.3. *Suppose that $p(\cdot) \in \mathcal{P}_0 \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ satisfies $p_- > \frac{n}{n+1}$. Let T be a generalized local singular integral operator, and let $w \in A_\infty^{\rm loc}$. Then T is bounded from $h^{p(\cdot)}(w)$ to $L^{p(\cdot)}(w)$ with the norm estimate*

$$\|T\|_{h^{p(\cdot)}(w)\to L^{p(\cdot)}(w)} \lesssim \|T\|_{L^2\to L^2} + D_1(T) + D_2(T).$$

Proof. We assume that $w \in L^1(\mathbb{R}^n)$: If $w \notin L^1(\mathbb{R}^n)$, then we can modify the proof below. It suffices to show that

$$\|Tf\|_{L^{p(\cdot)}(w)} \lesssim (\|T\|_{L^2\to L^2} + D_1(T) + D_2(T))\|f\|_{h^{p(\cdot)}(w)}$$

for all $f \in h^{p(\cdot)}(w) \cap L^{p_++q_w}(w) \cap L^\infty(\mathbb{R}^n)$ thanks to Lemma 5.12. Let q satisfy $p_+ + q_w + 1 < q < \infty$ and $L \gg 1$. Let $f \in h^{p(\cdot)}(w) \cap L^{p_++q_w}(w) \cap L^\infty(\mathbb{R}^n)$. Due to Theorem 1.5, there exist $\{a_j\}_{j=0}^\infty \subset L^0(\mathbb{R}^n)$, $\{b_j\}_{j=1}^\infty \subset L^0(\mathbb{R}^n)$, $\{\lambda_j\}_{j=0}^\infty \subset [0,\infty)$ and $\{\lambda'_j\}_{j=1}^\infty \subset [0,\infty)$ such that a_0 is a single $(p(\cdot),q)_w$-atom, that each a_j, $j \in \mathbb{N}$ is a $(p(\cdot),q,L)_w$-atom supported on a cube Q_j with $|Q_j| < 1$, that each b_j, $j \in \mathbb{N}$ is a $(p(\cdot),q,L)_w$-atom supported on a cube R_j with $|R_j| = 1$, that $f = \sum\limits_{j=0}^\infty \lambda_j a_j + \sum\limits_{j=1}^\infty \lambda'_j b_j$ holds in the topology of $h^{p(\cdot)}(w)$ and that

$$|\lambda_0| + \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^\infty;\{Q_j\}_{j=1}^\infty) + \mathcal{A}_{p(\cdot),w,v}(\{\lambda'_j\}_{j=1}^\infty;\{R_j\}_{j=1}^\infty) \lesssim \|f\|_{h^{p(\cdot)}(w)}.$$

Again using Theorem 1.5, we have $f = \sum\limits_{j=0}^\infty \lambda_j a_j + \sum\limits_{j=1}^\infty \lambda'_j b_j$ holds in the topology of $h^{p_++q_w}(w)$, especially in the topology of $L^{p_++q_w}(w)$ by Theorem 1.6.

We know that T maps $L^{p_++q_w}(w)$ continuously to itself by Proposition 6.1. Thus, $Tf = \sum\limits_{j=0}^\infty \lambda_j Ta_j + \sum\limits_{j=1}^\infty \lambda'_j Tb_j$ holds in the topology of $L^{p_++q_w}(w)$. Due to Lemma 6.2 and the triangle inequality, we have

$$\begin{aligned}\|Tf\|_{L^{p(\cdot)}(w)} &\lesssim \left\|\sum_{j=1}^\infty(\lambda_j|Ta_j|\chi_{3Q_j} + D_2(T)(M^{\rm loc}\chi_{Q_j})^{\frac{n+1}{n}})\right\|_{L^{p(\cdot)}(w)}\\ &\quad+\left\|\sum_{j=1}^\infty \lambda'_j|Tb_j|\right\|_{L^{p(\cdot)}(w)} + |\lambda_0|\|Ta_0\|_{L^{p(\cdot)}(w)}\\ &\lesssim \left\|\sum_{j=1}^\infty \lambda_j|Ta_j|\chi_{3Q_j}\right\|_{L^{p(\cdot)}(w)} + \left\|\sum_{j=1}^\infty \lambda'_j|Tb_j|\right\|_{L^{p(\cdot)}(w)}\\ &\quad+D_2(T)\left\|\sum_{j=1}^\infty \lambda_j(M^{\rm loc}\chi_{Q_j})^{\frac{n+1}{n}}\right\|_{L^{p(\cdot)}(w)} + |\lambda_0|\|Ta_0\|_{L^{p(\cdot)}(w)}.\end{aligned} \tag{6.5}$$

We choose $u, v > 0$ and $L \in \mathbb{Z}$ so that

$$v < \min(1,p_-), \quad L > \left[n\left(\frac{q_w}{v}-1\right)\right], \quad \max(q_w,p_+) < uv < q.$$

For the first term we use Theorem 5.1 to give

$$\left\|\sum_{j=1}^{\infty}\lambda_j|Ta_j|\chi_{3Q_j}\right\|_{L^{p(\cdot)}(w)} \lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j m^{(uv)}_{3Q_j,w}(Ta_j)\}_{j=1}^{\infty};\{3Q_j\}_{j=1}^{\infty}).$$

Recall that $uv > q_w$ and that $w \in \bigcup\limits_{\tilde{q}>q_w} A^{\rm loc}_{\tilde{q}}$. Since T is bounded on $L^{uv}(w)$ (see Proposition 6.1),

$$\begin{aligned}&\mathcal{A}_{p(\cdot),w,v}(\{\lambda_j m^{(uv)}_{3Q_j,w}(Ta_j)\}_{j=1}^{\infty};\{3Q_j\}_{j=1}^{\infty})\\ &\lesssim (\|T\|_{L^2\to L^2}+D_1(T)+D_2(T))\mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^{\infty};\{3Q_j\}_{j=1}^{\infty}).\end{aligned}$$

Using $\chi_{3Q_j} \lesssim M^{\rm loc}\chi_{Q_j}$ and the vector-valued inequality (Lemma 2.11), we have

$$|\lambda_0|+\mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^{\infty};\{3Q_j\}_{j=1}^{\infty}) \lesssim |\lambda_0|+\mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^{\infty};\{Q_j\}_{j=1}^{\infty}) \lesssim \|f\|_{h^{p(\cdot)}(w)}. \tag{6.6}$$

Thus, the estimate for the first term of (6.5) is valid.

The second term of (6.5) can be handled similarly to the first term. The result is

$$\left\|\sum_{j=1}^{\infty}\lambda'_j|Tb_j|\right\|_{L^{p(\cdot)}(w)} \lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda'_j\}_{j=1}^{\infty};\{R_j\}_{j=1}^{\infty}) \lesssim \|f\|_{h^{p(\cdot)}(w)}. \tag{6.7}$$

The third term of (6.5) is easy to deal with. As before, by the condition $0 < v \le 1$ and the vector-valued inequality (Lemma 2.11)

$$\begin{aligned}\left\|\sum_{j=1}^{\infty}\lambda_j(M^{\rm loc}\chi_{Q_j})^{\frac{n+1}{n}}\right\|_{L^{p(\cdot)}(w)} &\lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^{\infty};\{Q_j\}_{j=1}^{\infty})\\ &\le \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^{\infty};\{Q_j\}_{j=1}^{\infty}) \lesssim \|f\|_{h^{p(\cdot)}(w)}.\end{aligned} \tag{6.8}$$

Combining (6.6), (6.7) and (6.8) with $\|Ta_0\|_{L^{p(\cdot)}(w)} \lesssim 1$, we obtain the desired result. □

6.2. **Singular integral operators of the convolution type.** Theorem 6.3 estimates the integral kernel K only up to order 1. Here we consider the case where the kernel is smoother. To avoid the bothersome argument of justifying the definition of $Tf = k*f$ for $f \in h^{p(\cdot)}(w)$, we assume that $k \in C^{\infty}_{\rm c}(\mathbb{R}^n)$. Nevertheless, this assumption can be removed by a routine limiting argument, which we omit. Here we establish the following theorem.

Theorem 6.4. *Let $p(\cdot) \in \mathcal{P}_0 \cap {\rm LH}_0 \cap {\rm LH}_\infty$ and $w \in A^{\rm loc}_\infty$. Let $\{B_m\}_{m=0}^{\infty}$ be a positive sequence and $\gamma_0 > 0$. Let $k \in \mathcal{S}(\mathbb{R}^n)$ satisfy*

$$|x|^{n+m}|\nabla^m k(x)| \le B_m\chi_{[-\gamma_0,\gamma_0]^n}(x) \quad (x \in \mathbb{R}^n, m \in \mathbb{N}_0).$$

Define a convolution operator T by

$$Tf \equiv k*f \quad (f \in L^2(\mathbb{R}^n)).$$

Then T is an $h^{p(\cdot)}(w)$-bounded operator and the norm depends only on $\|\mathcal{F}k\|_{L^\infty}$ and a finite number of collections $B_0, B_1, \dots, B_N$ with $N \in \mathbb{N}$ depending only on $p(\cdot)$.

As we did in [57, §2.5.8] and [43, §5.3], the boundedness property of singular integral operators is useful for the Littlewood–Paley characterization. It matters that the estimate does not depend on $\|k\|_{L^1}$. This is absolutely necessary in Proposition 7.3.

The proof of Theorem 6.4 uses the following observations:

Lemma 6.5. *Let $p(\cdot) \in \mathcal{P}_0 \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and $w \in A_\infty^{\mathrm{loc}}$. Also let $L \in \mathbb{N}$ and T be the bounded linear operator on $L^2(\mathbb{R}^n)$ as in Theorem 6.4. Then any $(p(\cdot), \infty, L)_w$-atom a supported on Q satisfies $Ta \in \mathcal{P}_L^\perp(\mathbb{R}^n)$ and*

$$|Ta(x)| \lesssim |Ta(x)|\chi_{3Q}(x) + \left(\sum_{j=0}^{L} B_j\right) M^{\mathrm{loc}}\chi_Q(x)^{\frac{n+L+1}{n}} \quad (x \in \mathbb{R}^n).$$

Proof. It can be easily verified that $Ta \in \mathcal{P}_L^\perp(\mathbb{R}^n)$ using the moment condition of a. Due to [43, Propositions 5.3 and 5.4], we have

$$|Ta(x)| \lesssim |Ta(x)|\chi_{3Q}(x) + \left(\sum_{j=0}^{L} B_j\right) M\chi_Q(x)^{\frac{n+L+1}{n}}$$

for some $L \in \mathbb{N}$ depending only on $p(\cdot)$. Since a is supported on a cube with $|Q| \le 1$ and $\mathrm{supp}(k) \subset [-\gamma_0, \gamma_0]^n$, we see that Tf is supported on a cube with a volume less than or equal to $(2+2\gamma_0)^n$. Thus, we can replace the maximal operator by M^{loc}. □

Lemma 6.6. *Let $p(\cdot) \in \mathcal{P}_0 \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and $w \in A_\infty^{\mathrm{loc}}$. Also let $L \in \mathbb{N}$ and T be the bounded linear operator on $L^2(\mathbb{R}^n)$ as in Theorem 6.4. Assume that a is a $(p(\cdot), \infty, L)_w$-atom supported on a cube Q with $|Q| \le 1$. Then*

$$|Ta|\chi_{\mathbb{R}^n \setminus 5Q} \lesssim (M^{\mathrm{loc}}\chi_Q)^{\frac{n+L+1}{n}}$$

Proof. Let $x \in \mathbb{R}^n \setminus 5Q$. Then

$$Ta(x) = \int_{\mathbb{R}^n} \left(k(x-y) - \sum_{\alpha \in \mathbb{N}_0{}^n, |\alpha| \le L} \frac{\partial^\alpha k(x-x_0)}{\alpha!}(x_0-y)^\alpha\right) a(y)dy.$$

By the mean value theorem, there exist $\theta \in (0,1)$, which depends on x, y, x_0, L such that

$$\begin{aligned} &k(x-y) - \sum_{\alpha \in \mathbb{N}_0{}^n, |\alpha| \le L} \frac{\partial^\alpha k(x-x_0)}{\alpha!}(x_0-y)^\alpha \\ &= \sum_{\alpha \in \mathbb{N}_0{}^n, |\alpha| = L+1} \frac{\partial^\alpha k(x-x_0+\theta(x_0-y))}{\alpha!}(x_0-y)^\alpha. \end{aligned}$$

Hence

$$\begin{aligned} &|x-y|^{n+L+1}\left|k(x-y) - \sum_{\alpha \in \mathbb{N}_0{}^n, |\alpha| \le L} \frac{\partial^\alpha k(x-x_0)}{\alpha!}(x_0-y)^\alpha\right| \\ &\lesssim \ell(Q)^{L+1} \sup_{\alpha \in \mathbb{N}_0{}^n, |\alpha| = L+1} |z|^{n+L+1}|\partial^\alpha k(z)| \end{aligned}$$

for some $z \in \mathbb{R}^n$. Thus, we obtain

$$|Ta(x)| \lesssim \ell(Q)^{n+L+1}|x-c(Q)|^{-n-L-1}$$

Since Ta is supported on $Q(x_0, 2+2\gamma_0)$, we have the desired result. □

Fix $L \in \mathbb{N}$, a cube Q and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We define $P_{L,Q}f$ to be the unique polynomial of order L such that

$$\int_Q x^\beta(f(x) - P_{L,Q}f(x))\mathrm{d}x = 0$$

for all $\beta \in \mathbb{N}_0$ with $|\beta| \leq L$. If $Q = a + [0,r]^n$ for some $a \in \mathbb{R}^n$ and $r > 0$, then

$$P_{L,Q}f = P_{L,[0,1]^n}[f(a + r\cdot)]\left(\frac{\cdot - a}{r}\right).$$

Thus, we have

$$\|P_{L,Q}f\|_{L^\infty(Q)} \lesssim |Q|^{-\frac{1}{2}}\|f\|_{L^2(Q)}. \tag{6.9}$$

Similar to Lemma 5.4, we have the following pointwise estimate:

Lemma 6.7. *Let $p(\cdot) \in \mathcal{P}_0 \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and $w \in A_\infty^{\mathrm{loc}}$. Also let $L \in \mathbb{N}$ and T be the bounded linear operator on $L^2(\mathbb{R}^n)$ as in Theorem 6.4. Let a be a $(p(\cdot), \infty, 2L+2n+2)_w$-atom supported on Q. Under the assumption of Theorem 6.4,*

$$\mathcal{M}^0_{N_{p(\cdot),w}}[k * a](x) \lesssim \mathcal{M}^0_{N_{p(\cdot),w}}[\chi_{5Q}(k * a - P_{2L+2n+2,5Q}[k * a])](x) + M^{\mathrm{loc}}\chi_Q(x)^{\frac{n+L+1}{n}}$$

for all $x \in \mathbb{R}^n \setminus 5Q$.

Proof. Denote by x_0 the center of Q. Let $t \in (0,1)$ be fixed. Let $\varphi \in \mathcal{D}^0_{N_{p(\cdot),w}}(\mathbb{R}^n)$.

We have

$$k * a = \chi_{5Q}(k * a - P_{2L+2n+2,5Q}[k * a]) + \chi_{5Q}P_{2L+2n+2,5Q}[k * a] + \chi_{\mathbb{R}^n \setminus 5Q}k * a.$$

Therefore, it suffices to show that

$$\mathcal{M}^0_{N_{p(\cdot),w}}[\chi_{5Q}P_{2L+2n+2,5Q}[k * a] + \chi_{\mathbb{R}^n \setminus 5Q}k * a](x) \lesssim M^{\mathrm{loc}}\chi_Q(x)^{\frac{n+L+1}{n}}.$$

Note that

$$\chi_{5Q}P_{2L+2n+2,5Q}[k * a] + \chi_{\mathbb{R}^n \setminus 5Q}k * a \in \mathcal{P}^\perp_{2L+2n+2}(\mathbb{R}^n).$$

By using Lemma 2.22 twice (for the case $t \leq \ell(Q)$ and $t \geq \ell(Q)$) and Lemmas 6.6 and (6.9), we have

$$\begin{aligned}
&|\varphi_t * (\chi_{5Q}P_{2L+2n+2,5Q}[k * a] + \chi_{\mathbb{R}^n \setminus 5Q}k * a)(x)| \\
&\lesssim \min\left(1, \frac{\ell(Q)}{t}\right)^{2L+2n+3} \frac{\ell(Q)^n \max(t, \ell(Q))^{-n}}{1 + \max(t, \ell(Q))^{-L-n-1}|x - x_0|^{n+L+1}} \\
&= \min\left(1, \frac{\ell(Q)}{t}\right)^{2L+2n+3} \frac{\ell(Q)^n \max(t, \ell(Q))^{-n}}{1 + \min(1, t^{-1}\ell(Q))^{n+L+1}\ell(Q)^{-n-L-1}|x - x_0|^{n+L+1}} \\
&\leq \min\left(1, \frac{\ell(Q)}{t}\right)^{2L+2n+3} \frac{\ell(Q)^n \max(t, \ell(Q))^{-n}}{\min(1, t^{-1}\ell(Q))^{n+L+1}(1 + \ell(Q)^{-n-L-1}|x - x_0|^{n+L+1})} \\
&= \min\left(1, \frac{\ell(Q)}{t}\right)^{L+n+2} \frac{\ell(Q)^n \max(t, \ell(Q))^{-n}}{1 + \ell(Q)^{-n-L-1}|x - x_0|^{n+L+1}} \\
&\leq \frac{1}{1 + \ell(Q)^{-n-L-1}|x - x_0|^{n+L+1}}.
\end{aligned}$$

Therefore,

$$\sup_{0<t<1} |\varphi_t * (\chi_{5Q}P_{2L+2n+2,5Q}[k * a] + \chi_{\mathbb{R}^n \setminus 5Q}k * a)(x)| \lesssim \frac{1}{1 + \ell(Q)^{-n-L-1}|x - x_0|^{n+L+1}}.$$

Since

$$\mathrm{supp}(\mathcal{M}^0_{N_{p(\cdot),w}}[\chi_{5Q}P_{2L+2n+2,5Q}[k * a] + \chi_{\mathbb{R}^n \setminus 5Q}k * a]) \subset Q(x_0, 2 + 2\gamma_0)$$

and

$$\frac{\chi_{Q(x_0,2+2\gamma_0)}(x)}{1 + \ell(Q)^{-n-L-1}|x - x_0|^{n+L+1}} \lesssim M^{\mathrm{loc}}\chi_Q(x)^{\frac{n+L+1}{n}},$$

we obtain the desired result. □

Proof of Theorem 6.4. We assume $w \in L^1(\mathbb{R}^n)$; otherwise we can readily modify the proof below. It suffices to show that

$$\|Tf\|_{h^{p(\cdot)}(w)} \lesssim \left(\|\mathcal{F}k\|_{L^\infty} + \sum_{j=0}^{L} B_j\right) \|f\|_{h^{p(\cdot)}(w)}$$

for all $f \in h^{p(\cdot)}(w) \cap L^{p_+ + q_w}(w) \cap L^\infty(\mathbb{R}^n)$ thanks to Lemma 5.12.

Let $f \in h^{p(\cdot)}(w) \cap L^{p_+ + q_w}(w) \cap L^\infty(\mathbb{R}^n)$ and fix $L \gg n+1$. Due to Theorem 1.5, there exist $\{a_j\}_{j=0}^\infty \subset L^0(\mathbb{R}^n)$, $\{b_j\}_{j=1}^\infty \subset L^0(\mathbb{R}^n)$, $\{\lambda_j\}_{j=0}^\infty \subset [0,\infty)$ and $\{\lambda'_j\}_{j=1}^\infty \subset [0,\infty)$ such that a_0 is a single $(p(\cdot),q)_w$-atom, that each $a_j, j \in \mathbb{N}$ is a $(p(\cdot),\infty,L)_w$-atom supported on a cube Q_j with $|Q_j| < 1$, that each $b_j, j \in \mathbb{N}$ is a $(p(\cdot),\infty,L)_w$-atom supported on a cube R_j with $|R_j| = 1$, that $f = \sum\limits_{j=0}^\infty \lambda_j a_j + \sum\limits_{j=1}^\infty \lambda'_j b_j$ holds in the topology of $h^{p(\cdot)}(w) \cap L^{p_+ + q_w}(w)$, and that

$$|\lambda_0| + \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^\infty; \{Q_j\}_{j=1}^\infty) + \mathcal{A}_{p(\cdot),w,v}(\{\lambda'_j\}_{j=1}^\infty; \{R_j\}_{j=1}^\infty) \lesssim \|f\|_{h^{p(\cdot)}(w)}. \tag{6.10}$$

According to the famous Calderón–Zygmund theory ([21, 22, 49, 52]),

$$\|T\|_{L^2 \to L^2} + D_1(T) + D_2(T) \lesssim \|\mathcal{F}k\|_{L^\infty} + \sum_{j=0}^{L} B_j$$

and that T maps $L^{p_+ + q_w}(w)$ continuously to itself and satisfies

$$\|T\|_{L^{p_+ + q_w}(w) \to L^{p_+ + q_w}(w)} \lesssim \|\mathcal{F}k\|_{L^\infty} + \sum_{j=0}^{L} B_j$$

thanks to Lemma 6.1. Thus, $Tf = \sum\limits_{j=0}^\infty \lambda_j T a_j + \sum\limits_{j=1}^\infty \lambda'_j T b_j$ holds in the topology of $L^{p_+ + q_w}(w)$.

We put

$$\mathrm{I}_1 := \left\|\sum_{j=0}^\infty \lambda_j \chi_{10Q_j} \mathcal{M}^0_{N_{p(\cdot),w}}[Ta_j]\right\|_{L^{p(\cdot)}(w)}, \quad \mathrm{I}_2 := \left\|\sum_{j=1}^\infty \lambda'_j \mathcal{M}^0_{N_{p(\cdot),w}}[Tb_j]\right\|_{L^{p(\cdot)}(w)}$$

and

$$\mathrm{II} := \left\|\sum_{j=0}^\infty \lambda_j \chi_{\mathbb{R}^n \setminus 10Q_j} \mathcal{M}^0_{N_{p(\cdot),w}}[Ta_j]\right\|_{h^{p(\cdot)}(w)}.$$

Set

$$v := \frac{\min(1,p_-)}{2}, \quad uv > \max(p_+, q_w).$$

Then,

$$\begin{aligned}
\|Tf\|_{h^{p(\cdot)}(w)} &= \left\|\sum_{j=0}^\infty \lambda_j T a_j + \sum_{j=1}^\infty \lambda'_j T b_j\right\|_{h^{p(\cdot)}(w)} \\
&\lesssim \left\|\sum_{j=0}^\infty \lambda_j T a_j\right\|_{h^{p(\cdot)}(w)} + \left\|\sum_{j=1}^\infty \lambda'_j T b_j\right\|_{h^{p(\cdot)}(w)} \\
&= \left\|\sum_{j=0}^\infty \lambda_j \mathcal{M}^0_{N_{p(\cdot),w}}[Ta_j]\right\|_{L^{p(\cdot)}(w)} + \left\|\sum_{j=1}^\infty \lambda'_j \mathcal{M}^0_{N_{p(\cdot),w}}[Tb_j]\right\|_{L^{p(\cdot)}(w)} \\
&\lesssim \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{II}.
\end{aligned}$$

We employ Lemma 2.11 and Theorem 5.1 for I_1 to give

$$\begin{aligned}\mathrm{I}_1 &\le \left\| \left(\sum_{j=0}^{\infty} (\lambda_j \chi_{10Q_j} \mathcal{M}^0_{N_{p(\cdot),w}}[Ta_j])^v \right)^{\frac{1}{v}} \right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j m^{(uv)}_{Q_j,w}(\chi_{10Q_j}\mathcal{M}^0_{N_{p(\cdot),w}}[Ta_j])\}_{j=0}^{\infty}; \{Q_j\}_{j=0}^{\infty}).\end{aligned}$$

Similar to Lemma 5.3, since $|a_j| \le \chi_{Q_j}$ and $w \in A^{\rm loc}_{uv}$, we have

$$m^{(uv)}_{Q_j,w}(\chi_{10Q_j}\mathcal{M}^0_{N_{p(\cdot),w}}[Ta_j]) \lesssim \frac{\|a_j\|_{L^{uv}(w)}}{\|\chi_{Q_j}\|_{L^{uv}(w)}} \lesssim 1.$$

Hence,

$$\mathrm{I}_1 \lesssim |\lambda_0| + \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^{\infty}; \{Q_j\}_{j=1}^{\infty}).$$

Meanwhile, by Theorem 5.1 and Lemma 6.5, I_2 is estimated as

$$\begin{aligned}\mathrm{I}_2 &\lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda'_j m^{(uv)}_{R_j,w}(\mathcal{M}^0_{N_{p(\cdot),w}}[Tb_j])\}_{j=0}^{\infty}; \{R_j\}_{j=0}^{\infty}) \\ &\lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda'_j m^{(uv)}_{R_j,w}(M^{\rm loc}[Tb_j])\}_{j=0}^{\infty}; \{R_j\}_{j=0}^{\infty}) \\ &\lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda'_j m^{(uv)}_{R_j,w}(M^{\rm loc}[(Tb_j)\chi_{3R_j}])\}_{j=1}^{\infty}; \{R_j\}_{j=1}^{\infty}) \\ &\quad + \mathcal{A}_{p(\cdot),w,v}(\{\lambda'_j m^{(uv)}_{R_j,w}(M^{\rm loc}[(M^{\rm loc}\chi_{R_j})^{\frac{n+L+1}{n}}])\}_{j=1}^{\infty}; \{R_j\}_{j=1}^{\infty}) \\ &\lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda'_j\}_{j=1}^{\infty}; \{R_j\}_{j=1}^{\infty}).\end{aligned}$$

Consequently, we have

$$\mathrm{I}_1 + \mathrm{I}_2 \lesssim |\lambda_0| + \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=1}^{\infty}; \{Q_j\}_{j=1}^{\infty}) + \mathcal{A}_{p(\cdot),w,v}(\{\lambda'_j\}_{j=1}^{\infty}; \{R_j\}_{j=1}^{\infty}). \tag{6.11}$$

We employ Lemmas 2.11, 5.4 and 6.7 for II to give

$$\begin{aligned}\mathrm{II} &\lesssim \left\| \sum_{j=0}^{\infty} \lambda_j \chi_{\mathbb{R}^n \setminus 10Q_j} (M^{\rm loc}\chi_{Q_j})^{\frac{n+L+1}{n}} \right\|_{L^{p(\cdot)}(w)} \\ &\quad + \left\| \sum_{j=0}^{\infty} \lambda_j \chi_{\mathbb{R}^n \setminus 10Q_j} \mathcal{M}^0_{N_{p(\cdot),w}}[\chi_{5Q_j}(k * a - P_{2L+2n+2,5Q_j}[k * a_j])] \right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\| \sum_{j=0}^{\infty} \lambda_j \chi_{\mathbb{R}^n \setminus 10Q_j} (M^{\rm loc}\chi_{Q_j})^{\frac{n+L+1}{n}} \right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \mathcal{A}_{p(\cdot),w,v}(\{\lambda_j\}_{j=0}^{\infty}; \{Q_j\}_{j=0}^{\infty}).\end{aligned} \tag{6.12}$$

If we combine (6.10)–(6.12), we obtain the desired result. □

7. Littlewood–Paley characterization

Section 7 considers the Littlewood–Paley characterization of $h^{p(\cdot)}(w)$ as an application of the results in Section 6. The result of this section will be a natural extension to the weighted case of the result in [43]. What differs from [43] is that the Plancherel–Polya-Nikolskií inequality is not available in this weighted setting. To overcome this difficulty, we use Corollary 2.16. Section 7.1 modifies the idea in [43], where we refine what we obtained in Section 6. Under this modification, we combine the idea obtained in [43] with Corollary 2.16 in Section 7.2. Section 7.3 is devoted to the Littlewood–Paley characterization of $h^{p(\cdot)}(w)$ as a preparatory step in Section 8.

7.1. **Vector-valued extension of Theorem 6.4.** Theorem 7.1 is a natural extension of Theorem 6.4 in which $|\cdot|$ in the definition of $\mathcal{M}^0_{N_{p(\cdot),w}}f$ is replaced by $\ell^2(\mathbb{N}_0)$. We introduce the $\ell^2(\mathbb{N}_0)$-valued function space $h^{p(\cdot)}(w;\ell^2(\mathbb{N}_0))$. Suppose that we are given a sequence $\{f_j\}_{j=0}^\infty \subset \mathcal{D}'(\mathbb{R}^n)$.

Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ be a function such that $\chi_{[-1,1]^n} \le \psi \le \chi_{[-2,2]^n}$. We set $\psi^k \equiv 2^{kn}\psi(2^k\cdot)$ for $k \in \mathbb{N}$. With this in mind, we define

$$\|\{f_j\}_{j=0}^\infty\|_{h^{p(\cdot)}(w,\ell^2)} \equiv \left\| \sup_{k\in\mathbb{N}_0} \left(\sum_{j=0}^\infty |\psi^k * f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)}.$$

Observe that this is a natural vector-valued extension of the quasi-norm equivalence

$$\|f\|_{h^{p(\cdot)}(w)} \sim \left\| \sup_{k\in\mathbb{N}_0} |\psi^k * f| \right\|_{L^{p(\cdot)}(w)} \quad (f \in h^{p(\cdot)}(w)).$$

The $\ell^2(\mathbb{N}_0)$-valued function space $h^{p(\cdot)}(w,\ell^2(\mathbb{N}_0))$ is the set of all $\{f_j\}_{j=0}^\infty \subset \mathcal{D}'(\mathbb{R}^n)$ for which $\|\{f_j\}_{j=0}^\infty\|_{h^{p(\cdot)}(w,\ell^2)}$ is finite.

Then the next theorem is analogous to Theorem 6.4. We omit the proof due to similarity.

Theorem 7.1 (cf. [43, Theorem 5.6])**.** *Let $p(\cdot) \in \mathcal{P}_0 \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and $w \in A_\infty^{\mathrm{loc}}$. If $T = \{T_k\}_{k\in\mathbb{N}_0}$ is a collection of $L^2(\mathbb{R}^n;\ell^2(\mathbb{N}_0))$-$L^2(\mathbb{R}^n)$ bounded operators such that there exists a collection $\{k_{ij}\}_{i,j\in\mathbb{N}_0} \subset \mathcal{D}(\mathbb{R}^n)$ with the following properties*:

(1) *There exists a constant $\gamma_0 > 0$ such that*
$$|x|^{n+m}\|\{\nabla^m k_{ij}(x)\}_{i,j\in\mathbb{N}_0}\|_{\ell^2(\mathbb{N}_0)\to\ell^2(\mathbb{N}_0)} \lesssim \chi_{[-\gamma_0,\gamma_0]^n}(x) \quad (x \in \mathbb{R}^n).$$
for every $m \in \mathbb{N}_0$.

(2) *If $\{f_j\}_{j=0}^\infty$ is a sequence of compactly supported $L^2(\mathbb{R}^n)$-functions, then*
$$T_i[\{f_j\}_{j=0}^\infty](x) = \sum_{j=0}^\infty k_{ij} * f_j(x), \quad i \in \mathbb{N}_0$$
for $x \in \mathbb{R}^n$.

(3) *$k_{ij} \equiv 0$ if $|i|+|j|$ is large enough.*

Then

$$\left\|\{T_i[\{f_j\}_{j=0}^\infty]\}_{i=0}^\infty\right\|_{h^{p(\cdot)}(w,\ell^2)} \lesssim \left\|\{f_j\}_{j=0}^\infty\right\|_{h^{p(\cdot)}(w,\ell^2)} \tag{7.1}$$

for all $\{f_j\}_{j=1}^\infty \in h^{p(\cdot)}(w,\ell^2(\mathbb{N}_0))$.

7.2. **A vector-valued inequality.** We will use the following vector-valued inequality to obtain the Littlewood–Paley characterization of $h^{p(\cdot)}(w)$.

Since the Fourier transform of non-zero compactly supported functions is not compactly supported, we must taylor some auxiliary estimate without using the Plancherel–Polya–Nikolskií inequality. See [49] for the Plancherel–Polya–Nikolskií inequality for example.

Lemma 7.2. *Let $p(\cdot) \in \mathcal{P}_0 \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and $w \in A_\infty^{\mathrm{loc}}$. Assume that $L \gg 1$. Let $\phi, \phi^* \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ satisfy $\phi^* \in \mathcal{P}_{2L}^\perp(\mathbb{R}^n)$, that*

$$|\phi|, |\phi^*| \le \chi_{[-1,1]^n} \tag{7.2}$$

and that

$$\phi^* = \phi - 2^{-n}\phi\left(\frac{\cdot}{2}\right). \tag{7.3}$$

Then, we have

$$\left\|\sup_{k\in\mathbb{N}_0}\left(\sum_{j=1}^{\infty}|\Phi_{2^{-k}}*\phi^*_{2^{-j}}*f|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}\lesssim\|\phi*f\|_{L^{p(\cdot)}(w)}+\left\|\left(\sum_{j=1}^{\infty}|\phi^*_{2^{-j}}*f|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}$$

for all $f\in\mathcal{D}'(\mathbb{R}^n)$ *and* $\Phi\in C^\infty_{\rm c}(\mathbb{R}^n)$ *with* $\operatorname{supp}(\Phi)\subset[-1,1]^n$. *In particular,*

$$\left\|\{\phi^*_{2^{-j}}*f\}_{j=0}^{\infty}\right\|_{h^{p(\cdot)}(w,\ell^2)}\lesssim\|\phi*f\|_{L^{p(\cdot)}(w)}+\left\|\left(\sum_{j=1}^{\infty}|\phi^*_{2^{-j}}*f|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}.$$

We remark that the couple (ϕ,ϕ^*) exists according to [31, Lemma 6.5].

Proof. We employ Lemma 2.23. Choose $\psi,\psi^*\in C^\infty_{\rm c}(\mathbb{R}^n)$ so that

$$|\psi|,|\psi^*|\le\chi_{[-1,1]^n}\tag{7.4}$$

that

$$\psi^*\in\mathcal{P}_{2L}^{\perp}(\mathbb{R}^n),\tag{7.5}$$

and that

$$\phi*\psi+\sum_{j=1}^{\infty}\phi^*_{2^{-j}}*\psi^*_{2^{-j}}=\delta\tag{7.6}$$

in the topology of $\mathcal{D}'(\mathbb{R}^n)$. Fix k and j for now. We decompose

$$\Phi_{2^{-k}}*\phi^*_{2^{-j}}*f=\Phi_{2^{-k}}*\psi*\phi^*_{2^{-j}}*\phi*f+\sum_{l=1}^{\infty}\Phi_{2^{-k}}*\phi^*_{2^{-j}}*\psi^*_{2^{-l}}*\phi^*_{2^{-l}}*f$$

It follows from Lemma 2.22 that

$$|\Phi_{2^{-k}}*\psi*\phi^*_{2^{-j}}|\lesssim2^{-2Lj}\chi_{[-3,3]^n}\tag{7.7}$$

and that

$$\begin{aligned}|\Phi_{2^{-k}}*\phi^*_{2^{-j}}*\psi^*_{2^{-l}}|&\lesssim2^{n\min(j,k,l)-2L\min(k-j,k-l,l-j,j-l)}\chi_{[-2^{2-\min(j,k,l)},2^{2-\min(j,k,l)}]^n}\\&\lesssim2^{n\min(j,k,l)+2L|l-j|}\chi_{[-2^{2-\min(j,k,l)},2^{2-\min(j,k,l)}]^n}.\end{aligned}\tag{7.8}$$

Not that $\min(k-j,k-l,l-j,j-l)\le-|l-j|$. Let $L\gg2A>A>B\gg1$. Let r be a constant, which is slightly less than $\frac{\min(1,p_-)}{q_w}(<1)$. Thanks to Lemma 2.24, (7.7) and Hölder's inequality for l, we have

$$\begin{aligned}
&|\Phi_{2^{-k}} * \psi * \phi^*_{2^{-j}} * \phi * f(x)|\\
&\lesssim 2^{-2Lj}\int_{[-3,3]^n} |\phi * f(x-z)|\mathrm{d}z\\
&\lesssim 2^{-2Lj}\int_{[-3,3]^n} \left(\int_{\mathbb{R}^n} \frac{|\phi * f(x-z-y)|^r}{m_{0,Ar,Br}(y)}\mathrm{d}y\right)^{\frac{1}{r}}\mathrm{d}z\\
&\quad + 2^{-2Lj}\int_{[-3,3]^n} \left(\sum_{l=1}^{\infty} 2^{ln-lLr}\int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l}} * f(x-z-y)|^r}{m_{0,Ar,Br}(y)}\mathrm{d}y\right)^{\frac{1}{r}}\mathrm{d}z\\
&\lesssim 2^{-2Lj}\int_{[-3,3]^n} \left(\int_{\mathbb{R}^n} \frac{|\phi * f(x-z-y)|^r}{m_{0,Ar,Br}(y)}\mathrm{d}y\right)^{\frac{1}{r}}\mathrm{d}z\\
&\quad + 2^{-2Lj}\int_{[-3,3]^n} \sum_{l=1}^{\infty} 2^{-2lA}\left(2^{ln}\int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l}} * f(x-z-y)|^r}{m_{0,Ar,Br}(y)}\mathrm{d}y\right)^{\frac{1}{r}}\mathrm{d}z.
\end{aligned}$$

Since

$$m_{0,Ar,Br}(y) \sim m_{0,Ar,Br}(y+z)$$

for all $y \in \mathbb{R}^n$ and $z \in [-3,3]^n$ and

$$2^{lAr} m_{0,Ar,Br}(y) \geq m_{l,Ar,Br}(y)$$

for all $y \in \mathbb{R}^n$, we see that

$$\begin{aligned}
(7.9)\quad &|\Phi_{2^{-k}} * \psi * \phi^*_{2^{-j}} * \phi * f(x)|\\
&\lesssim 2^{-2Lj}\left(\int_{\mathbb{R}^n} \frac{|\phi * f(x-y)|^r}{m_{0,Ar,Br}(y)}\mathrm{d}y\right)^{\frac{1}{r}} + 2^{-2Lj}\sum_{l=1}^{\infty} 2^{-2lA}\left(2^{ln}\int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l}} * f(x-y)|^r}{m_{0,Ar,Br}(y)}\mathrm{d}y\right)^{\frac{1}{r}}\\
&\lesssim 2^{-2Lj}\left(\int_{\mathbb{R}^n} \frac{|\phi * f(x-y)|^r}{m_{0,Ar,Br}(y)}\mathrm{d}y\right)^{\frac{1}{r}} + 2^{-2Lj}\left\{\sum_{l=1}^{\infty}\left(2^{ln}\int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l}} * f(x-y)|^r}{m_{l,Ar,Br}(y)}\mathrm{d}y\right)^{\frac{2}{r}}\right\}^{\frac{1}{2}}
\end{aligned}$$

by the Hölder inequality. Likewise,

$$\begin{aligned}
&|\Phi_{2^{-k}} * \phi^*_{2^{-j}} * \psi^*_{2^{-l}} * \phi^*_{2^{-l}} * f(x)|\\
&\lesssim 2^{n\min(j,k,l)+2L|l-j|}\int_{[-2^{2-\min(j,k,l)},2^{2-\min(j,k,l)}]^n} |\phi^*_{2^{-l}} * f(x-z)|\mathrm{d}z
\end{aligned}$$

by using (7.8). Since

$$\begin{aligned}
m_{l,Ar,Br}(y+z) &= (1+2^l|y+z|)^{Ar} e^{|y+z|Br}\\
&\leq (1+2^l|z|)^{Ar}(1+2^l|y|)^{Ar} e^{|y|Br+|z|Br}\\
&\lesssim 2^{Ar\max(0,l-\min(l,j,k))} m_{l,Ar,Br}(y)
\end{aligned}$$

for all $z \in [-2^{2-\min(j,k,l)}, 2^{2-\min(j,k,l)}]^n$, thanks to Lemmas 2.22 and 2.24, we obtain

$$\begin{aligned}
&|\Phi_{2^{-k}} * \phi^*_{2^{-j}} * \psi^*_{2^{-l}} * \phi^*_{2^{-l}} * f(x)| \\
&\lesssim 2^{n\min(j,k,l)+2L|l-j|} \\
&\quad \times \int_{[-2^{2-\min(j,k,l)}, 2^{2-\min(j,k,l)}]^n} \left(\sum_{l'=l}^{\infty} 2^{l'n+(l-l')Lr} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l'}} * f(x-z-y)|^r}{m_{l,Ar,Br}(y)} \mathrm{d}y \right)^{\frac{1}{r}} \mathrm{d}z \\
&\lesssim 2^{n\min(j,k,l)+2L|l-j|+A\max(0,l-\min(l,j,k))} \\
&\quad \times \int_{[-2^{2-\min(j,k,l)}, 2^{2-\min(j,k,l)}]^n} \sum_{l'=l}^{\infty} 2^{2(l-l')A} \left(2^{l'n} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l'}} * f(x-y)|^r}{m_{l,Ar,Br}(y)} \mathrm{d}y \right)^{\frac{1}{r}} \mathrm{d}z.
\end{aligned}$$

Due to the fact that $L \gg A \gg 1$, we have

$$\begin{aligned}
&|\Phi_{2^{-k}} * \phi^*_{2^{-j}} * \psi^*_{2^{-l}} * \phi^*_{2^{-l}} * f(x)| \\
&\lesssim \sum_{l'=l}^{\infty} 2^{L|l-j|+2A(l-l')} \left(2^{l'n} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l'}} * f(x-y)|^r}{m_{l,Ar,Br}(y)} \mathrm{d}y \right)^{\frac{1}{r}} \\
&\lesssim \sum_{l'=l}^{\infty} 2^{-L|l-j|+2A(l-l')} \left(2^{l'n} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l'}} * f(x-y)|^r}{m_{l,Ar,Br}(y)} \mathrm{d}y \right)^{\frac{1}{r}}.
\end{aligned}$$

Since $2^{-Ar(l-l')} m_{l,Ar,Br} \ge m_{l',Ar,Br}$,

$$\begin{aligned}
&|\Phi_{2^{-k}} * \phi^*_{2^{-j}} * \psi^*_{2^{-l}} * \phi^*_{2^{-l}} * f(x)| \\
&\lesssim \sum_{l'=l}^{\infty} 2^{-L|l-j|+A(l-l')} \left(2^{l'n} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l'}} * f(x-y)|^r}{m_{l',Ar,Br}(y)} \mathrm{d}y \right)^{\frac{1}{r}}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\sum_{l=1}^{\infty} |\Phi_{2^{-k}} * \phi^*_{2^{-j}} * \psi^*_{2^{-l}} * \phi^*_{2^{-l}} * f(x)| \\
&\lesssim \sum_{l=1}^{\infty} \sum_{l'=l}^{\infty} 2^{-L|l-j|+A(l-l')} \left(2^{l'n} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l'}} * f(x-y)|^r}{m_{l',Ar,Br}(y)} \mathrm{d}y \right)^{\frac{1}{r}} \\
&\le \sum_{l'=1}^{\infty} \sum_{l=-\infty}^{l'} 2^{-L|l-j|+A(l-l')} \left(2^{l'n} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l'}} * f(x-y)|^r}{m_{l',Ar,Br}(y)} \mathrm{d}y \right)^{\frac{1}{r}},
\end{aligned}$$

Since

$$\sum_{l=-\infty}^{l'} 2^{-L|l-j|+A(l-l')} \sim 2^{-L|l'-j|}$$

thanks to the fact that $L > A > 0$, we have

$$\sum_{l=1}^{\infty} |\Phi_{2^{-k}} * \phi^*_{2^{-j}} * \psi^*_{2^{-l}} * \phi^*_{2^{-l}} * f(x)| \lesssim \sum_{l'=1}^{\infty} 2^{-L|l'-j|} \left(2^{l'n} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l'}} * f(x-y)|^r}{m_{l',Ar,Br}(y)} \mathrm{d}y \right)^{\frac{1}{r}}. \tag{7.10}$$

Therefore, from (7.9) and (7.10), we have

$$
\begin{aligned}
(7.11)\quad & |\Phi_{2^{-k}} * \phi^*_{2^{-j}} * f(x)| \\
& \lesssim 2^{-2Lj} \left(\int_{\mathbb{R}^n} \frac{|\phi * f(x-y)|^r}{m_{0,Ar,Br}(y)} \mathrm{d}y \right)^{\frac{1}{r}} + 2^{-2Lj} \left\{ \sum_{l=1}^{\infty} \left(2^{ln} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l}} * f(x-y)|^r}{m_{l,Ar,Br}(y)} \mathrm{d}y \right)^{\frac{2}{r}} \right\}^{\frac{1}{2}} \\
& \quad + \sum_{l'=1}^{\infty} 2^{-L|l'-j|} \left(2^{l'n} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l'}} * f(x-y)|^r}{m_{l',Ar,Br}(y)} \mathrm{d}y \right)^{\frac{1}{r}}.
\end{aligned}
$$

We observe

$$
\begin{aligned}
& \sum_{j=1}^{\infty} \left\{ \sum_{l'=1}^{\infty} 2^{-L|l'-j|} \left(2^{l'n} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l'}} * f(x-y)|^r}{m_{l',Ar,Br}(y)} \mathrm{d}y \right)^{\frac{1}{r}} \right\}^2 \\
& \le \sum_{j=1}^{\infty} \left[\sum_{l'=-\infty}^{\infty} 2^{-L|l'-j|} \times \sum_{l'=1}^{\infty} 2^{-L|l'-j|} \left(2^{l'n} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l'}} * f(x-y)|^r}{m_{l',Ar,Br}(y)} \mathrm{d}y \right)^{\frac{2}{r}} \right] \\
& \lesssim \sum_{j=1}^{\infty} \sum_{l'=1}^{\infty} 2^{-L|l'-j|} \left(2^{l'n} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l'}} * f(x-y)|^r}{m_{l',Ar,Br}(y)} \mathrm{d}y \right)^{\frac{2}{r}} \\
& = \sum_{l'=1}^{\infty} \sum_{j=1}^{\infty} 2^{-L|l'-j|} \left(2^{l'n} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l'}} * f(x-y)|^r}{m_{l',Ar,Br}(y)} \mathrm{d}y \right)^{\frac{2}{r}}.
\end{aligned}
$$

Note that

$$
\sum_{j=1}^{\infty} 2^{-L|l'-j|} \lesssim \sum_{j=-\infty}^{\infty} 2^{-L|l'-j|} \sim 1.
$$

Therefore,

$$
\begin{aligned}
(7.12)\qquad & \sum_{j=1}^{\infty} \left\{ \sum_{l'=l}^{\infty} 2^{-L|l'-j|} \left(2^{l'n} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l'}} * f(x-y)|^r}{m_{l',Ar,Br}(y)} \mathrm{d}y \right)^{\frac{1}{r}} \right\}^2 \\
& \lesssim \sum_{l'=1}^{\infty} \left(2^{l'n} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l'}} * f(x-y)|^r}{m_{l',Ar,Br}(y)} \mathrm{d}y \right)^{\frac{2}{r}}.
\end{aligned}
$$

Likewise we can prove

$$
\begin{aligned}
(7.13)\qquad & |\Phi_{2^{-k}} * \phi * f(x)| \\
& \lesssim \left(\int_{\mathbb{R}^n} \frac{|\phi * f(x-y)|^r}{m_{0,Ar,Br}(y)} \mathrm{d}y \right)^{\frac{1}{r}} + \left(\sum_{l=1}^{\infty} \left(2^{ln} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l}} * f(x-y)|^r}{m_{l,Ar,Br}(y)} \mathrm{d}y \right)^{\frac{2}{r}} \right)^{\frac{1}{2}}
\end{aligned}
$$

for all $k \in \mathbb{N}_0$ with the implicit constant independent of k. If we take the supremum over $k \in \mathbb{N}_0$ and the ℓ^2-norm for j, we obtain

$$
\begin{aligned}
& \sup_{k \in \mathbb{N}_0} |\Phi_{2^{-k}} * \phi * f(x)| + \left(\sum_{j=1}^{\infty} \sup_{k \in \mathbb{N}_0} |\Phi_{2^{-k}} * \phi^*_{2^{-j}} * f(x)|^2 \right)^{\frac{1}{2}} \\
& \lesssim \left(\int_{\mathbb{R}^n} \frac{|\phi * f(x-y)|^r}{m_{0,Ar,Br}(y)} \mathrm{d}y \right)^{\frac{1}{r}} + \left(\sum_{l=1}^{\infty} \left(2^{ln} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-l}} * f(x-y)|^r}{m_{l,Ar,Br}(y)} \mathrm{d}y \right)^{\frac{2}{r}} \right)^{\frac{1}{2}}
\end{aligned}
$$

from (7.11), (7.12) and (7.13). To complete, invoke Corollary 2.16. □

7.3. **Littlewood–Paley characterization of $h^{p(\cdot)}(w)$.** An important consequence of Theorem 7.1 is the Littlewood–Paley characterization of $h^{p(\cdot)}(w)$. We obtain it under a strong assumption of L.

Proposition 7.3. *Let $p(\cdot) \in \mathcal{P}_0 \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and $w \in A_\infty^{\mathrm{loc}}$. Assume $L \gg 1$. Let $\phi, \phi^* \in \mathcal{D}(\mathbb{R}^n)$ satisfy $\phi^* \in \mathcal{P}_L^\perp(\mathbb{R}^n)$, (7.2) and (7.3). Then a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $h^{p(\cdot)}(w)$ if and only if*

$$\|\phi * f\|_{L^{p(\cdot)}(w)} + \left\| \left(\sum_{j=1}^\infty |\phi^*_{2^{-j}} * f|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)} < \infty$$

In this case, we have

$$\|f\|_{h^{p(\cdot)}(w)} \sim \|\phi * f\|_{L^{p(\cdot)}(w)} + \left\| \left(\sum_{j=1}^\infty |\phi^*_{2^{-j}} * f|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)}. \tag{7.14}$$

Proof. Assuming that $f \in h^{p(\cdot)}(w)$, we first show that

$$\|\phi * f\|_{L^{p(\cdot)}(w)} + \left\| \left(\sum_{j=1}^\infty |\phi^*_{2^{-j}} * f|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)} \lesssim \|f\|_{h^{p(\cdot)}(w)}.$$

The definition of the grand maximal function $\mathcal{M}^0_{N_{p(\cdot),w}} f$ easily gives that

$$\|\phi * f\|_{L^{p(\cdot)}(w)} \lesssim \|\mathcal{M}^0_{N_{p(\cdot),w}} f\|_{L^{p(\cdot)}(w)} \lesssim \|f\|_{h^{p(\cdot)}(w)}.$$

Therefore, we must show that

$$\left\| \left(\sum_{j=1}^\infty |\phi^*_{2^{-j}} * f|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)} \lesssim \|f\|_{h^{p(\cdot)}(w)}.$$

By the monotone convergence theorem, the matters are reduced to showing that

$$\left\| \left(\sum_{j=1}^N |\phi^*_{2^{-j}} * f|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)} \lesssim \|f\|_{h^{p(\cdot)}(w)}$$

with the implicit constant independent of N. By the Khinchine inequality, we have only to show that

$$\left\| \sum_{j=1}^N a_j \phi^*_{2^{-j}} * f \right\|_{L^{p(\cdot)}(w)} \lesssim \|f\|_{h^{p(\cdot)}(w)}$$

for any sequences $\{a_j\}_{j=1}^N \subset \{-1,1\}^N$. However, this is a direct consequence of Theorem 6.4.

Let us move on to the proof of

$$\|f\|_{h^{p(\cdot)}(w)} \lesssim \|\phi * f\|_{L^{p(\cdot)}(w)} + \left\| \left(\sum_{j=1}^\infty |\phi^*_{2^{-j}} * f|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)}.$$

Choose $\psi, \psi^* \in C_c^\infty(\mathbb{R}^n)$ so that (7.4)–(7.6) hold. Consider the operator

$$\{g_j\}_{j=0}^\infty \in h^{p(\cdot)}(w, \ell^2(\mathbb{N}_0)) \mapsto \psi * g_0 + \sum_{j=1}^\infty \psi^*_{2^{-j}} * g_j \in h^{p(\cdot)}(w).$$

This operator is $h^{p(\cdot)}(w,\ell^2(\mathbb{N}_0))$-$h^{p(\cdot)}(w)$ bounded thanks to Theorem 7.1. As a result,

$$\left\|\psi * f_0 + \sum_{j=1}^{\infty} \psi^*_{2^{-j}} * g_j\right\|_{h^{p(\cdot)}(w)} \lesssim \left\|\{g_j\}_{j=0}^{\infty}\right\|_{h^{p(\cdot)}(w,\ell^2)}.$$

The right-hand side must be written out fully. Use (7.6) and Lemma 7.2. If we let $g_0 = \phi * f$, $g_j = \phi^*_{2^{-j}} * f$ $(j \geq 1)$, then

$$\|f\|_{h^{p(\cdot)}(w)} \lesssim \left\|\{g_j\}_{j=0}^{\infty}\right\|_{h^{p(\cdot)}(w,\ell^2)} \lesssim \|\phi * f\|_{L^{p(\cdot)}(w)} + \left\|\left(\sum_{j=1}^{\infty} |\phi^*_{2^{-j}} * f|^2\right)^{\frac{1}{2}}\right\|_{L^{p(\cdot)}(w)},$$

since

$$f = \psi * \phi * f + \sum_{j=1}^{\infty} \psi^*_{2^{-j}} * \phi_{2^{-j}} * f.$$

Thus, the proof is complete. □

Let us relax the assumption on L in Proposition 7.3.

Theorem 7.4. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. In Proposition 7.3, the same conclusion holds if $L = 0$.*

Proof. We start with the set-up. Assume $L^\dagger \gg s_0 \gg 1$. Let $\zeta, \zeta^* \in C^\infty_{\rm c}(\mathbb{R}^n)$ satisfy $\zeta^* \in \mathcal{P}^\perp_{L^\dagger}(\mathbb{R}^n)$,

$$|\zeta|, |\zeta^*| \leq \chi_{[-1,1]^n}.$$

and

$$\zeta^* = \zeta - \frac{1}{2^n}\zeta\left(\frac{\cdot}{2}\right).$$

Let $\phi, \phi^* \in C^\infty_{\rm c}(\mathbb{R}^n)$ satisfy $\phi^* \in \mathcal{P}^\perp_0(\mathbb{R}^n)$, (7.2) and (7.3).

It suffices to show that

$$\|\phi * f\|_{L^{p(\cdot)}(w)} + \left\|\left(\sum_{j=1}^{\infty} |\phi^*_{2^{-j}} * f|^2\right)^{\frac{1}{2}}\right\|_{L^{p(\cdot)}(w)}$$

$$\sim \|\zeta * f\|_{L^{p(\cdot)}(w)} + \left\|\left(\sum_{j=1}^{\infty} |\zeta^*_{2^{-j}} * f|^2\right)^{\frac{1}{2}}\right\|_{L^{p(\cdot)}(w)}, \tag{7.15}$$

since we proved in Proposition 7.3 that $f \in h^{p(\cdot)}(w)$ holds if and only if

$$\|\zeta * f\|_{L^{p(\cdot)}(w)} + \left\|\left(\sum_{j=1}^{\infty} |\zeta^*_{2^{-j}} * f|^2\right)^{\frac{1}{2}}\right\|_{L^{p(\cdot)}(w)} < \infty$$

and that the norm equivalence (7.14) holds.

We content ourselves with proving

$$\left\|\left(\sum_{j=1}^{\infty} |\zeta^*_{2^{-j}} * f|^2\right)^{\frac{1}{2}}\right\|_{L^{p(\cdot)}(w)} \lesssim \|\phi * f\|_{L^{p(\cdot)}(w)} + \left\|\left(\sum_{j=1}^{\infty} |\phi^*_{2^{-j}} * f|^2\right)^{\frac{1}{2}}\right\|_{L^{p(\cdot)}(w)},$$

since the remaining estimates needed to establish (7.15) can be proved similarly.

Let $r>0$ be a constant which is slightly less than $\frac{p_-}{q_w}$. Fix $x \in \mathbb{R}^n$. We assume that $A<L^\dagger$ and $Br>8n+6\log D$. Let j be fixed. Then we have

$$|\zeta^*_{2^{-j}}*f(x)| \lesssim \left(\sum_{l=j}^\infty 2^{L^\dagger(j-l)r}2^{ln}\int_{\mathbb{R}^n}\frac{|\phi^*_{2^{-l}}*f(x-y)|^r}{m_{l,Ar,Br}(y)}dy\right)^{\frac{1}{r}}$$
$$\lesssim \sum_{l=j}^\infty 2^{A(j-l)r}\left(2^{ln}\int_{\mathbb{R}^n}\frac{|\phi^*_{2^{-l}}*f(x-y)|^r}{m_{l,Ar,Br}(y)}dy\right)^{\frac{1}{r}}$$

thanks to Lemma 2.25. Consequently,

$$\left(\sum_{j=1}^\infty |\zeta^*_{2^{-j}}*f(x)|^2\right)^{\frac{1}{2}} \lesssim \left(\sum_{j=1}^\infty \left(2^{ln}\int_{\mathbb{R}^n}\frac{|\phi^*_{2^{-l}}*f(x-y)|^r}{m_{l,Ar,Br}(y)}dy\right)^{\frac{2}{r}}\right)^{\frac{1}{2}}.$$

It remains to use Corollary 2.16. □

8. Wavelet characterization

As a further application of Theorem 7.4, we consider the wavelet expansion.

Choose compactly supported C^r-functions for large enough $r \in \mathbb{N}$

$$\varphi \text{ and } \psi^l \quad (l=1,2,\ldots,2^n-1) \tag{8.1}$$

so that the following conditions are satisfied:

(1) For any $J \in \mathbb{Z}$, the system

$$\left\{\varphi_{J,k},\, \psi^l_{j,k} \,:\, k \in \mathbb{Z}^n,\, j \ge J,\, l=1,2,\ldots,2^n-1\right\}$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$. Here, given a function F defined on $\mathbb{R}^n$, we write

$$F_{j,k} \equiv 2^{\frac{jn}{2}}F(2^j\cdot -k)$$

for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$.

(2) Fix a large integer $L \in \mathbb{N}$ for now. We have

$$\psi^l \in \mathcal{P}_L^\perp(\mathbb{R}^n) \quad (l=1,2,\ldots,2^n-1). \tag{8.2}$$

In addition, they are real-valued and compactly supported with

$$\operatorname{supp}(\varphi)=\operatorname{supp}(\psi^l)=[0,2N-1]^n \tag{8.3}$$

for some $N \in \mathbb{N}$. See [38] for example.

We also define $\chi_{j,k} \equiv 2^{\frac{jn}{2}}\chi_{Q_{j,k}}$ and $\chi^*_{j,k} \equiv 2^{\frac{jn}{2}}\chi_{Q^*_{j,k}}$ for $j \in \mathbb{Z}$ and $k=(k_1,k_2,\ldots,k_n) \in \mathbb{Z}^n$, where $Q_{j,k}$ and $Q^*_{j,k}$ are the dyadic cube and its expansion which are given by

$$Q_{j,k} \equiv \prod_{m=1}^n \left[2^{-j}k_m, 2^{-j}(k_m+1)\right] \tag{8.4}$$

and

$$Q^*_{j,k} \equiv \prod_{m=1}^n \left[2^{-j}k_m, 2^{-j}(k_m+2N-1)\right], \tag{8.5}$$

respectively. Then using the L^2-inner product $\langle \cdot, \cdot \rangle$, for $f \in L^1_{\rm loc}$, we define two square functions Vf, $W_s f$ by

$$V f \equiv V^{\varphi} f \equiv \left(\sum_{k \in \mathbb{Z}^n} |\langle f, \varphi_{J,k} \rangle \chi_{J,k}|^2 \right)^{\frac{1}{2}},$$

$$W f \equiv W^{\psi^l} f \equiv \left(\sum_{l=1}^{2^n-1} \sum_{j=J}^{\infty} \sum_{k \in \mathbb{Z}^n} \left|\langle f, \psi^l_{j,k} \rangle \chi_{j,k}\right|^2 \right)^{\frac{1}{2}}.$$

Here, J is a fixed integer. In 1994, Lemarié-Rieusset commented that the class Muckenhoupt has a lot to do with the wavelet characterization [37]. We remark that Kopaliani considered the wavelet characterization of $L^{p(t)}(\mathbb{R})$ in 2008 (see [34]).

Based on these works, we will prove the following theorem:

Theorem 8.1. *Let $p(\cdot) \in \mathcal{P}_0 \cap {\rm LH}_0 \cap {\rm LH}_\infty$ and $w \in A^{\rm loc}_\infty$. Assume that*

$$L \geq \max\left(-1, \left[n\left(\frac{q_w}{\min(1, p_-)} - 1\right)\right]\right) \tag{8.6}$$

in (8.2).

(1) *Let $f \in h^{p(\cdot)}(w)$. Then*
$$\|f\|_{h^{p(\cdot)}(w)} \sim \|Vf\|_{L^{p(\cdot)}(w)} + \|Wf\|_{L^{p(\cdot)}(w)}.$$
(2) *If $f \in L^\infty_{\rm c}(\mathbb{R}^n)$ satisfies $Vf + Wf \in L^{p(\cdot)}(w)$, then $f \in h^{p(\cdot)}(w)$.*

The proof of Theorem 8.1 consists of several steps. We start with a setup. Let $\phi \in C^\infty_{\rm c}(\mathbb{R}^n) \setminus \mathcal{P}_0(\mathbb{R}^n)^\perp$ and $\phi^* \in C^\infty_{\rm c}(\mathbb{R}^n)$ satisfy

$$\operatorname{supp}(\phi) \subset [-1,1]^n, \quad \phi^* = \phi - \frac{1}{2^n}\phi\left(\frac{\cdot}{2}\right).$$

Choose ψ, ψ^* according to Lemma 2.23. In the light of the construction in [47], we can arrange ϕ, ϕ^*, ψ and ψ^* so that they are even functions satisfying

$$\phi * \psi + \sum_{j=1}^{\infty} \phi^*_{2^{-j}} * \psi^*_{2^{-j}} = \delta$$

in $\mathcal{D}'(\mathbb{R}^n)$ and that $\phi^*, \psi^* \in \mathcal{P}_{n+L}(\mathbb{R}^n)^\perp$ where L is in (8.6). We must justify the definition of the couplings $\langle f, \psi^l_{j,k} \rangle$ and $\langle f, \varphi_{J,k} \rangle$. To this end, we will prove the following estimate:

Lemma 8.2. *For all $f \in L^{p_+ + q_w}(w) \cap h^{p(\cdot)}(w)$,*

$$\|f\|_{h^{p(\cdot)}(w)} \gtrsim \|Vf\|_{L^{p(\cdot)}(w)} + \|Wf\|_{L^{p(\cdot)}(w)}. \tag{8.7}$$

Before the proof, we offer a word about Lemma 8.2. Fu and Yang obtained a similar estimate in [18, Theorem 1.9]. However, we cannot use [18, Theorem 1.9] directly due to the presence of $w \in A^{\rm loc}_\infty$. As such, we must establish an estimate from scratch.

Proof. It suffices to prove that

$$Vf + Wf \lesssim \sup_{z \in \mathbb{R}^n} \frac{|\phi_{2^{-J}} * f(\cdot - z)|}{m_{J,A,B}(z)} + \left(\sum_{j'=J+1}^{\infty} \sup_{z \in \mathbb{R}^n} \frac{|\phi^*_{2^{-j'}} * f(\cdot - z)|^2}{m_{j',A,B}(z)^2} \right)^{\frac{1}{2}}. \tag{8.8}$$

Once this estimate is shown, we obtain the conclusion as follows. Fix $r > 0$ slightly less than $\frac{\min(1,p_-)}{q_w}$. Then $(n/r, n+L+1) \neq \emptyset$, so we can take $A \in (n/r, n+L+1) \cap \mathbb{N}$. For the first term, by Lemma 2.24 and Hölder's inequality, we have

$$\begin{aligned}\sup_{z\in\mathbb{R}^n} \frac{|\phi_{2^{-J}} * f(\cdot - z)|}{m_{J,A,B}(z)} &\lesssim \left(\sum_{k=J}^{\infty} 2^{kn+(J-k)(n+L+1)r} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-k}} * f(x-y)|^r}{m_{J,Ar,Br}(y)} \mathrm{d}y\right)^{\frac{1}{r}} \\ &\lesssim \left(\sum_{k=J}^{\infty} \left(2^{kn} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-k}} * f(x-y)|^r}{m_{k,Ar,Br}(y)} \mathrm{d}y\right)^{\frac{2}{r}}\right)^{\frac{1}{2}}.\end{aligned}$$

For the second term, since $A > n/r$, again by Lemma 2.24 and Hölder's inequality, we have

$$\begin{aligned}\sup_{z\in\mathbb{R}^n} \frac{|\phi^*_{2^{-j'}} * f(\cdot - z)|^2}{m_{j',A,B}(z)^2} &\lesssim \left(\sum_{k=j'}^{\infty} 2^{kn+(j'-k)(n+L+1)r} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-k}} * f(x-y)|^r}{m_{j',Ar,Br}(y)} \mathrm{d}y\right)^{\frac{2}{r}} \\ &\lesssim \sum_{k=j'}^{\infty} 2^{2(j'-k)(n+L+1-\varepsilon_0)} \left(2^{kn} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-k}} * f(x-y)|^r}{m_{j',Ar,Br}(y)} \mathrm{d}y\right)^{\frac{2}{r}}.\end{aligned}$$

Here, we choose $0 < \varepsilon_0 < n+L+1-A$. Using the estimate $m_{j',Ar,Br} \geq 2^{Ar(j'-k)} m_{k,Ar,Br}$, we have

$$\begin{aligned}&\sum_{j'=J+1}^{\infty} \sup_{z\in\mathbb{R}^n} \frac{|\phi^*_{2^{-j'}} * f(\cdot - z)|^2}{m_{j',A,B}(z)^2} \\ &\lesssim \sum_{k=J}^{\infty} \sum_{j'=J}^{k} 2^{2(j'-k)(n+L+1-\varepsilon_0)} \left(2^{kn} 2^{-Ar(j'-k)} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-k}} * f(x-y)|^r}{m_{k,Ar,Br}(y)} \mathrm{d}y\right)^{\frac{2}{r}} \\ &\lesssim \sum_{k=J}^{\infty} \left(\sum_{j'=-\infty}^{k} 2^{2(j'-k)(n+L+1-\varepsilon_0-A)}\right) \left(2^{kn} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-k}} * f(x-y)|^r}{m_{k,Ar,Br}(y)} \mathrm{d}y\right)^{\frac{2}{r}} \\ &\sim \sum_{k=J}^{\infty} \left(2^{kn} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-k}} * f(x-y)|^r}{m_{k,Ar,Br}(y)} \mathrm{d}y\right)^{\frac{2}{r}}\end{aligned}$$

Hence,

$$Vf + Wf \lesssim \left(\sum_{k=J}^{\infty} \left(2^{kn} \int_{\mathbb{R}^n} \frac{|\phi^*_{2^{-k}} * f(x-y)|^r}{m_{k,Ar,Br}(y)} \mathrm{d}y\right)^{\frac{2}{r}}\right)^{\frac{1}{2}}$$

Finally, we can resort to the vector-valued boundedness of the operators (Corollary 2.16).

So, we move on to estimate (8.8). However, since we can handle Vf similar to Wf, we content ourselves with the proof of

$$Wf \lesssim \sup_{z\in\mathbb{R}^n} \frac{|\phi_{2^{-J}} * f(\cdot - z)|}{m_{J,A,B}(z)} + \left(\sum_{j'=J+1}^{\infty} \sup_{z\in\mathbb{R}^n} \frac{|\phi^*_{2^{-j'}} * f(\cdot - z)|^2}{m_{j',A,B}(z)^2}\right)^{\frac{1}{2}}. \tag{8.9}$$

instead of (8.8).

Fix $r > 0$ slightly less than $\frac{\min(1,p_-)}{q_w}$. Then $(n/r, n+L+1) \neq \emptyset$, so we can take $A \in (n/r, n+L+1)$. Let $j \in \mathbb{N}$ be fixed. Since ψ, ψ^* are radial,

$$\begin{aligned}\left|\langle f, \psi^l_{j,k}\rangle \chi_{j,k}\right| &\le \left|\langle \phi_{2^{-J}} * \psi_{2^{-J}} * f, \psi^l_{j,k}\rangle \chi_{j,k}\right| + \sum_{j'=J+1}^{\infty} \left|\langle \phi^*_{2^{-j'}} * \psi^*_{2^{-j'}} * f, \psi^l_{j,k}\rangle \chi_{j,k}\right| \\ &= \left|\langle \phi_{2^{-J}} * f, \psi_{2^{-J}} * \psi^l_{j,k}\rangle \chi_{j,k}\right| + \sum_{j'=J+1}^{\infty} \left|\langle \phi^*_{2^{-j'}} * f, \psi^*_{2^{-j'}} * \psi^l_{j,k}\rangle \chi_{j,k}\right|.\end{aligned} \tag{8.10}$$

By the moment condition,

$$\begin{aligned} 2^{\frac{jn}{2}}|\psi_{2^{-J}} * \psi^l_{j,k}| &\lesssim 2^{Jn-(j-J)(n+L+1)}\chi_{2^{4+j-J}Q^*_{j,k}}, \\ 2^{\frac{jn}{2}}|\psi^*_{2^{-j'}} * \psi^l_{j,k}| &\lesssim 2^{\min(j,j')n - n\max(j'-j,0)-|j-j'|(L+1)}\chi_{16Q^*_{j,k}\cup 2^{4+j-j'}Q^*_{j,k}}.\end{aligned} \tag{8.11}$$

By inserting (8.11) into (8.10), we obtain

$$\begin{aligned} &\left|\langle f, \psi^l_{j,k}\rangle \chi_{j,k}\right| \\ &\lesssim 2^{-(j-J)(n+L+1)}\left(\int_{2^{4+j-J}Q^*_{j,k}} |\phi_{2^{-J}} * f(y)|\mathrm{d}y\right)\chi_{Q_{j,k}} \\ &\quad + \sum_{j'=J+1}^{\infty} 2^{\min(j,j')n - n\max(j'-j,0)-|j-j'|(L+1)}\left(\int_{16Q^*_{j,k}\cup 2^{4+j-j'}Q^*_{j,k}} |\phi^*_{2^{-j'}} * f(y)|\mathrm{d}y\right)\chi_{Q_{j,k}}.\end{aligned}$$

Let $x \in Q_{j,k}$. Using the function $m_{J,A,B}$, we estimate

$$\left(\int_{2^{4+j-J}Q^*_{j,k}} |\phi_{2^{-J}} * f(y)|\mathrm{d}y\right) \lesssim 2^{-Jn}\sup_{z\in\mathbb{R}^n}\frac{|\phi_{2^{-J}} * f(x-z)|}{m_{J,A,B}(z)}.$$

Likewise, for $x \in Q_{j,k}$,

$$\int_{16Q^*_{j,k}\cup 2^{4+j-j'}Q^*_{j,k}} |\phi^*_{2^{-j'}} * f(y)|\mathrm{d}y \lesssim 2^{-\min(j,j')n + A\max(j'-j,0)}\sup_{z\in\mathbb{R}^n}\frac{|\phi^*_{2^{-j'}} * f(x-z)|}{m_{j',A,B}(z)}.$$

Consequently,

$$\begin{aligned} &\left(\sum_{k\in\mathbb{Z}^n}\left|\langle f, \psi^l_{j,k}\rangle \chi_{j,k}\right|^2\right)^{\frac{1}{2}} \\ &\lesssim 2^{-(j-J)(n+L+1)}\sup_{z\in\mathbb{R}^n}\frac{|\phi_{2^{-J}} * f(\cdot - z)|}{m_{J,A,B}(z)} + \sum_{j'=J+1}^{\infty} 2^{-|j'-j|(L+1+n-A)}\sup_{z\in\mathbb{R}^n}\frac{|\phi^*_{2^{-j'}} * f(\cdot - z)|}{m_{j',A,B}(z)}.\end{aligned}$$

Recall that A satisfies $L+1+n>A$. Taking the ℓ^2-norm over $j=J,J+1,J+2,\dots$ and $\ell=1,2,\dots,2^n-1$, we obtain

$$
\begin{aligned}
&\left(\sum_{\ell=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|\langle f,\psi^l_{j,k}\rangle\chi_{j,k}\right|^2\right)^{\frac12}\\
&\lesssim\left(\sum_{j=J}^{\infty}2^{-2(j-J)(n+L+1)}\right)^{\frac12}\sup_{z\in\mathbb{R}^n}\frac{|\phi_{2^{-J}}*f(\cdot-z)|}{m_{J,A,B}(z)}\\
&\quad+\left(\sum_{j=J}^{\infty}\left(\sum_{j'=J}^{\infty}2^{-|j'-j|(L+1+n-A)}\sup_{z\in\mathbb{R}^n}\frac{|\phi^*_{2^{-j'}}*f(\cdot-z)|}{m_{j',A,B}(z)}\right)^2\right)^{\frac12}\\
&\lesssim\sup_{z\in\mathbb{R}^n}\frac{|\phi_{2^{-J}}*f(\cdot-z)|}{m_{J,A,B}(z)}+\left(\sum_{j'=J}^{\infty}\sup_{z\in\mathbb{R}^n}\frac{|\phi^*_{2^{-j'}}*f(\cdot-z)|^2}{m_{j',A,B}(z)^2}\right)^{\frac12}.
\end{aligned}
$$

Thus, the proof is complete. □

Lemma 8.2 has an important consequence. First, since $L^\infty_{\rm c}(\mathbb{R}^n)\cap h^{p(\cdot)}(w)$ is dense in $h^{p(\cdot)}(w)$ (see Lemma 5.12), we can extend couplings $\langle f,\psi^l_{j,k}\rangle$ and $\langle f,\varphi_{J,k}\rangle$, which are initially defined for $f\in L^\infty_{\rm c}(\mathbb{R}^n)\cap h^{p(\cdot)}(w)$, to bounded linear functionals from $L^\infty_{\rm c}(\mathbb{R}^n)\cap h^{p(\cdot)}(w)$. We still write $\langle f,\psi^l_{j,k}\rangle$ and $\langle f,\varphi_{J,k}\rangle$ for these extended functionals. By the Fatou lemma, we have (8.7) for all $f\in h^{p(\cdot)}(w)$,

Corollary 8.3. *The conclusion of Lemma 8.2 remains valid for all $f\in h^{p(\cdot)}(w)$.*

With Corollary 8.3 in mind, we complete the proof of Theorem 8.1. Let us prove

$$
\|Vf+Wf\|_{L^{p(\cdot)}(w)}\sim\|\phi*f\|_{L^{p(\cdot)}(w)}+\left\|\left(\sum_{j=1}^{\infty}|\phi^*_{2^{-j}}*f|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}.
$$

In view of Corollary 8.3, it remains to establish

$$
\|Vf+Wf\|_{L^{p(\cdot)}(w)}\gtrsim\|\phi*f\|_{L^{p(\cdot)}(w)}+\left\|\left(\sum_{j=1}^{\infty}|\phi^*_{2^{-j}}*f|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}\tag{8.12}
$$

for all $f\in L^{p_++q_w}(w)$. As before, since the remaining estimates needed for (8.12) are easier to prove, we content ourselves with the proof of

$$
\|Vf+Wf\|_{L^{p(\cdot)}(w)}\gtrsim\left\|\left(\sum_{j=1}^{\infty}|\phi^*_{2^{-j}}*f|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}\tag{8.13}
$$

instead of (8.12).

We prove (8.13). Let $j\in\mathbb{N}\cap[J,\infty)$ be fixed. Since $f\in L^{p_++q_w}(w)$, we can use the wavelet decomposition obtained in [30]. and estimate each term of the decomposition:

$$
\phi^*_{2^{-j}}*f=\sum_{k\in\mathbb{Z}^n}\langle f,\varphi_{J,k}\rangle\phi^*_{2^{-j}}*\varphi_{J,k}+\sum_{l=1}^{2^n-1}\sum_{j'=J}^{\infty}\sum_{k\in\mathbb{Z}^n}\langle f,\psi^l_{j',k}\rangle\phi^*_{2^{-j}}*\psi^l_{j',k}.
$$

We have

$$2^{\frac{Jn}{2}}|\phi^*_{2^{-j}} * \varphi_{J,k}| \lesssim 2^{Jn-(j-J)(n+L+1)}\chi_{16Q^*_{J,k}}$$

and

$$2^{\frac{j'n}{2}}|\phi^*_{2^{-j}} * \psi^l_{j',k}| \lesssim \begin{cases} 2^{j'n-(j-j')(n+L+1)}\chi_{16Q^*_{j',k}\cup 2^{4+j'-j}Q^*_{j',k}} & (j' \le j), \\ 2^{jn-(j'-j)(L+1)}\chi_{16Q^*_{j',k}\cup 2^{4+j'-j}Q^*_{j',k}} & (j' \ge j). \end{cases}$$

As a result,

$$\sum_{k\in\mathbb{Z}^n} |\langle f, \varphi_{J,k}\rangle \phi^*_{2^{-j}} * \varphi_{J,k}| \lesssim \sum_{k\in\mathbb{Z}^n} 2^{\frac{Jn}{2}-(j-J)(n+L+1)}|\langle f, \varphi_{J,k}\rangle|\chi_{16Q^*_{J,k}}$$

and

$$\sum_{j'=J}^{\infty}\sum_{k\in\mathbb{Z}^n} |\langle f, \psi^l_{j',k}\rangle \phi^*_{2^{-j}} * \psi^l_{j',k}| \lesssim \sum_{j'=J}^{\infty}\sum_{k\in\mathbb{Z}^n} 2^{\frac{j'n}{2}-|j-j'|(n+L+1)}|\langle f, \psi^l_{j',k}\rangle|\chi_{16Q^*_{j',k}\cup 2^{4+j'-j}Q^*_{j',k}}.$$

Choose $r \in (0,1)$ so that

$$n + L + 1 - \frac{n}{r} > 0, \quad \frac{p_-}{r} > q_w = \inf\{u \in [1,\infty) \,:\, w \in A^{\rm loc}_u\}.$$

Then

$$\begin{aligned}\sum_{j=J}^{\infty}\left(\sum_{k\in\mathbb{Z}^n} |\langle f, \varphi_{J,k}\rangle \phi^*_{2^{-j}} * \varphi_{J,k}|\right)^2 &\lesssim \left(\sum_{j=J}^{\infty} 2^{-2(j-J)(n+L+1)}\right)\left(\sum_{k\in\mathbb{Z}^n} 2^{\frac{Jn}{2}}|\langle f, \varphi_{J,k}\rangle|\chi_{16Q^*_{J,k}}\right)^2 \\ &\lesssim \left(\sum_{k\in\mathbb{Z}^n} |\langle f, \varphi_{J,k}\rangle| M^{\rm loc}[(\chi_{J,k})^r]^{\frac{1}{r}}\right)^2\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=J}^{\infty}\left(\sum_{j'=J}^{\infty}\sum_{k\in\mathbb{Z}^n} |\langle f, \psi^l_{j',k}\rangle \phi^*_{2^{-j}} * \psi^l_{j',k}|\right)^2 \\ &\lesssim \sum_{j=J}^{\infty}\left(\sum_{j'=J}^{\infty}\sum_{k\in\mathbb{Z}^n} 2^{\frac{j'n}{2}-|j-j'|(n+L+1)}|\langle f, \psi^l_{j',k}\rangle|\chi_{16Q^*_{j',k}\cup 2^{4+j'-j}Q^*_{j',k}}\right)^2 \\ &\lesssim \sum_{j=J}^{\infty}\left(\sum_{j'=J}^{\infty}\sum_{k\in\mathbb{Z}^n} 2^{\frac{j'n}{2}-|j-j'|(n+L+1-n/r)}|\langle f, \psi^l_{j',k}\rangle|(M^{\rm loc}\chi_{Q_{j',k}})^{\frac{1}{r}}\right)^2.\end{aligned}$$

By Proposition 2.12

$$\begin{aligned}&\left\|\left\{\sum_{j=J}^{\infty}\left(\sum_{j'=J}^{\infty}\sum_{k\in\mathbb{Z}^n} |\langle f, \psi^l_{j',k}\rangle \phi^*_{2^{-j}} * \psi^l_{j',k}|\right)^2\right\}^{\frac{1}{2}}\right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\|\left\{\sum_{j=J}^{\infty}\left(\sum_{j'=J}^{\infty}\sum_{k\in\mathbb{Z}^n} 2^{\frac{j'n}{2}-|j-j'|(n+L+1-n/r)}|\langle f, \psi^l_{j',k}\rangle|(M^{\rm loc}\chi_{Q_{j',k}})^{\frac{1}{r}}\right)^2\right\}^{\frac{1}{2}}\right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\|\left\{\sum_{j=J}^{\infty}\left(\sum_{j'=J}^{\infty}\sum_{k\in\mathbb{Z}^n} 2^{\frac{j'n}{2}-|j-j'|(n+L+1-n/r)}|\langle f, \psi^l_{j',k}\rangle|\chi_{Q_{j',k}}\right)^2\right\}^{\frac{1}{2}}\right\|_{L^{p(\cdot)}(w)}.\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
&\left\| \left\{ \sum_{j=J}^{\infty} \left(\sum_{j'=J}^{\infty} \sum_{k \in \mathbb{Z}^n} |\langle f, \psi^l_{j',k} \rangle \phi^*_{2^{-j}} * \psi^l_{j',k}| \right)^2 \right\}^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)} \\
&\lesssim \left\| \left\{ \sum_{j=J}^{\infty} \sum_{j'=J}^{\infty} \sum_{k \in \mathbb{Z}^n} \left(2^{\frac{j'n}{2} - \frac{1}{2}|j-j'|(n+L+1-n/r)} |\langle f, \psi^l_{j',k} \rangle| \chi_{Q_{j',k}} \right)^2 \right\}^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)} \\
&\lesssim \left\| \left\{ \sum_{j'=J}^{\infty} \sum_{k \in \mathbb{Z}^n} \left(|\langle f, \psi^l_{j',k} \rangle| \chi_{j',k} \right)^2 \right\}^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)},
\end{aligned}$$

as required.

9. Examples and relations to other function spaces

In this section we give some examples of the weighted local Hardy spaces with variable exponents and weights. One of the significant example is the Dirac Delta. We consider the condition to belong to $h^{p(\cdot)}(w)$ in Section 9.1. Next we provide the examples of weights. We handle the power weights in Section 9.2 and the exponential weights in Section 9.3, respectively. Finally, Section 9.4 and 9.5 is devoted to consider the relation to other function spaces.

9.1. **Dirac Delta.** Let $w \in A_\infty^{\rm loc}$. Also let $p(\cdot) \in \mathcal{P}_0 \cap {\rm LH}_0 \cap {\rm LH}_\infty$. Let δ be the Dirac Delta. Then $\mathcal{M}^0_{N_{p(\cdot)},w}\delta(x) \sim |x|^{-n}$ near the origin and $\mathcal{M}^0_{N_{p(\cdot)},w}\delta$ is bounded away from the origin and supported on a bounded set. Therefore, if $p(\cdot)$ and w satisfy

$$\int_{B(1)} |x|^{-np(x)} w(x) {\rm d}x < \infty, \tag{9.1}$$

then $\delta \in h^{p(\cdot)}(w)$.

Example 9.1. The following couples satisfy (9.1) and falls within the scope of this paper.

(1) $w(x) = 1$, $p(x) = \max(2^{-1}, \min(1, |x|))$, $x \in \mathbb{R}^n$.
(2) $w(x) = \frac{|x|^{n+1}}{1+|x|^{2n+1}}$, $p(x) = 2$, $x \in \mathbb{R}^n$.
(3) $w(x) = |x|^{n+1}\exp(|x|)$, $p(x) = 2$, $x \in \mathbb{R}^n$.

9.2. **Case of power weights.** Let $\mu \in \mathbb{R}$ and define $w_\mu(x) \equiv (1+|x|)^\mu$. Also let $p(\cdot) \in \mathcal{P}_0 \cap {\rm LH}_0 \cap {\rm LH}_\infty$.

Proposition 9.2. $\mathcal{S}(\mathbb{R}^n) \subset h^{p(\cdot)}(w_\mu)$.

Proof. This is a consequence of the atomic decomposition. We can decompose any $f \in \mathcal{S}(\mathbb{R}^n)$ into the sum of $(p(\cdot), q, -1)_{w_\mu}$-atoms:

$$f = \sum_{m \in \mathbb{Z}^n} \lambda_m a_m,$$

where $\lambda_m = {\rm O}((1+|m|)^{-N})$ for any $N \in \mathbb{N}$ and each a_m is a $(p(\cdot), q, -1)_{w_\mu}$-atom supported on $Q(m, 1/2)$. □

Proposition 9.3. *Any element $f \in h^{p(\cdot)}(w_\mu)$, which is initially defined as an element in $\mathcal{D}'(\mathbb{R}^n)$ can be extended uniquely to the continuous functional over $\mathcal{S}(\mathbb{R}^n)$, that is, $h^{p(\cdot)}(w_\mu) \subset \mathcal{S}'(\mathbb{R}^n)$.*

Proof. In fact, if we argue as in [55, Proposition 3.1] and write $\check{\varphi} = \varphi(-\cdot)$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we obtain

$$\begin{aligned}|\langle f, \varphi(\cdot - x)\rangle| &= |f * \check{\varphi}(x)| \\ &\lesssim \frac{1}{\|\chi_{B(x,1)}\|_{L^{p(\cdot)}(w_\mu)}} \|\check{\varphi}\|_{\mathcal{D}_{N_{p(\cdot)},w_\mu}} \|f\|_{h^{p(\cdot)}(w_\mu)}\end{aligned}$$

for any $\varphi \in \mathcal{D}_{N_{p(\cdot)},w_\mu}(\mathbb{R}^n)$. Notice that $\|\chi_{B(x,1)}\|_{L^{p(\cdot)}(w_\mu)} \gtrsim (1+|x|)^K$ for some $K > 0$.

By the use of the partition of unity, any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ has the following decomposition:

$$\varphi = \sum_{m \in \mathbb{Z}^n} a_m \varphi_m(\cdot - m),$$

where each $\varphi_m \in \mathcal{D}_N(\mathbb{R}^n)$ depends linearly on φ and $|a_m| \lesssim (1+|m|)^{-N}$ for any $N \in \mathbb{N}$. Therefore, we can define

$$\langle f, \varphi \rangle = \sum_{m \in \mathbb{Z}^n} a_m \langle f, \varphi_m(\cdot - m)\rangle$$

for $f \in h^{p(\cdot)}(w_\mu)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, where the convergence takes place absolutely. Thus, $h^{p(\cdot)}(w_\mu) \subset \mathcal{S}'(\mathbb{R}^n)$. □

Proposition 9.4. *Let $\kappa \in \mathbb{R}$. Then $f \mapsto (1+|\cdot|^2)^{\frac{\kappa}{2}} f$ is an isomorphism from $h^{p(\cdot)}(w_\mu)$ to $h^{p(\cdot)}(w_{\mu-\kappa})$.*

Proof. Simply observe $\mathcal{M}^0_{N_{p(\cdot)},w_\mu}[(1+|\cdot|^2)^{\frac{\kappa}{2}} f] \lesssim w_\kappa \mathcal{M}^0_{N_{p(\cdot)},w_\mu} f$. □

9.3. **Case of exponential weights.** We work in $\mathbb{R}$. Let $\mu \in \mathbb{R}$ and define $w^{(\mu)}(x) \equiv \exp(\mu x)$ for $x \in \mathbb{R}$. Also let $p(\cdot) \in \mathcal{P}_0 \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$.

Proposition 9.5. *Let $\kappa \in \mathbb{R}$. The mapping $f \mapsto w^{(\kappa)} f$ is an isomorphism from $h^{p(\cdot)}(w^{(\mu)})$ to $h^{p(\cdot)}(w^{(\mu-\kappa)})$.*

Proof. Simply observe $\mathcal{M}^0_{N_{p(\cdot)},w^{(\mu)}}(w^{(\kappa)} f) \sim w^{(\kappa)} \mathcal{M}^0_{N_{p(\cdot)},w^{(\mu)}} f$. □

Similar phenomena can be observed if $|\mu| \ll 1$. We omit further details.

9.4. **Periodic case.** Although the exponent $p(\cdot)$ must be constant in this subsection, it seems useful to discuss periodic function spaces. Let $0 < p < \infty$. Let $L^p(\mathbb{T}^n)$ be the set of all p-locally integrable functions f with period 1 for which

$$\|f\|_{L^p(\mathbb{T}^n)} = \left(\int_{[0,1]^n} |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

Similarly, the periodic local Hardy space $h^p(\mathbb{T}^n)$ is the set of all periodic distributions $f \in \mathcal{D}'(\mathbb{T}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ for which

$$\sup_{0<t\le 1} \sup_{\varphi \in \mathcal{D}_N} |\varphi_t * f| \in L^p(\mathbb{T}^n).$$

The norm is given by

$$\|f\|_{h^p(\mathbb{T}^n)} = \left\| \sup_{0<t\le 1} \sup_{\varphi \in \mathcal{D}_N} |\varphi_t * f| \right\|_{L^p(\mathbb{T}^n)}.$$

If a variable exponent $p(\cdot)$ is periodic and satisfies the global log-Hölder condition, then $p(\cdot)$ must be constant. Thus, we assume that $p(\cdot)$ is a constant here.

Lemma 9.6. *For any* $0 < p \le \infty$, $L^p(\mathbb{T}^n) \hookrightarrow L^p(w_{-n-1})$ *and*

$$\|f\|_{L^p(\mathbb{T}^n)} \sim \|f\|_{L^p(w_{-n-1})}.$$

In particular, $h^p(\mathbb{T}^n) \hookrightarrow h^p(w_{-n-1})$.

Proof. Note that

$$w_{-n-1}(x) = (1+|x|)^{-n-1} \sim M\chi_{Q_{0,0}}(x)^{\frac{n+1}{n}}$$

for $x \in \mathbb{R}^n$. Hence simply use $\sum\limits_{m \in \mathbb{Z}^n} (1+|m|)^{-n-1} < \infty$. □

9.5. **Weighted uniformly local Lebesgue spaces with variable exponents.** Let $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and $w \in A^{\rm loc}_{p(\cdot)}$. Then the weighted uniformly local Lebesgue space $L^{p(\cdot)}_{\rm uloc}(w)$ with a variable exponent is defined to be all $f \in L^1_{\rm loc}$ for which the norm $\|f\|_{L^{p(\cdot)}_{\rm uloc}(w)} = \sup\limits_{m \in \mathbb{Z}^n} \|\chi_{Q_{0,m}} f\|_{L^{p(\cdot)}(w)}$ is finite. This is a natural extension of the uniformly local Lebesgue space $L^p_{\rm uloc}$, which considers a substitute of L^∞. If we replace the supremum by the ℓ^r-norm, then the weighted amalgam space $(\ell^r, L^{p(\cdot)}_{\rm uloc}(w))$ with a variable exponent is obtained as an extension of the amalgam space (ℓ^r, L^p) considered in [2, 4, 17, 26, 33]. Although our results are applicable to amalgam spaces, to simplify the argument, we consider uniformly local Lebesgue spaces with variable exponents.

For $w \in A^{\rm loc}_{p(\cdot)}$, we write $w_m(x) = w(x)(1+|x-m|)^{-p_+(1+n)}$. Then by the triangle inequality, we can check that

$$\|f\|_{L^{p(\cdot)}_{\rm uloc}(w)} \sim \sup_{m \in \mathbb{Z}^n} \|f\|_{L^{p(\cdot)}(w_m)}. \tag{9.2}$$

Therefore, if we define the weighted uniformly locally integrable local Hardy spaces $h^{p(\cdot)}_{\rm uloc}(w)$ with variable exponent $p(\cdot)$ and weight w to be the set of all distributions $f \in \mathcal{D}'(\mathbb{R}^n)$ for which

$$\|f\|_{h^{p(\cdot)}_{\rm uloc}(w)} := \|\mathcal{M}^0_{N_{p(\cdot)},w} f\|_{L^{p(\cdot)}_{\rm uloc}(w)}$$

is finite, then we can apply the results obtained in this paper to $h^{p(\cdot)}_{\rm uloc}(w)$. For example, as in this paper, we can obtain the Litttlewood–Paley characterization.

10. Appendix–Proof of Proposition 2.9

Let w be a weight. It is known that $w \in A^{\rm loc}_{p(\cdot)}$ if and only if $M^{\rm loc}$ is bounded on $L^{p(\cdot)}(w)$. In this section, we characterize the class $A^{\rm loc}_{p(\cdot)}$ motivated by reference [14]. As a corollary of this characterization, which is stated in Proposition 2.9, we show that $A^{\rm loc}_{p(\cdot)}$ is monotone in $p(\cdot)$. That is, if $p(\cdot), q(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ satisfy $p(\cdot) \le q(\cdot)$, then $A^{\rm loc}_{p(\cdot)} \subset A^{\rm loc}_{q(\cdot)}$. Similar to Section 2.1, the matters are reduced to the maximal operator generated by dyadic grids and let $\mathfrak{D} = \mathcal{D}_{(1,1,\dots,1)}$. We can handle $\mathcal{D}_{\mathbf{a}}$ for other values of $\mathbf{a} \in \{0,1,2\}^n$. Define $M^{\mathfrak{D}} f$ as the maximal function of $f \in L^0(\mathbb{R}^n)$ with respect to $\mathfrak{D}$. That is,

$$M^{\mathfrak{D}} f(x) = \sup_{Q \in \mathfrak{D}} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| {\rm d}y \quad (x \in \mathbb{R}^n).$$

For $R_0 > 0$, let $M^{\mathfrak{D}}_{\le R_0} f$ be the maximal function of $f \in L^0(\mathbb{R}^n)$, where the supremum is taken over all cubes $Q \in \mathfrak{D}$ with $\ell(Q) \le R_0$, while $M^{\mathfrak{D}}_{\ge R_0} f$ stands for the maximal function of $f \in L^0(\mathbb{R}^n)$ with respect to $\mathfrak{D}$, where the supremum is taken over all cubes $Q \in \mathfrak{D}$ with $\ell(Q) \ge R_0$. Thus,

$$M^{\mathfrak{D}}_{\le R_0} f(x) = \sup_{Q \in \mathfrak{D}, \ell(Q) \le R_0} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| \mathrm{d}y$$

$$M^{\mathfrak{D}}_{\ge R_0} f(x) = \sup_{Q \in \mathfrak{D}, \ell(Q) \ge R_0} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| \mathrm{d}y.$$

Let $p(\cdot)$ be a variable exponent. We define the index p_E by

$$\frac{1}{p_E} = \frac{1}{|E|} \int_E \frac{\mathrm{d}x}{p(x)} \tag{10.1}$$

for all measurable sets E with $|E| > 0$. We also define the norm $\| \cdot \|_{L^{p(\cdot)}(E)}$ by

$$\|f\|_{L^{p(\cdot)}(E)} = \|\chi_E f\|_{L^{p(\cdot)}}$$

for all measurable functions f. Although it is an abuse of the notation, we write $L^{p(\cdot)}(w)$ and $L^{p(\cdot)}(E)$ for a weight w and a measurable set E.

Definition 10.1. Let $p(\cdot) \in \mathrm{LH}_0 \cap \mathrm{LH}_\infty \cap \mathcal{P}$. Define $\tilde{A}^{\mathfrak{D}}_{p(\cdot)}$ as the class of weights w satisfying

$$[w]_{\tilde{A}^{\mathfrak{D}}_{p(\cdot)}} \equiv \sup_{Q \in \mathfrak{D}} |Q|^{-p_Q} \|w\|_{L^1(Q)} \|w^{-1}\|_{L^{p'(\cdot)/p(\cdot)}(Q)} < \infty.$$

Recall [45] shows that $M^{\mathfrak{D}}$ is bounded on $L^{p(\cdot)}(w)$ if and only if $w \in A^{\mathfrak{D}}_{p(\cdot)}$. Here, we show that $\tilde{A}^{\mathfrak{D}}_{p(\cdot)}$ enjoys the same property.

Theorem 10.2. *The maximal operator $M^{\mathfrak{D}}$ is bounded on $L^{p(\cdot)}(w)$ if and only if $w \in \tilde{A}^{\mathfrak{D}}_{p(\cdot)}$. This is equivalent to $\tilde{A}^{\mathfrak{D}}_{p(\cdot)} = A^{\mathfrak{D}}_{p(\cdot)}$.*

The following property has been frequently used in this paper.

Corollary 10.3. *If $q(\cdot) \ge p(\cdot)$, then $A^{\mathfrak{D}}_{q(\cdot)} \supset A^{\mathfrak{D}}_{p(\cdot)}$.*

We remark that Proposition 2.9 follows from the corresponding assertion to the generalized dyadic grid $\mathfrak{D}$. Corollary 10.3 remains true for other grids. Due to Lemma 10.6 below as well as Theorem 10.2, $A^{\mathfrak{D}}_{q(\cdot)} = \tilde{A}^{\mathfrak{D}}_{q(\cdot)} \supset \tilde{A}^{\mathfrak{D}}_{p(\cdot)} = A^{\mathfrak{D}}_{p(\cdot)}$, which proves Corollary 10.3. Thus, along with the technique of constructing a weight in $A^{\mathfrak{D}}_{p(\cdot)}$ from $A^{\mathrm{loc}}_{p(\cdot)}$, this relation of weights proves Proposition 2.9. Another corollary of Corollary 10.3 and (2.4) is the monotonicity of the class of $A_{p(\cdot)}$ considered in [10].

Corollary 10.4. *If $q(\cdot) \ge p(\cdot)$, then $A_{q(\cdot)} \supset A_{p(\cdot)}$.*

10.1. **Sufficiency in Theorem 10.2.** We also need the local versions of $A^{\mathfrak{D}}_{p(\cdot)}$: For a measurable subset E of $\mathbb{R}^n$, we define

$$\mathfrak{D}(E) \equiv \{Q \in \mathfrak{D} \, : \, Q \subset E\}.$$

Definition 10.5. Let E be a subset of $\mathbb{R}^n$.

(1) Define $\tilde{A}^{\mathfrak{D}}_{p(\cdot)}(E)$ as the class of weights w satisfying

$$[w]_{\tilde{A}^{\mathfrak{D}}_{p(\cdot)}(E)} \equiv \sup_{Q \in \mathfrak{D}(E)} |Q|^{-p_Q} \|w\|_{L^1(Q)} \|w^{-1}\|_{L^{p'(\cdot)/p(\cdot)}(Q)} < \infty.$$

(2) Define $A^{\mathfrak{D}}_p(E)$ for $E \in \mathfrak{D}$ and $1 < p < \infty$ analogously.

The following lemma is easy to prove:

Lemma 10.6. [14, Lemma 3.1] *Let $p(\cdot), q(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Assume $p(\cdot) \le q(\cdot)$ everywhere. Then there exists a constant $C_0 > 0$, which depends on $p_\pm$, $q_\pm$, $c_*(p(\cdot))$, $c_*(q(\cdot))$, $c^*(p(\cdot))$ and $c^*(q(\cdot))$, such that*

$$[w]_{\tilde{A}^{\mathfrak{D}}_{q(\cdot)}} \le C_0 [w]_{\tilde{A}^{\mathfrak{D}}_{p(\cdot)}}.$$

Proof. Fix $Q \in \mathfrak{D}$. Let $\alpha(\cdot)$ satisfy

$$\frac{1}{\alpha(\cdot)} = \frac{q(\cdot)}{q'(\cdot)} - \frac{p(\cdot)}{p'(\cdot)} = q(\cdot) - p(\cdot) \ge 0.$$

Since $\alpha(\cdot) \in \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, we have

$$\|\chi_Q\|_{L^{\alpha(\cdot)}} \sim |Q|^{\frac{1}{\alpha_Q}} = |Q|^{\frac{1}{|Q|}\int_Q q(x)\mathrm{d}x - \frac{1}{|Q|}\int_Q p(x)\mathrm{d}x} \sim |Q|^{q_Q - p_Q} \sim 1.$$

Here, the third equivalence follows from [14, Lemma 2.1] (see also Lemma 2.2). Thus, from the Hölder inequality for Lebesgue spaces with variable exponents, we have

$$[w]_{\tilde{A}^{\mathfrak{D}}_{q(\cdot)}} \le C_0 [w]_{\tilde{A}^{\mathfrak{D}}_{p(\cdot)}}.$$

Thus, the proof is complete. □

We have a local counterpart.

Corollary 10.7. *Let $p(\cdot), q(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and let $R \in \mathfrak{D}$. Assume $p(\cdot) \le q(\cdot)$ everywhere. Then there exists a constant $C_0 > 0$, which depends on $p_\pm(R)$, $q_\pm(R)$, $c_*(p(\cdot)|R)$, $c_*(q(\cdot)|R)$, $c^*(p(\cdot)|R)$ and $c^*(q(\cdot)|R)$ such that $[w]_{\tilde{A}^{\mathfrak{D}}_{q(\cdot)}(R)} \le C_0 [w]_{\tilde{A}^{\mathfrak{D}}_{p(\cdot)}(R)}$ for all $R \in \mathfrak{D}$.*

Although the following estimate is crude, it is important.

Lemma 10.8. [14, Lemma 3.3] *Let $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. If $w \in \tilde{A}^{\mathfrak{D}}_{p(\cdot)}$, then*

$$w(Q) \gtrsim \min\left(1, \frac{|Q|}{|S|}\right)^{p_+} w(S)$$

for all cubes $Q, S \in \mathfrak{D}$ with $Q \cap S \ne \emptyset$.

Proof. When $Q, S \in \mathfrak{D}$ with $Q \cap S \ne \emptyset$, there are four cases;

(1) $Q \subset S$,
(2) $Q \supset S$,
(3) $|Q| \le |S|$ and there is a unique cube $\widetilde{S} \supset S$ such that $|\widetilde{S}| = 2^n |S|$ and $Q \subset \widetilde{S}$,
(4) $|S| \le |Q|$ and there is a unique cube $\widetilde{Q} \supset Q$ such that $|\widetilde{Q}| = 2^n |Q|$ and $S \subset \widetilde{Q}$.

First, we assume (1) and (3). We know that $w \in \tilde{A}^{\mathfrak{D}}_{p_+}$ thanks to Lemma 10.6. Since

$$M^{\mathfrak{D}} \chi_Q \gtrsim \frac{|Q|}{|S|} \chi_{\widetilde{S}},$$

we have

$$\begin{aligned} w(Q) = \int_{\mathbb{R}^n} \chi_Q(z)^{p_+} w(z)\mathrm{d}z &\gtrsim \int_{\mathbb{R}^n} M^{\mathfrak{D}} \chi_Q(z)^{p_+} w(z)\mathrm{d}z \\ &\gtrsim \int_{\mathbb{R}^n} \chi_{\widetilde{S}}(z) \min\left(1, \frac{|Q|}{|S|}\right)^{p_+} w(z)\mathrm{d}z \\ &= \min\left(1, \frac{|Q|}{|S|}\right)^{p_+} w(\widetilde{S}) \ge \min\left(1, \frac{|Q|}{|S|}\right)^{p_+} w(S). \end{aligned}$$

If (2) and (4) hold, then this is clear since w is a non-negative function. When we consider the case (4), note that using the above argument, we can show that $w(Q) \gtrsim w(\tilde{Q})$. Thus, the proof is complete. □

Since we assume that w is (globally) in $\tilde{A}^{\mathfrak{D}}_{p(\cdot)}$, w has at most polynomial growth. Here and below, we let $Q^\dagger_k \in \mathfrak{D}$ be the unique cube in $\mathfrak{D}_k$ containing 0 for $k \in \mathbb{Z}$. It is noteworthy that $\{Q^\dagger_{-k}\}_{k\in\mathbb{Z}}$ is an increasing family of cubes.

Corollary 10.9. *Let $w \in \tilde{A}^{\mathfrak{D}}_{p(\cdot)}$ and $Q \in \mathfrak{D}$. Assume that $Q \neq Q^\dagger_{-k}$ for any $k \in \mathbb{N}$. Then $w(Q) \lesssim (1+|x|)^{p_+ n}$ for all $x \in Q$.*

Proof. Fix $Q \in \mathfrak{D}$. Let $k_0 \in \mathbb{Z}$ be the largest integer such that $Q \subset Q^\dagger_{k_0}$. If $k_0 \geq -1$, then

$$w(Q) \leq w(Q^\dagger_{k_0}) \leq w(Q^\dagger_{-1}) \lesssim 1 \lesssim (1+|x|)^{p_+ n}.$$

If $k_0 \leq -2$, then $Q \subsetneq Q^\dagger_{k_0}$ by assumption. Note that Q and $Q^\dagger_{k_0+2}$ do not intersect; otherwise $Q \subset Q^\dagger_{k_0+2}$ or $Q \supsetneq Q^\dagger_{k_0+2}$. The first possibility never occurs in view of the maximality of k_0. Meanwhile, since $Q \subsetneq Q^\dagger_{k_0}$ and $Q, Q^\dagger_{k_0} \in \mathfrak{D}$, $|Q^\dagger_{k_0}| \geq 4^n|Q|$ and $|Q^\dagger_{k_0+2}| = \frac{1}{4^n}|Q^\dagger_{k_0}| \geq |Q|$. Hence, the latter case is also impossible. Thus, $|x| \sim \ell(Q^\dagger_{k_0})$ for all $x \in Q (\subset \mathbb{R}^n \setminus Q^\dagger_{k_0+2})$. Consequently, thanks to Lemma 10.8

$$w(Q) \leq w(Q^\dagger_{k_0}) \lesssim \left(\frac{|Q^\dagger_{k_0}|}{|Q^\dagger_0|}\right)^{p_+} w(Q^\dagger_0) \lesssim (1+|x|)^{p_+ n},$$

as required. □

We have various quantities equivalent to $\|\chi_Q\|_{L^{p(\cdot)}(w)}$ if the cube $Q \in \mathfrak{D}$ is small.

Lemma 10.10. *Let $w \in \tilde{A}^{\mathfrak{D}}_{p(\cdot)}$ and $Q \in \mathfrak{D}_k$ with $k \geq -1$. Then*

$$\|\chi_Q\|_{L^{p(\cdot)}(w)} \sim w(Q)^{\frac{1}{p_+(Q)}} \sim w(Q)^{\frac{1}{p_-(Q)}} \sim w(Q)^{\frac{1}{p(x)}} \sim w(Q)^{\frac{1}{p_Q}} \tag{10.2}$$

for all $x \in Q$.

Proof. We concentrate on the proof of $w(Q)^{\frac{1}{p_+(Q)}} \sim w(Q)^{\frac{1}{p_-(Q)}}$; other equivalences are clear, since other quantities are between $w(Q)^{\frac{1}{p_+(Q)}}$ and $w(Q)^{\frac{1}{p_-(Q)}}$.

First, assume that $Q \cap Q^\dagger_{-1} = \emptyset$. In this case, we choose the largest integer $k_0 \in \mathbb{Z}$ such that $Q \subset Q^\dagger_{k_0}$. We have $k_0 \leq -2$. Then Q and $Q^\dagger_{k_0+2}$ do not intersect. Otherwise we have either $Q \subset Q^\dagger_{k_0+2}$ or $Q_{k_0+2} \subsetneq Q \subsetneq Q^\dagger_{k_0}$. As in the proof for Corollary 10.9, neither of these cases occurs.

Since $w \in \tilde{A}^{\mathfrak{D}}_{p(\cdot)}$ and $|x| \sim \ell(Q^\dagger_{k_0}) \sim \ell(Q^\dagger_{k_0+2}) \gtrsim 1$ for all $x \in Q$,

$$\left(\frac{r}{1+|x|}\right)^{np_+} w(Q^\dagger_{-1}) \leq \left(\frac{r}{1+|x|}\right)^{np_+} w(Q^\dagger_{k_0}) \lesssim w(Q) \leq w(Q^\dagger_{k_0}) \lesssim (1+|x|)^{np_+} w(Q^\dagger_{-1}) \tag{10.3}$$

for all $x \in Q$ thanks to Lemma 10.8, where $r = -\log_2 \ell(Q)$. If we use the global/local log-Hölder conditions, then $w(Q)^{\frac{1}{p_+(Q)}} \sim w(Q)^{\frac{1}{p_-(Q)}}$ and hence (10.2).

Let us deal with the case $Q \cap Q^\dagger_{-1} \neq \emptyset$. Then we have $Q \subset Q^\dagger_{-2}$. In fact, if $Q = Q^\dagger_k (k \geq -1)$, this claim is clear since $\{Q^\dagger_j\}_j$ is decreasing. Otherwise, let $Q \in \mathfrak{D}_k \setminus \{Q^\dagger_k\} (k \geq 0)$. If k is even, then $Q \subset Q_0^\dagger$ thanks to the construction of the dyadic grids. In particular $Q \subset Q_{-2}^\dagger$. Similarly, if k is odd, then $Q \subset Q_{-1}^\dagger \subset Q_{-2}^\dagger$. Hence

$$w(Q_{-2}^\dagger) \geq w(Q) \gtrsim \left(\frac{|Q|}{|Q_{-2}^\dagger|}\right)^{p_+} w(Q_{-2}^\dagger)$$

thanks to Lemma 10.8. Thus,

$$w(Q)^{\left|\frac{1}{p_+(Q)}-\frac{1}{p_-(Q)}\right|} \gtrsim w(Q_{-2}^\dagger)^{\left|\frac{1}{p_+(Q)}-\frac{1}{p_-(Q)}\right|} \left(\frac{|Q_{-2}^\dagger|}{|Q|}\right)^{-p_+\left|\frac{1}{p_+(Q)}-\frac{1}{p_-(Q)}\right|} \gtrsim 1$$

and

$$w(Q)^{\left|\frac{1}{p_+(Q)}-\frac{1}{p_-(Q)}\right|} \leq w(Q_{-2}^\dagger)^{\left|\frac{1}{p_+(Q)}-\frac{1}{p_-(Q)}\right|} \lesssim 1.$$

Thus, $w(Q)^{\frac{1}{p_+(Q)}} \sim w(Q)^{\frac{1}{p_-(Q)}}$. □

Using a similar argument to [14, Proposition 3.8], we have the following equivalence:

Lemma 10.11. *If $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and $w \in \tilde{A}_{p(\cdot)}^{\mathfrak{D}}$, then, for all $Q \in \mathfrak{D}$,*

$$|Q|^{-p_Q} w(Q) \left\|w^{-1}\right\|_{L^{\frac{p'(\cdot)}{p(\cdot)}}(Q)} \sim \frac{w(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^{p_Q-1}.$$

Proof. Let $w \in \tilde{A}_{p(\cdot)}^{\mathfrak{D}}$ and suppose that $Q \in \bigcup_{k=0}^{\infty} \mathfrak{D}_k$. By the definition of $\|w\|_{\tilde{A}_{p(\cdot)}^{\mathfrak{D}}}$, we have

$$\frac{w(Q)}{|Q|^{p_Q}} \|w^{-1}\|_{L^{\frac{p'(\cdot)}{p(\cdot)}}(Q)} \leq \|w\|_{\tilde{A}_{p(\cdot)}^{\mathfrak{D}}}.$$

Due to Lemma 10.10, $w(Q)^{\frac{1}{p_Q}} \sim \|\chi_Q\|_{L^{p(\cdot)}(w)} = \|w^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(Q)}$. By virtue of the Hölder inequality, we have

$$\begin{aligned}|Q| = \int_Q w(y)^{\frac{1}{p(y)}} w(y)^{-\frac{1}{p(y)}} \mathrm{d}y &\leq 2\|w^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(Q)} \|w^{-\frac{1}{p(\cdot)}}\|_{L^{p'(\cdot)}(Q)} \\ &\lesssim \|w(Q)^{\frac{1}{p_Q}} w^{-\frac{1}{p(\cdot)}}\|_{L^{p'(\cdot)}(Q)}.\end{aligned}$$

This means that

$$\int_Q \left(\frac{w(Q)^{\frac{1}{p_Q}}}{|Q|}\right)^{p'(y)} w(y)^{-\frac{p'(y)}{p(y)}} \mathrm{d}y \gtrsim 1.$$

Again, using Lemma 10.10, we have $w(Q)^{\frac{1}{p_Q}} \sim w(Q)^{\frac{1}{p(y)}}$ for all $y \in Q$. Since $|Q| \leq 1$, we obtain

$$\int_Q \left(\frac{w(Q)}{|Q|^{p_Q}}\right)^{\frac{p'(y)}{p(y)}} w(y)^{-\frac{p'(y)}{p(y)}} \mathrm{d}y \gtrsim 1.$$

Therefore,

$$\frac{w(Q)}{|Q|^{p_Q}} \left\|w^{-1}\right\|_{L^{\frac{p'(\cdot)}{p(\cdot)}}(Q)} \gtrsim 1.$$

From the definition of the quantity $[w]_{\tilde{A}_{p(\cdot)}^{\mathfrak{D}}}$, we conclude

$$\frac{w(Q)}{|Q|^{p_Q}} \left\|w^{-1}\right\|_{L^{\frac{p'(\cdot)}{p(\cdot)}}(Q)} \sim 1.$$

Hence

$$\left\|w^{-1}\right\|_{L^{\frac{p'(\cdot)}{p(\cdot)}}(Q)} \sim \frac{|Q|^{p_Q}}{w(Q)}.$$

This implies that by Lemma 10.10

$$\sigma(Q)^{p_Q-1} \sim \frac{|Q|^{p_Q}}{w(Q)}.$$

Thus, if $|Q| \leq 1$, then we have the desired property.

This means that $\sigma \in A_\infty^{\mathrm{loc},\mathfrak{D}}$. Furthermore, if R satisfies $|R| = 1$, then by Remark 2.4, and

$$\min(\|\chi_R\|_{L^{\frac{1}{p(\cdot)-1}}(\sigma)}^{\frac{1}{p_+(R)-1}}, \|\chi_R\|_{L^{\frac{1}{p(\cdot)-1}}(\sigma)}^{\frac{1}{p_-(R)-1}}) \leq \sigma(R) \leq \max(\|\chi_R\|_{L^{\frac{1}{p(\cdot)-1}}(\sigma)}^{\frac{1}{p_+(R)-1}}, \|\chi_R\|_{L^{\frac{1}{p(\cdot)-1}}(\sigma)}^{\frac{1}{p_-(R)-1}}),$$

we have

$$\|\chi_R\|_{L^{\frac{p'(\cdot)}{p(\cdot)}}(\sigma)} \sim \sigma(R)^{p_\infty - 1}.$$

By the localization property (Lemma 2.5), we have

$$\|w^{-1}\|_{L^{\frac{p'(\cdot)}{p(\cdot)}}(Q)} = \|\chi_Q\|_{L^{\frac{p'(\cdot)}{p(\cdot)}}(\sigma)} \sim \sigma(Q)^{p_\infty - 1}$$

for all cubes Q with $|Q| \geq 1$. Since we know that $|Q|^{\frac{1}{p_\infty} - \frac{1}{p_Q}} \sim w(Q)^{\frac{1}{p_\infty} - \frac{1}{p_Q}} \sim 1$, we see that $\sigma(Q)^{\frac{1}{p_\infty} - \frac{1}{p_Q}} \sim 1$. Thus, the proof is complete.

□

Under these preparations, we establish the boundedness of the local maximal operator.

Lemma 10.12. [14, Lemma 5.1] *Let $w \in \tilde{A}_{p(\cdot)}^{\mathfrak{D}}$. Then there exists $r_0 = r_0([w]_{\tilde{A}_{p(\cdot)}^{\mathfrak{D}}}, p(\cdot)) \in (0,1)$ such that $-\log_2 r_0$ is an integer and that*

$$\|\chi_Q M_{\leq r_0}^{\mathfrak{D}}[\chi_Q f]\|_{L^{p(\cdot)}(w)} \lesssim \|\chi_{3Q} f\|_{L^{p(\cdot)}(w)}$$

for all $f \in L^{p(\cdot)}(w)$ and $Q \in \mathfrak{D}_j$ with $\ell(Q) < r_0 \leq \frac{1}{2}$. Here $j = 1 - \log_2 r_0$.

Proof. For now, let $r_0 \in (0,1)$ be small enough. We will specify it shortly. Then, there exists $j \in \mathbb{N}$ such that $2^{-j} < r_0 \leq 2^{-j+1}$. Fix $Q \in \mathfrak{D}_j$. Note that $\ell(Q) < r_0$ by the definition of j. Let C_0 be the constant from Lemma 10.6. Write $c_1 \equiv C_0[w]_{\tilde{A}_{p(\cdot)}^{\mathfrak{D}}}$. Then there exist $c_2 > 0$ and $\varepsilon \in (0,1)$, which is independent of $S \in \mathfrak{D}$ such that $[\sigma]_{A_q(S)} \leq c_1$ implies $[\sigma]_{A_{q-\varepsilon}(S)} \leq c_2$ for all $\sigma \in A_q(S)$ and $q \in [p_-, p_+ + 1]$ by the openness property established by Hytönen and Pérez [27] (see also Lemma 2.13).

Next using the log-Hölder continuity, we can choose $r_0 < \frac{1}{2}n^{-1/2}$ so that $p_+(3S) - \varepsilon < p_-(3S)$ for all $S \in \mathfrak{D}_{-\log_2 r_0}$ and $j \equiv -\log_2 r_0 + 1 \in \mathbb{N}$. By virtue of Lemma 10.6,

$$[w]_{A_{p_+(3Q)}^{\mathfrak{D}}(3Q)} \leq C_0[w]_{\tilde{A}_{p(\cdot)}^{\mathfrak{D}}(3Q)} \leq C_0[w]_{\tilde{A}_{p(\cdot)}^{\mathfrak{D}}} = c_1.$$

By the property of c_2, we have

$$[w]_{A_{p_-(3Q)}^{\mathfrak{D}}(3Q)} \leq C_0[w]_{A_{p_+(3Q)-\varepsilon}^{\mathfrak{D}}(3Q)} \leq c_2 C_0.$$

Let $f \in L^{p(\cdot)}(w)$ with $\|\chi_{3Q} f\|_{L^{p(\cdot)}(w)} \leq 1$. Set $g \equiv \chi_Q f$ and

$$q(\cdot) \equiv \frac{p(\cdot)}{p_-(3Q)}. \tag{10.4}$$

Fix $x \in Q$ and choose a cube $R \in \bigcup_{k=j}^{\infty} \mathfrak{D}_k$ with $x \in R$. Note that $\ell(R) \le 2^{-j} < r_0$. Then for all $\beta > 0$,

$$\left(\frac{1}{|R|}\int_R |g(y)|\mathrm{d}y\right)^{q(x)} \le \left(\frac{1}{|R|}\int_R |g(y)|^{q_-(R)}\mathrm{d}y\right)^{\frac{q(x)}{q_-(R)}} = \left(\frac{1}{\beta|R|}\int_R \beta|g(y)|^{q_-(R)}\mathrm{d}y\right)^{\frac{q(x)}{q_-(R)}}$$

by Hölder's inequality. By the inequality $t \le 1 + t^{\frac{q(\cdot)}{q_-(R)}}$, we have

$$\begin{aligned}\left(\frac{1}{|R|}\int_R |g(y)|\mathrm{d}y\right)^{q(x)} &\le \left(\frac{1}{\beta|R|}\int_R \left(1+\beta^{\frac{q(y)}{q_-(R)}}|g(y)|^{q(y)}\right)\mathrm{d}y\right)^{\frac{q(x)}{q_-(R)}}\\ &= \left(\frac{1}{\beta}+\frac{1}{|R|}\int_R \beta^{\frac{q(y)}{q_-(R)}-1}|g(y)|^{q(y)}\mathrm{d}y\right)^{\frac{q(x)}{q_-(R)}}.\end{aligned}$$

We choose $\beta \equiv \max(1, w(3Q)^{\frac{1}{p_-(3Q)}})$. We suppose that $0 \notin 5Q$. Decompose equally $3Q$ into 3^n cubes $Q_1, Q_2, \ldots, Q_{3^n}$. By Corollary 10.9, we have

$$w(Q_k) \lesssim (1+|y|)^{p_+n}$$

for all $y \in Q_k$ and $k = 1, 2, \ldots, 3^n$. Note that for all $y \in Q_k$ and $z \in 3Q$, $|y| \sim |z|$. Thus,

$$w(3Q) = \sum_{k=1}^{3^n} w(Q_k) \lesssim (1+|y|)^{p_+n} \sim (1+|z|)^{p_+n} \tag{10.5}$$

for all $z \in 3Q$. Meanwhile, if $5Q \ni 0$, then $w(3Q) \lesssim w([-10,10]^n) \lesssim 1 \lesssim (1+|z|)^{p_+n}$ for all $z \in 3Q$. Since $Q \in \mathfrak{D}_j$ for $j \in \mathbb{N}$, $R \subset 3Q$. Hence, estimate (10.5) still holds for all $y \in R$.

Thus, since $q(\cdot) \in \mathrm{LH}_\infty$,

$$\beta^{\frac{q(y)}{q_-(R)}-1} = \max(1, w(3Q)^{\frac{1}{p_-(3Q)}})^{\frac{q(y)}{q_-(R)}-1} \lesssim ((1+|y|)^{p_+n})^{q(y)-q_-(R)} \lesssim 1$$

for all $y \in R$, where the implicit constant depends on $p(\cdot)$. Since $q(x) \ge q_-(R)$ and $\beta \ge 1$, we obtain

$$\begin{aligned}&\left(\frac{1}{|R|}\int_R |g(y)|\mathrm{d}y\right)^{q(x)}\\ &\lesssim \frac{1}{\beta} + \left(\frac{1}{|R|}\int_R |g(y)|^{q(y)}\mathrm{d}y\right)^{\frac{q(x)}{q_-(R)}}\\ &= \min(1, w(3Q)^{-\frac{1}{p_-(3Q)}}) + \left(\frac{1}{|R|}\int_R |g(y)|^{q(y)}\mathrm{d}y\right)^{\frac{q(x)}{q_-(R)}-1} \times \frac{1}{|R|}\int_R |g(y)|^{q(y)}\mathrm{d}y.\end{aligned} \tag{10.6}$$

Thanks to the Young inequality, the definition of the quantity $A^{\mathfrak{D}}_{p_-(3Q)}(3Q)$ and the fact that $\|\chi_{3Q}f\|_{L^{p(\cdot)}(w)} \le 1$, we have

$$\begin{aligned}\int_R |g(y)|^{q(y)}\mathrm{d}y &\le \int_R |g(y)|^{p(y)}w(y)\mathrm{d}y + \int_R w(y)^{-\frac{1}{p_-(3Q)-1}}\mathrm{d}x\\ &= \int_R |g(y)|^{p(y)}w(y)\mathrm{d}y + \left\{\left(\int_R w(y)^{-\frac{1}{p_-(3Q)-1}}\mathrm{d}x\right)^{p_-(3Q)-1}\right\}^{\frac{1}{p_-(3Q)-1}}\\ &\lesssim 1 + ([w]_{A^{\mathfrak{D}}_{p_-(3Q)}(3Q)}|R|^{p_-(3Q)}w(R)^{-1})^{-\frac{1}{p_-(3Q)-1}}.\end{aligned}$$

Recall that $\ell(R) \le \frac{1}{2}$. Since $\frac{q(x)}{q_-(R)} - 1 \ge 0$, thanks to the log-Hölder continuity of $q(\cdot)$ and Lemma 10.10, we have

$$\left(\frac{1}{|R|}\int_R |g(y)|^{q(y)}\mathrm{d}y\right)^{\frac{q(x)}{q_-(R)}-1} \lesssim \left(1 + ([w]_{A^{\mathfrak{D}}_{p_-(3Q)}(3Q)}|R|^{p_-(3Q)}w(R)^{-1})^{-\frac{1}{p_-(3Q)-1}}\right)^{\frac{q(x)}{q_-(R)}-1} \sim 1.$$

From (10.6), we obtain

$$\left(\frac{1}{|R|}\int_R |g(y)|\mathrm{d}y\right)^{q(x)} \lesssim \min(1, w(3Q)^{-\frac{1}{p_-(3Q)}}) + \frac{1}{|R|}\int_R |g(y)|^{q(y)}\mathrm{d}y.$$

Recall that R is a cube with $\ell(R) \le r_0 \le 1/2$. Therefore from the definition of $M^{\mathfrak{D}}_{\le r_0}$, we deduce

$$M^{\mathfrak{D}}_{\le r_0}g(x)^{q(x)} \lesssim M^{\mathfrak{D}}_{\le r_0}[|g|^{q(\cdot)}](x) + \min(1, w(3Q)^{-\frac{1}{p_-(3Q)}}) \tag{10.7}$$

Recall that $q(\cdot)$ is given by (10.4). Inserting the definition of $q(\cdot)$ into (10.7), we obtain

$$M^{\mathfrak{D}}_{\le r_0}g(x)^{p(x)} \lesssim (M^{\mathfrak{D}}_{\le r_0}[|g|^{q(\cdot)}](x))^{p_-(3Q)} + \frac{1}{\max(1, w(3Q))}.$$

Since $w \in A_{p_-(3Q)}(3Q)$, integrating the above inequality for the measure $w(x)\mathrm{d}x$ over Q gives

$$\begin{aligned}\int_Q M^{\mathfrak{D}}_{\le r_0}g(x)^{p(x)}w(x)\mathrm{d}x &\lesssim \int_Q \left[(M^{\mathfrak{D}}_{\le r_0}[|g|^{q(\cdot)}](x))^{p_-(3Q)} + \frac{1}{\max(1, w(3Q))}\right]w(x)\mathrm{d}x\\ &\lesssim \int_Q |f(x)|^{q(\cdot)p_-(3Q)}w(x)\mathrm{d}x + 1 \sim 1.\end{aligned}$$

Hence, we have $\|\chi_Q M^{\mathfrak{D}}_{\le r_0}g\|_{L^{p(\cdot)}(w)} = \|\chi_Q M^{\mathfrak{D}}_{\le r_0}[\chi_Q f]\|_{L^{p(\cdot)}(w)} \le 1$, as desired. □

By the localization argument (see Lemma 2.5), we can prove $M^{\mathfrak{D}}_{\le r_0}$ is bounded on $L^{p(\cdot)}(w)$.

Lemma 10.13. [14, Lemma 5.3] *Let $w \in \tilde{A}^{\mathfrak{D}}_{p(\cdot)}$. Then there exists $r_0 = r_0([w]_{\tilde{A}^{\mathfrak{D}}_{p(\cdot)}}, p(\cdot)) \in (0,1)$ such that*

$$\|M^{\mathfrak{D}}_{\le r_0}f\|_{L^{p(\cdot)}(w)} \lesssim \|f\|_{L^{p(\cdot)}(w)}$$

for all $f \in L^{p(\cdot)}(w)$.

Let $w \in \tilde{A}^{\mathfrak{D}}_{p(\cdot)}$. Define

$$E_k f \equiv \sum_{Q\in\mathfrak{D}_k} \chi_Q m_Q(f) \quad (k \in \mathbb{Z})$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We harvest a corollary of Lemmas 10.12 and 10.13.

Corollary 10.14. *Let $w \in \tilde{A}^{\mathfrak{D}}_{p(\cdot)}$. If $k \gg 1$, then E_k is bounded on $L^{p(\cdot)}(w)$.*

Proof. Simply observe that $|E_k f| \le M^{\mathfrak{D}}_{\le r_0}f$ for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. □

We obtain another corollary of Lemmas 10.12 and 10.13.

Corollary 10.15. *Let $w \in \tilde{A}^{\mathfrak{D}}_{p(\cdot)}$ and $r_0 \in (0,1)$ be the same as in Lemma 10.12. Then*

$$\|M^{\mathfrak{D}}_{\ge r_0}[\chi_R f]\|_{L^{p(\cdot)}(w)} \lesssim \|f\|_{L^{p(\cdot)}(w)} \tag{10.8}$$

for all $f \in L^{p(\cdot)}(w)$ and $R \in \mathfrak{D}$ with $\ell(R) = r_0$ with the implicit constant dependent on r_0.

Proof. Let $k \equiv 1 - \log_2 r_0$. Fix $R \in \mathfrak{D}$ with $\ell(R) = r_0$.

Let $x \in Q_k^\dagger$. Then, a geometric observation shows that there exists the smallest cube $Q_i^\dagger$ such that $|\ell(Q_i)| \geq r_0, Q_k^\dagger, R \subset Q_i^\dagger$ and $Q_k^\dagger \cap Q = \emptyset$ for $Q \in \bigcup_{j=-\infty}^{i} (\mathfrak{D}_j \setminus \{Q_j^\dagger\})$. Thus,

$$\chi_{Q_k^\dagger}(x) M_{\geq r_0}^{\mathfrak{D}}[\chi_R f](x) = \frac{\chi_{Q_k^\dagger}(x)}{|Q_i^\dagger|} \int_{Q_i^\dagger} \chi_R(y)|f(y)|\mathrm{d}y.$$

From this pointwise estimate, we have

$$\begin{aligned}
\|\chi_{Q_k^\dagger} M_{\geq r_0}^{\mathfrak{D}}[\chi_R f]\|_{L^{p(\cdot)}(w)} &\leq \left\| \frac{\chi_{Q_k^\dagger}}{|R|} \int_{Q_i^\dagger} \chi_R(y)|f(y)|\mathrm{d}y \right\|_{L^{p(\cdot)}(w)} \\
&= \frac{\|\chi_{Q_k^\dagger}\|_{L^{p(\cdot)}(w)}}{\|\chi_R\|_{L^{p(\cdot)}(w)}} \left\| \frac{\chi_R}{|R|} \int_{Q_i^\dagger} \chi_R(y)|f(y)|\mathrm{d}y \right\|_{L^{p(\cdot)}(w)} \\
&\lesssim \|M_{\leq r_0}^{\mathfrak{D}} f\|_{L^{p(\cdot)}(w)} \\
&\lesssim \|f\|_{L^{p(\cdot)}(w)}.
\end{aligned}$$

Thus, we obtain

$$\left\|\chi_{Q_k^\dagger} M_{\geq r_0}^{\mathfrak{D}}[\chi_{Q_k^\dagger} f]\right\|_{L^{p(\cdot)}(w)} \leq \left\|M_{\leq r_0}^{\mathfrak{D}} f\right\|_{L^{p(\cdot)}(w)} \lesssim \|f\|_{L^{p(\cdot)}(w)} \tag{10.9}$$

by Lemma 10.13.

Let $x \in \mathbb{R}^n \setminus Q_k^\dagger$. Then, a geometric observation shows that there exists the largest number $\ell < k$ such that $x \in Q_\ell^\dagger$. Then, by the maximality of ℓ, we have $|x| \sim |Q_\ell^\dagger|$. Since $Q_k^\dagger \subset Q_\ell^\dagger$, we obtain

$$\begin{aligned}
&\chi_{\mathbb{R}^n \setminus Q_k^\dagger}(x) M_{\geq r_0}^{\mathfrak{D}}[\chi_{Q_k^\dagger} f](x) \\
&= \frac{\chi_{Q_\ell^\dagger \setminus Q_k^\dagger}(x)}{|Q_\ell^\dagger|} \int_{Q_\ell^\dagger} \chi_{Q_k^\dagger}(y)|f(y)|\mathrm{d}y \sim \frac{\chi_{Q_\ell^\dagger \setminus Q_k^\dagger}(x)}{|x|^n} \int_{Q_k^\dagger} |f(y)|\mathrm{d}y
\end{aligned}$$

Then, by Hölder's inequality and Lemma 10.10, we have

$$\begin{aligned}
\int_{Q_k^\dagger} |f(y)|\mathrm{d}y &\lesssim \|\chi_{Q_k^\dagger} f\|_{L^{p(\cdot)}(w)} \|w^{-1}\chi_{Q_k^\dagger}\|_{L^{p'(\cdot)}(w)} \\
&\leq \|\chi_{Q_k^\dagger}\|_{L^{p'(\cdot)}(\sigma)} \|f\|_{L^{p(\cdot)}(w)} \lesssim \sigma(Q_k^\dagger)^{\frac{1}{p'_{Q_k^\dagger}}} \|f\|_{L^{p(\cdot)}(w)},
\end{aligned}$$

where $\sigma \equiv w^{-\frac{1}{p(\cdot)-1}}$ stands for the dual weight. We obtain

$$\chi_{\mathbb{R}^n \setminus Q_k^\dagger}(x) M_{\geq r_0}^{\mathfrak{D}}[\chi_{Q_k^\dagger} f](x) \lesssim \chi_{\mathbb{R}^n \setminus Q_k^\dagger}(x)|x|^{-n} \sigma(Q_k^\dagger)^{\frac{1}{p'_{Q_k^\dagger}}} \|f\|_{L^{p(\cdot)}(w)}.$$

Thus,

$$\left\|\chi_{\mathbb{R}^n \setminus Q_k^\dagger} M_{\geq r_0}^{\mathfrak{D}}[\chi_{Q_k^\dagger} f]\right\|_{L^{p(\cdot)}(w)} \lesssim \||\cdot|^{-n}\chi_{\mathbb{R}^n \setminus Q_k^\dagger}\|_{L^{p(\cdot)}(w)} \|f\|_{L^{p(\cdot)}(w)}. \tag{10.10}$$

So we will estimate $\||\cdot|^{-n}\chi_{\mathbb{R}^n \setminus Q_k^\dagger}\|_{L^{p(\cdot)}(w)}$ using the modular. Let C_0 be the constant from Lemma 10.6. Write $c_1 \equiv {C_0}^2[w]_{\tilde{A}_{p(\cdot)}^{\mathfrak{D}}}$. Then there exist $c_2 > 0$ and $\varepsilon \in (0,1)$ independent of the set E such that $[\sigma]_{A_q^{\mathfrak{D}}(E)} \leq c_1$ implies $[\sigma]_{A_{q-\varepsilon}^{\mathfrak{D}}(E)} \leq c_2$ for all $\sigma \in A_q^{\mathfrak{D}}(E)$ and $q \in [p_-, p_+ + 1]$ for all sets E again by the openness property established by Hytönen and Pérez [27]. Let $k' \ll -1$ be an integer so that $p_+(\mathbb{R}^n \setminus Q^\dagger_{k'}) - \frac{1}{3}\varepsilon \le p_\infty$. Then $w \in A_{p_\infty + \frac{1}{3}\varepsilon}(\mathbb{R}^n \setminus Q^\dagger_{k'})$ with

$$[w]_{A^{\mathfrak{D}}_{p_\infty + \frac{1}{3}\varepsilon}(\mathbb{R}^n \setminus Q^\dagger_{k'})} \le C_0 [w]_{A^{\mathfrak{D}}_{p_+(\mathbb{R}^n \setminus Q^\dagger_{k'})}(\mathbb{R}^n \setminus Q^\dagger_{k'})} \le {C_0}^2 [w]_{\tilde{A}^{\mathfrak{D}}_{p(\cdot)}} = c_1$$

from Lemma 10.6. As a result,

$$[w]_{A^{\mathfrak{D}}_{p_\infty - \frac{2}{3}\varepsilon}(\mathbb{R}^n \setminus Q^\dagger_{k'})} \le c_2,$$

or equivalently, $w \in A^{\mathfrak{D}}_{p_\infty - \frac{2}{3}\varepsilon}(\mathbb{R}^n \setminus Q^\dagger_{k'})$. Thus, we have

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus Q^\dagger_k} \frac{w(x)\mathrm{d}x}{|x|^{np(x)}} &\lesssim \int_{Q^\dagger_{k'} \setminus Q^\dagger_k} \frac{w(x)\mathrm{d}x}{(1+|x|)^{np(x)}} + \int_{\mathbb{R}^n \setminus Q^\dagger_{k'}} \frac{w(x)\mathrm{d}x}{|x|^{np(x)}} \\
&\lesssim \int_{Q^\dagger_{k'}} \frac{w(x)\mathrm{d}x}{(1+|x|)^{np(x)}} + \int_{\mathbb{R}^n \setminus Q^\dagger_{k'}} \frac{w(x)\mathrm{d}x}{|x|^{n\left(p_\infty - \frac{2}{3}\varepsilon\right)}} \\
&\lesssim \int_{Q^\dagger_{k'}} w(x)\mathrm{d}x + \int_{\mathbb{R}^n \setminus Q^\dagger_{k'}} \left[M^{\mathfrak{D}}\chi_{[-1,1]^n}(x)\right]^{p_\infty - \frac{2}{3}\varepsilon} w(x)\mathrm{d}x \\
&\lesssim \int_{Q^\dagger_{k'}} w(x)\mathrm{d}x + \int_{[-1,1]^n} w(x)\mathrm{d}x \lesssim 1.
\end{aligned}$$

Consequently, $\| |\cdot|^{-n}\chi_{\mathbb{R}^n \setminus Q^\dagger_k}\|_{L^{p(\cdot)}(w)} \lesssim 1$. By combining estimates (10.9) and (10.10), we conclude that (10.8) holds.

□

From this corollary, we can obtain the boundedness property of $M^{\mathfrak{D}}_{\ge r_0}$ for the function supported on the cube with comparative ease. Next, consider this operator for the function supported on the outside of the cube. However, this case is very complicated. We must prepare some lemmas.

Lemma 10.16. *Let $w \in \tilde{A}^{\mathfrak{D}}_{p(\cdot)}$. Then*

$$\sup_{R \in \mathfrak{D}, |R \setminus Q^\dagger_{k'}| > 0} |R \setminus Q^\dagger_{k'}|^{-p_\infty} \|w\|_{L^1(R \setminus Q^\dagger_{k'})} \|w^{-1}\|_{L^{\frac{1}{p_\infty - 1}}(R \setminus Q^\dagger_{k'})} < \infty$$

for some $k' \le -1$.

For the proof, we invoke the following fact: from [14, Corollary 3.7] (see also Remark 10.21 below), $\|\chi_Q\|_{L^{p(\cdot)}(w)} \sim w(Q)^{\frac{1}{p_Q}}$ for all cubes Q as long as $\omega \in A_\infty$ and $p(\cdot) \in \mathcal{P}_0 \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$.

Proof. Since $w \in \tilde{A}^{\mathfrak{D}}_{p(\cdot)}$, the weight w satisfies the condition

$$\sup_{R \in \mathfrak{D}} |R|^{-p_R} \|w\|_{L^1(R)} \|w^{-1}\|_{L^{\frac{p'(\cdot)}{p(\cdot)}}(R)} < \infty.$$

Since

$$|R|^{-p_R} \sim |R \setminus Q^\dagger_{k'}|^{-p_R} \sim |R \setminus Q^\dagger_{k'}|^{-p_{R \setminus Q^\dagger_{k'}}}$$

for any cube $R \in \mathfrak{D}$ such that $R \subset Q^\dagger_{k'}$ fails, we have

$$\sup_{R \in \mathfrak{D}, |R \setminus Q^\dagger_{k'}| > 0} |R \setminus Q^\dagger_{k'}|^{-p_{R \setminus Q^\dagger_{k'}}} \|w\|_{L^1(R \setminus Q^\dagger_{k'})} \|w^{-1}\|_{L^{\frac{p'(\cdot)}{p(\cdot)}}(R \setminus Q^\dagger_{k'})} < \infty.$$

Let $\tau \in (0,1)$ be small enough. By the Hölder inequality for variable exponent Lebesgue spaces,

$$\|w^{-1}\|_{L^{\frac{1}{p_\infty - 1 + \tau}}(R \setminus Q^\dagger_{k'})} \le 2\|w^{-1}\|_{L^{\frac{1}{p(\cdot) - 1}}(R \setminus Q^\dagger_{k'})} \|\chi_{R \setminus Q^\dagger_{k'}}\|_{L^{\frac{1}{p_\infty + \tau - p(\cdot)}}}.$$

By [14, Corollary 3.7 and Lemma 2.1] (see also Lemma 2.2 and Remark 10.21 below), $\|\chi_{R\setminus Q^\dagger_{k'}}\|_{L^{\frac{1}{p_\infty+\tau-p(\cdot)}}} \sim |R\setminus Q^\dagger_{k'}|^{p_\infty+\tau-p_{R\setminus Q^\dagger_{k'}}}$. Thus,

$$\begin{aligned}&\sup_{R\in\mathfrak{D},|R\setminus Q^\dagger_{k'}|>0}|R\setminus Q^\dagger_{k'}|^{-p_\infty-\tau}\|w\|_{L^1(R\setminus Q^\dagger_{k'})}\|w^{-1}\|_{L^{\frac{1}{p_\infty-1+\tau}}(R\setminus Q^\dagger_{k'})}\\ &\lesssim \sup_{R\in\mathfrak{D},|R\setminus Q^\dagger_{k'}|>0}|R\setminus Q^\dagger_{k'}|^{-p_{R\setminus Q^\dagger_{k'}}}\|w\|_{L^1(R\setminus Q^\dagger_{k'})}\|w^{-1}\|_{L^{\frac{p'(\cdot)}{p(\cdot)}}(R\setminus Q^\dagger_{k'})}<\infty,\end{aligned}$$

or equivalently,

$$\sup_{R\in\mathfrak{D},|R\setminus Q^\dagger_{k'}|>0}|R\setminus Q^\dagger_{k'}|^{-\frac{p_\infty+\tau}{p_\infty+\tau-1}}\|w^{\frac{1}{p_\infty-1+\tau}}\|_{L^{p_\infty+\tau-1}(R\setminus Q^\dagger_{k'})}\|w^{-\frac{1}{p_\infty-1+\tau}}\|_{L^1(R\setminus Q^\dagger_{k'})}<\infty.$$

This means that $w^{-\frac{1}{p_\infty-1+\tau}}\in A_{1+\frac{1}{p_\infty+\tau-1}}\subset A_{1+\frac{1}{p_\infty-1}}$. By virtue of Lemma 2.13 with $\varepsilon=(4^{n+6}[w]_{A^{\mathfrak{D}}_{\infty,R^\times}})^{-1}$,

$$\sup_{R\in\mathfrak{D},|R\setminus Q^\dagger_{k'}|>0}|R\setminus Q^\dagger_{k'}|^{-\frac{p_\infty+\tau}{p_\infty+\tau-1}-\frac{1}{1+\varepsilon}}\|w^{\frac{1}{p_\infty-1+\tau}}\|_{L^{p_\infty+\tau-1}(R\setminus Q^\dagger_{k'})}(\|w^{-\frac{1+\varepsilon}{p_\infty-1+\tau}}\|_{L^1(R\setminus Q^\dagger_{k'})})^{\frac{1}{1+\varepsilon}}<\infty.$$

Since

$$1+\frac{p_\infty-1+\tau}{1+\varepsilon}<p_\infty,$$

as long as τ is small enough, we obtain

$$\sup_{R\in\mathfrak{D},|R\setminus Q^\dagger_{k'}|>0}|R\setminus Q^\dagger_{k'}|^{-p_\infty}\|w\|_{L^1(R\setminus Q^\dagger_{k'})}\|w^{-1}\|_{L^{\frac{1}{p_\infty-1}}(R\setminus Q^\dagger_{k'})}<\infty$$

by Hölder's inequality. □

Assuming $f=E_kf$, we obtain some growth information of f.

Lemma 10.17. *Let r_0 be the same as in Lemma 10.12. Suppose that $f\in L^{p(\cdot)}(w)$ satisfies $f=E_kf$ for some $k\in\mathbb{Z}$ such that $2^{-k}<r_0$. Let $w\in\tilde{A}^{\mathfrak{D}}_{p(\cdot)}$. Then*

$$|f(x)|=|E_kf(x)|\lesssim M^{\mathfrak{D}}_{\geq r_0}f(x)\lesssim(1+|x|)^{\frac{p_+n}{p_-}}\|f\|_{L^{p(\cdot)}(w)} \tag{10.11}$$

In particular, we have

$$\|f\|_{L^{p_\infty}(w)}\lesssim\|f\|_{L^{p(\cdot)}(w)}.$$

Proof. Fix $x\in\mathbb{R}^n$ and a cube $Q\in\mathfrak{D}$ satisfying $x\in Q$ and $\ell(Q)\geq r_0$. Then by [14, Corollary 3.7] and Lemma 10.11 (see also Remark 10.21 below),

$$\begin{aligned}\frac{1}{|Q|}\int_Q|f(y)|\mathrm{d}y&\leq\frac{2}{|Q|}\|\chi_Qf\|_{L^{p(\cdot)}(w)}\|w^{-1}\chi_Q\|_{L^{p'(\cdot)}(w)}\\ &\leq\frac{2}{|Q|}\|\chi_Q\|_{L^{p'(\cdot)}(\sigma)}\|f\|_{L^{p(\cdot)}(w)}\\ &\leq\frac{2\sigma(Q)^{\frac{1}{p'_Q}}}{|Q|}\|f\|_{L^{p(\cdot)}(w)}\\ &\lesssim\left(\frac{[w]_{A^{\mathfrak{D}}_{p_-}}}{w(Q)}\right)^{\frac{1}{p_-}}\|f\|_{L^{p(\cdot)}(w)},\end{aligned}$$

where $\sigma\equiv w^{-\frac{1}{p(\cdot)-1}}$ stands for the dual weight. Thus,

$$\frac{1}{|Q|}\int_Q|f(y)|\mathrm{d}y\lesssim\left(\frac{[w]_{A^{\mathfrak{D}}_{p_-}}}{w(Q)}\right)^{\frac{1}{p_-}}\|f\|_{L^{p(\cdot)}(w)}. \tag{10.12}$$

If $0 \notin 2Q$, then there exists the largest number $\ell \leq -1$ such that $Q \subset Q_\ell^\dagger$. By the geometric observation, we have $|x| \sim \ell(Q_\ell^\dagger)$. By Lemma 10.8, since

$$w(Q) \gtrsim w(Q_\ell^\dagger) \min\left(1, \frac{|Q|}{|Q_\ell^\dagger|}\right)^{p_+},$$

we have

$$\frac{1}{w(Q)} \lesssim \frac{1}{w(Q_\ell^\dagger)} \left(1 + \frac{|Q_\ell^\dagger|}{|Q|}\right)^{p_+} \lesssim \frac{1}{w(Q_\ell^\dagger)} (1+|x|)^{p_+ n} \leq \frac{1}{w(Q_0^\dagger)} (1+|x|)^{p_+ n}$$

Meanwhile, if $0 \in 2Q$, then $64Q \supset [-r_0, r_0]^n$. By the doubling condition, we have

$$\frac{1}{w(Q)} \lesssim \frac{1}{w(64Q)} \leq \frac{1}{w([-r_0, r_0]^n)} \lesssim 1. \tag{10.13}$$

Inserting this estimate into (10.12), we obtain

$$M^{\mathfrak{D}}_{\geq r_0} f(x) \lesssim (1+|x|)^{\frac{p_+ n}{p_-}} \|f\|_{L^{p(\cdot)}(w)}. \tag{10.14}$$

Then we have

$$|f(x)| = |E_k f(x)| \lesssim M^{\mathfrak{D}}_{\geq r_0} f(x) \lesssim (1+|x|)^{\frac{p_+ n}{p_-}} \|f\|_{L^{p(\cdot)}(w)}.$$

Moreover, $C\|f\|_{L^{p(\cdot)}(w)}^{-1} f$ satisfies the assumption of Lemma 2.21 for some constant $C > 0$. By Lemma 2.21 (i), we have

$$\|f\|_{L^{p_\infty}(w)} \lesssim \|f\|_{L^{p(\cdot)}(w)}. \tag{10.15}$$

□

Lemma 10.18. *Let r_0 be the same as in Lemma 10.12. Let $w \in \tilde{A}^{\mathfrak{D}}_{p(\cdot)}$. Then $M^{\mathfrak{D}}_{\geq r_0}$ is bounded on $L^{p(\cdot)}(w)$.*

Proof. Let c_1, c_2, and $k' \ll -1$ be the same constants in the proof of Corollary 10.15. Then, the integer k' satisfies $p_+(\mathbb{R}^n \setminus Q_{k'}^\dagger) - \varepsilon \leq p_+(\mathbb{R}^n \setminus Q_{k'}^\dagger) - \varepsilon/3 \leq p_\infty$. Thus we have $w \in A_{p_\infty+\varepsilon}(\mathbb{R}^n \setminus Q_{k'}^\dagger)$ with

$$[w]_{A^{\mathfrak{D}}_{p_\infty+\varepsilon}(\mathbb{R}^n \setminus Q_{k'}^\dagger)} \leq C_0 [w]_{A^{\mathfrak{D}}_{p_+(\mathbb{R}^n \setminus Q_{k'}^\dagger)}(\mathbb{R}^n \setminus Q_{k'}^\dagger)} \leq {C_0}^2 [w]_{\tilde{A}^{\mathfrak{D}}_{p(\cdot)}} = c_1$$

from Lemma 10.6. By the property of $\varepsilon > 0$, we have $[w]_{A^{\mathfrak{D}}_{p_\infty}(\mathbb{R}^n \setminus Q_{k'}^\dagger)} \leq c_2$.

Let $k \gg 1$ have the same parity as k'. Since $M^{\mathfrak{D}}_{\geq r_0} f = M^{\mathfrak{D}}_{\geq r_0} \circ E_k f$, we can assume $f = E_k f = \chi_{\mathbb{R}^n \setminus Q_{k'}^\dagger} E_k f$ thanks to Corollary 10.15. Let us establish

$$\|\chi_{Q_{k'}^\dagger} M^{\mathfrak{D}}_{\geq r_0} f\|_{L^{p(\cdot)}(w)} + \|\chi_{\mathbb{R}^n \setminus Q_{k'}^\dagger} M^{\mathfrak{D}}_{\geq r_0} f\|_{L^{p(\cdot)}(w)} \lesssim \|f\|_{L^{p(\cdot)}(w)}.$$

First, let $x \in Q_{k'}^\dagger$. Since $\ell(Q_{k'}^\dagger) \geq 2 > r_0$, all cubes $Q \in \mathfrak{D}$ satisfying $x \in Q$ and $\ell(Q) \geq r_0$ must either include the cube $Q_{k'}^\dagger$ or be a cube in $\bigcup_{j=k'}^{-\log_2 r_0} \mathfrak{D}_j$. Thus, we can write such cubes Q as $Q_\ell^\dagger$ for $\ell \leq k' - 1$. By virtue of these observations, we have

$$M^{\mathfrak{D}}_{\geq r_0} f(x) \lesssim \sup_{\ell \in \mathbb{Z}, \ell \leq k'-1} \frac{1}{|Q_\ell^\dagger|} \int_{Q_\ell^\dagger \setminus Q_{k'}^\dagger} |f(y)| \mathrm{d}y.$$

Then, using the Hölder inequality, Lemma 10.16, Lemma 10.17 and the fact $Q^\dagger_{k'} \subset Q^\dagger_\ell$, we have

$$\begin{aligned}
&\|\chi_{Q^\dagger_{k'}} M^{\mathfrak{D}}_{\geq r_0} f\|_{L^{p(\cdot)}(w)} \\
&\lesssim \sup_{\ell \in \mathbb{Z}, \ell \leq k'-1} m_{Q^\dagger_\ell \setminus Q^\dagger_{k'}}(|f|) \times \|\chi_{Q^\dagger_{k'}}\|_{L^{p(\cdot)}(w)} \\
&\sim \sup_{\ell \in \mathbb{Z}, \ell \leq k'-1} m_{Q^\dagger_\ell \setminus Q^\dagger_{k'}}(|f|) \times \|\chi_{Q^\dagger_{k'-1} \setminus Q^\dagger_{k'}}\|_{L^{p(\cdot)}(w)} \\
&\leq \sup_{\ell \in \mathbb{Z}, \ell \leq k'-1} \frac{2}{|Q^\dagger_\ell|} \|f\|_{L^{p_\infty}(w)} \|w^{-1}\chi_{Q^\dagger_\ell \setminus Q^\dagger_{k'}}\|_{L^{p'_\infty}(w)} \|\chi_{Q^\dagger_\ell \setminus Q^\dagger_{k'}}\|_{L^{p_\infty}(w)} \\
&\lesssim \sup_{\ell \in \mathbb{Z}, \ell \leq k'-1} \frac{1}{|Q^\dagger_\ell|} \|\chi_{Q^\dagger_\ell \setminus Q^\dagger_{k'}}\|_{L^{p_\infty}(w)} \|\chi_{Q^\dagger_\ell \setminus Q^\dagger_{k'}}\|_{L^{p'_\infty}(\sigma)} \|f\|_{L^{p_\infty}(w)} \\
&\lesssim \|f\|_{L^{p(\cdot)}(w)}.
\end{aligned} \tag{10.16}$$

Next, we define

$$\mathfrak{N}g(x) \equiv \sup_{R \in \mathfrak{D}, R \cap Q^\dagger_{k'} = \emptyset \text{ or } Q^\dagger_{k'} \subsetneq R} \frac{\chi_{R \setminus Q^\dagger_{k'}}(x)}{|R \setminus Q^\dagger_{k'}|} \int_{R \setminus Q^\dagger_{k'}} |f(y)| \mathrm{d}y.$$

By Lemma 10.16, w satisfies the condition

$$\sup_{R \in \mathfrak{D}, R \cap Q^\dagger_{k'} = \emptyset \text{ or } Q^\dagger_{k'} \subsetneq R} |R \setminus Q^\dagger_{k'}|^{-p_\infty} \|w\|_{L^1(R \setminus Q^\dagger_{k'})} \|w^{-1}\|_{L^{p'_\infty/p_\infty}(R \setminus Q^\dagger_{k'})} < \infty.$$

Therefore, we have

$$\|\mathfrak{N}g\|_{L^{p_\infty}(w)} \lesssim \|g\|_{L^{p_\infty}(w)}$$

thanks to [32, Theorem 1.1] and [39, Theorem B]. Here, we can verify that $M^{\mathfrak{D}}_{\geq r_0} f(z) \leq \mathfrak{N}f(z)$ for $z \in \mathbb{R}^n \setminus Q^\dagger_{k'}$. Since $\mathfrak{N}$ is bounded on $L^{p_\infty}(w)$, we deduce

$$\|\chi_{\mathbb{R}^n \setminus Q^\dagger_{k'}} M^{\mathfrak{D}}_{\geq r_0} f\|_{L^{p_\infty}(w)} \lesssim \|\chi_{\mathbb{R}^n \setminus Q^\dagger_{k'}} \mathfrak{N}f\|_{L^{p_\infty}(w)} \lesssim \|f\|_{L^{p_\infty}(w)} \lesssim \|f\|_{L^{p(\cdot)}(w)} \tag{10.17}$$

from Lemma 10.17. Meanwhile, thanks to (10.14), $C\|f\|^{-1}_{L^{p(\cdot)}(w)} \chi_{\mathbb{R}^n \setminus Q^\dagger_{k'}} M^{\mathfrak{D}}_{\geq r_0} f$ satisfies the assumption of Lemma 2.21 for some constant $C > 0$. By (10.17) and Lemma 2.21 (2), we obtain

$$\|\chi_{\mathbb{R}^n \setminus Q^\dagger_{k'}} M^{\mathfrak{D}}_{\geq r_0} f\|_{L^{p(\cdot)}(w)} \lesssim \|\chi_{\mathbb{R}^n \setminus Q^\dagger_{k'}} M^{\mathfrak{D}}_{\geq r_0} f\|_{L^{p_\infty}(w)}.$$

By combining (10.17) and (10.18), we obtain

$$\|\chi_{\mathbb{R}^n \setminus Q^\dagger_{k'}} M^{\mathfrak{D}}_{\geq r_0} f\|_{L^{p(\cdot)}(w)} \lesssim \|f\|_{L^{p(\cdot)}(w)}. \tag{10.18}$$

Thus, the desired result is given from (10.16) and (10.18). □

By combining Lemmas 10.12 and 10.18, we conclude that $M^{\mathfrak{D}}$ is bounded on $L^{p(\cdot)}(w)$, which implies that $\tilde{A}^{\mathfrak{D}}_{p(\cdot)} \subset A^{\mathfrak{D}}_{p(\cdot)}$.

10.2. **Necessity of Theorem 10.2.**

Now let us prove $\tilde{A}^{\mathfrak{D}}_{p(\cdot)} \supset A^{\mathfrak{D}}_{p(\cdot)}$.

Consider the converse. We suppose that we have a weight w such that $M^{\mathfrak{D}}$ is bounded on $L^{p(\cdot)}(w)$.

A weight w is doubling if $w(\tilde{\tilde{Q}}) \lesssim w(Q)$ for any $Q \in \mathfrak{D}$, where $\tilde{\tilde{Q}} \in \mathfrak{D}$ is the dyadic grand parent of Q, That is, $\tilde{\tilde{Q}}$ is a cube $R \in \mathfrak{D}$ with $|Q| = 4^{-n}|R|$ and $Q \subset R$. We will use the following observation:

Lemma 10.19. *Let a doubling weight w, $p(\cdot) \in \mathcal{P}$ and $C > 0$ satisfy*

$$\sup_{\lambda>0} \lambda \|\chi_{(\lambda,\infty]}(M^{\mathfrak{D}} f)\|_{L^{p(\cdot)}(w)} \le C\|f\|_{L^{p(\cdot)}(w)}, \tag{10.19}$$

or equivalently $M^{\mathfrak{D}}$ is weak bounded on $L^{p(\cdot)}(w)$. Then

$$\|\chi_Q\|_{L^{p(\cdot)}(w)} \gtrsim \min\left(1, \frac{|Q|}{|R|}\right) \|\chi_R\|_{L^{p(\cdot)}(w)}$$

for all $Q, R \in \mathfrak{D}$ satisfying $Q \cap R \neq \emptyset$.

Proof. We can assume $\ell(Q) \le \ell(R)$; otherwise the conclusion is trivial by the doubling property of w. If we denote by $\tilde{\tilde{R}} \in \mathfrak{D}$, which is the dyadic grand parent of R, then

$$M^{\mathfrak{D}}\chi_Q \ge \frac{|Q \cap \tilde{\tilde{R}}|}{|\tilde{\tilde{R}}|}\chi_{\tilde{\tilde{R}}} = \frac{|Q|}{4^n|R|}\chi_{\tilde{\tilde{R}}} \ge \frac{|Q|}{4^n|R|}\chi_R.$$

Therefore,

$$R \subset \left\{x \in \mathbb{R}^n \,:\, M^{\mathfrak{D}}\chi_Q(x) \ge \frac{|Q|}{4^n|R|}\chi_R(x)\right\}.$$

Hence from (10.19), we have

$$\|\chi_R\|_{L^{p(\cdot)}(w)} \le \left\|\chi_{(\frac{|Q|}{5^n|R|},\infty)}(M^{\mathfrak{D}}\chi_Q)\right\|_{L^{p(\cdot)}(w)} \le \frac{5^n C|R|}{|Q|}\|\chi_Q\|_{L^{p(\cdot)}(w)},$$

which proves Lemma 10.19. □

We prove the weighted analogy to Lemma 2.2.

Lemma 10.20. *Let $p(\cdot) \in \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and w be a variable exponent and a weight such that $M^{\mathfrak{D}}$ is weak bounded on $L^{p(\cdot)}(w)$. Then for all $Q \in \mathfrak{D}_k$ with $k \ge 0$,*

$$\|\chi_Q\|_{L^{p(\cdot)}(w)} \sim w(Q)^{\frac{1}{p_Q}} \sim w(Q)^{\frac{1}{p_-(Q)}} \sim w(Q)^{\frac{1}{p_+(Q)}}. \tag{10.20}$$

Before the proof, a couple of remarks may be in order.

Remark 10.21.

(1) Notice that $M^{\mathfrak{D}}$ is not assumed bounded on $L^{p(\cdot)}(w)$. However, it is absolutely necessary to assume that $M^{\mathfrak{D}}$ is weak bounded on $L^{p(\cdot)}(w)$ (see Lemma 10.25 below).
(2) As in [14, Lemma 3.4], the same conclusion holds for the case of $w \in A_\infty^{\mathrm{loc},\mathfrak{D}}$. In fact, assuming $w \in A_u^{\mathrm{loc},\mathfrak{D}}$, we have (10.3) with p_+ replaced by u, which corresponds to [14, (3.5)].
(3) As in [14, Corollary 3.7], if $w \in A_\infty^{\mathfrak{D}}$, (10.20) remains valid for any $Q \in \mathfrak{D}$. In fact, assuming $w \in A_u^{\mathrm{loc},\mathfrak{D}}$, we have (10.3) with p_+ replaced by u, which corresponds to the key inequality in the proof of [14, Corollary 3.7].

Proof. Let $k \ge 0$. Fix $Q \in \mathfrak{D}_k$. Recall that $Q_k^\dagger \in \mathfrak{D}$ is the unique cube in $\mathfrak{D}_k$ containing 0. Then, we can find the smallest cube $Q_\ell^\dagger (\ell \le k)$ such that $Q \subset Q_\ell^\dagger$. Due to Lemma 10.19,

$$\|\chi_{Q_\ell^\dagger}\|_{L^{p(\cdot)}(w)} \ge \|\chi_Q\|_{L^{p(\cdot)}(w)} \gtrsim \frac{|Q|}{|Q_\ell^\dagger|}\|\chi_{Q_\ell^\dagger}\|_{L^{p(\cdot)}(w)}.$$

By the log-Hölder condition, we obtain

$$(\|\chi_Q\|_{L^{p(\cdot)}(w)})^{\left|\frac{1}{p_-(Q)} - \frac{1}{p_+(Q)}\right|} \sim 1.$$

Due to Lemma 2.3, we have

$$\min(\|\chi_Q\|_{L^{p(\cdot)}(w)}^{p_-(Q)}, \|\chi_Q\|_{L^{p(\cdot)}(w)}^{p_+(Q)}) \sim \max(\|\chi_Q\|_{L^{p(\cdot)}(w)}^{p_-(Q)}, \|\chi_Q\|_{L^{p(\cdot)}(w)}^{p_+(Q)}) \sim w(Q).$$

Thus, we obtain (10.20). □

Now, let us investigate how fast w grows.

Lemma 10.22. *Let $p(\cdot) \in \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ be a variable exponent such that $M^{\mathfrak{D}}$ is weak bounded on $L^{p(\cdot)}(w)$. Then w has at most polynomial growth. More precisely,*

$$w(\{y \in \mathbb{R}^n \,:\, |y| \le |x|\}) \lesssim (1+|x|)^{p_+ n} \tag{10.21}$$

for all $x \in \mathbb{R}^n$.

Proof. By Remark 2.4 and Lemma 10.19,

$$\min\{w(Q_k^\dagger)^{\frac{1}{p_+(Q)}}, w(Q_k^\dagger)^{\frac{1}{p_-(Q)}}\} \le \|\chi_{Q_k^\dagger}\|_{L^{p(\cdot)}(w)} \lesssim \frac{|Q_k^\dagger|}{|Q_0^\dagger|}\|\chi_{Q_0^\dagger}\|_{L^{p(\cdot)}(w)} \lesssim 2^{-kn}.$$

As a result, $w(Q_k^\dagger) \lesssim 2^{-kp_+ n}$ for all $k \le 0$ or equivalently, (10.21) holds. □

We obtain a crude conclusion assuming that $M^{\mathfrak{D}}$ is weak bounded on $L^{p(\cdot)}(w)$.

Lemma 10.23 (c.f. [14, Lemma 6.3])**.** *Let $p(\cdot) \in \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ be a variable exponent such that $M^{\mathfrak{D}}$ is weak bounded on $L^{p(\cdot)}(w)$. Then $w \in A_\infty^{\mathfrak{D},\mathrm{loc}}$. That is, there exists $q>1$ such that*

$$\sup_{Q \in \mathfrak{D}, |Q| \le 1} \frac{1}{|Q|}\int_Q w(x)\mathrm{d}x \left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{q-1}}\mathrm{d}x\right)^{q-1} \lesssim 1$$

Proof. Let $Q \in \mathfrak{D}$ with $|Q| \le 1$ and $E \subset Q$ be a measurable set. Using

$$\|\chi_E\|_{L^{p(\cdot)}(w)} \le \max\{w(E)^{\frac{1}{p_+(Q)}}, w(E)^{\frac{1}{p_-(Q)}}\}, \quad Q \subset \{M^{\mathfrak{D}}\chi_E \ge |Q|^{-1}|E|\},$$

we will show that

$$\frac{w(E)}{w(Q)} \gtrsim \left(\frac{|E|}{|Q|}\right)^{p_+}. \tag{10.22}$$

Once this is achieved, we will have $q \in [1,\infty)$ such that $w \in A_q^{\mathfrak{D}}$. Then, due to Lemma 10.20,

$$\begin{aligned}
\max\{w(Q)^{\frac{1}{p_+(Q)}}, w(Q)^{\frac{1}{p_-(Q)}}\} &\sim \|\chi_Q\|_{L^{p(\cdot)}(w)} \\
&\le \|\chi_{\{M^{\mathfrak{D}}\chi_E \ge |Q|^{-1}|E|\}}\|_{L^{p(\cdot)}(w)} \\
&\lesssim \frac{|Q|}{|E|}\|\chi_E\|_{L^{p(\cdot)}(w)} \\
&\le \frac{|Q|}{|E|}\max\{w(E)^{\frac{1}{p_+(Q)}}, w(E)^{\frac{1}{p_-(Q)}}\}.
\end{aligned}$$

As a result

$$\frac{w(E)}{w(Q)} \gtrsim \min\left\{\left(\frac{|E|}{|Q|}\right)^{p_-(Q)}, \left(\frac{|E|}{|Q|}\right)^{p_+(Q)}\right\} = \left(\frac{|E|}{|Q|}\right)^{p_+(Q)} \ge \left(\frac{|E|}{|Q|}\right)^{p_+}.$$

Thus, the proof of (10.22) is complete. □

Corollary 10.24 ([14, Corollary 6.6])**.** *Let $p(\cdot), q(\cdot) \in \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ be variable exponents such that $M^{\mathfrak{D}}$ is weak bounded on $L^{p(\cdot)}(w)$. Then*

$$\|\chi_Q\|_{L^{q(\cdot)}(w)} \sim w(Q)^{\frac{1}{q_Q}} \tag{10.23}$$

for all cubes $Q \in \mathfrak{D}$.

This corollary seems to be the same as Lemma 10.20. However, we remark that the weak boundedness is assumed on $L^{p(\cdot)}(w)$ and the equivalence for a different exponent $q(\cdot)$ is obtained.

Proof. Simply combine Lemmas 2.21 and 10.22. In fact, as in [14, Lemma 3.4 and Corollary 3.7] (see also Remark 10.21), we have

$$\|\chi_Q\|_{L^{q(\cdot)}(w)} \sim w(Q)^{\frac{1}{q_Q}}, \tag{10.24}$$

where we use Lemma 10.23 if $|Q| \le 1$ but we use Lemma 2.5 and (10.24) for cubes having volume 1 if $|Q| \ge 1$ □

As the following lemma shows, the dual space inherits the boundedness of the operator $M^{\mathfrak{D}}$ from the original space.

Lemma 10.25. *Let $p(\cdot) \in \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ be a variable exponent such that $M^{\mathfrak{D}}$ is bounded on $L^{p(\cdot)}(w)$. Then $M^{\mathfrak{D}}$ is weak bounded on $L^{p'(\cdot)}(\sigma)$, where $\sigma \equiv w^{-\frac{1}{p(\cdot)-1}}$ stands for the dual weight.*

Proof. Let $\lambda > 0$ be fixed. Also, let $f \in L^{p'(\cdot)}(\sigma)$. By the duality $L^{p(\cdot)}(w)$-$L^{p'(\cdot)}(\sigma)$ and the Stein type dual inequality, we have

$$\begin{aligned}\|\lambda\chi_{[\lambda,\infty]}(M^{\mathfrak{D}}f)\|_{L^{p'(\cdot)}(\sigma)} &\sim \sup_{g\in L^{p(\cdot)}(w), \|g\|_{L^{p(\cdot)}(w)}=1} \int_{\mathbb{R}^n} \lambda\chi_{[\lambda,\infty]}(M^{\mathfrak{D}}f(x))|g(x)|\mathrm{d}x \\ &\lesssim \sup_{g\in L^{p(\cdot)}(w), \|g\|_{L^{p(\cdot)}(w)}=1} \int_{\mathbb{R}^n} |f(x)|M^{\mathfrak{D}}g(x)\mathrm{d}x.\end{aligned}$$

Finally, use the $L^{p(\cdot)}(w)$-boundedness of $M^{\mathfrak{D}}$ and the Hölder inequality. □

If we reexamine the proof of Lemma 10.25, then we see that Lemma 10.25 holds for a wider class of function spaces. We summarize our observation below.

Remark 10.26. A Banach lattice over $\mathbb{R}^n$ is a Banach space $(\mathcal{X}(\mathbb{R}^n), \|\cdot\|_{\mathcal{X}})$ contained in $L^0(\mathbb{R}^n)$ such that, for all $g \in \mathcal{X}(\mathbb{R}^n)$ and $f \in L^0(\mathbb{R}^n)$, the implication "$|f| \le |g| \Rightarrow f \in \mathcal{X}(\mathbb{R}^n)$ and $\|f\|_{\mathcal{X}} \le \|g\|_{\mathcal{X}}$" holds. The dual lattice $\mathcal{X}'(\mathbb{R}^n)$ of $\mathcal{X}(\mathbb{R}^n)$ is given by the set of all $g \in L^0(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{X}'} = \sup\{\|fg\|_{L^1} \,:\, g \in \mathcal{X}\}$$

is finite. According to [1], $\mathcal{X}'$ is a Banach lattice over $\mathbb{R}^n$. Lemma 10.25 is available for Banach lattices. Namely, if $\mathcal{X}$ is a Banach lattice over $\mathbb{R}^n$ $\tilde{M}^{\mathcal{D}(Q)}$ is bounded on $\mathcal{X}$. Then $\tilde{M}^{\mathcal{D}(Q)}$ is weak bounded on $\mathcal{X}'(\mathbb{R}^n)$. As the example of $\mathcal{X}(\mathbb{R}^n) = L^1(\mathbb{R}^n)$ shows, it can happen that $\tilde{M}^{\mathcal{D}(Q)}$ is not bounded on $\mathcal{X}(\mathbb{R}^n)$.

We conclude the proof of necessity. Thus, we suppose that there exists a constant $C > 0$ such that

$$\|M^{\mathfrak{D}}f\|_{L^{p(\cdot)}(w)} \le C\|f\|_{L^{p(\cdot)}(w)}.$$

Fix a cube $Q \in \mathfrak{D}$. Then we have

$$M^{\mathfrak{D}}[\sigma\chi_Q](x) \ge \frac{\sigma(Q)}{|Q|}\chi_Q(x).$$

As a result,

$$\frac{\sigma(Q)}{|Q|}\|\chi_Q\|_{L^{p(\cdot)}(w)} \le C\|\chi_Q\|_{L^{p(\cdot)}(\sigma)}. \tag{10.25}$$

Note that w has at most polynomial growth thanks to Lemma 10.22. Additionally it should be observed that $M^{\mathfrak{D}}$ is weak bounded on $L^{p'(\cdot)}(\sigma)$ thanks to Lemma 10.25. Thus, we conclude from Lemma 10.22 that σ has at most polynomial growth. Thus, Corollary 10.24 can be applied to both w and σ. Due to Corollary 10.24, we have

$$\|\chi_Q\|_{L^{p(\cdot)}(w)} \sim w(Q)^{\frac{1}{p_Q}}, \quad \|\chi_Q\|_{L^{p(\cdot)}(\sigma)} \sim \sigma(Q)^{\frac{1}{p_Q}}.$$

Inserting these estimates into (10.25), we obtain

$$\frac{\sigma(Q)}{|Q|} w(Q)^{\frac{1}{p_Q}} \leq C\sigma(Q)^{\frac{1}{p_Q}},$$

or equivalently,

$$|Q|^{-p_Q}\|w\|_{L^1(Q)}\sigma(Q)^{p_Q-1} \leq C,$$

where constant C is independent of Q. If we use Corollary 10.24 once again, we conclude

$$|Q|^{-p_Q}\|w\|_{L^1(Q)}\|\sigma\|_{L^{\frac{p(\cdot)}{p'(\cdot)}}(Q)} \leq C,$$

as required.

Acknowledgement

The authors thank Professor Lars Diening for the preprint [14], which motivated us to complete Section 10. Mitsuo Izuki was partially supported by Grand-in-Aid for Scientific Research (C), No. 15K04928, for Japan Society for the Promotion of Science. Toru Nogayama was supported financially by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists (20J10403 and 22J00614). Takahiro Noi was partially supported by Grand-in-Aid for Young Scientists (B), No. 17K14207, for Japan Society for the Promotion of Science. Yoshihiro Sawano was partially supported by Grand-in-Aid for Scientific Research (C), No. 19K03546, for Japan Society for the Promotion of Science. This work was partly supported by Osaka Central Advanced Mathematical Institute: MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849.

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