KIRILLOV-RESHETIKHIN MODULES AND QUANTUM K-MATRICES

HIROTO KUSANO, MASATO OKADO, AND HIDEYA WATANABE

Abstract. From a quantum K-matrix of the fundamental representation, we construct one for the Kirillov-Reshetikhin module by fusion construction. Using the $\imath$crystal theory by the last author, we also obtain combinatorial K-matrices corresponding to the symmetric tensor representations of affine type A for all quasi-split Satake diagrams.

1. Introduction

A quantum symmetric pair, is a pair $(\mathbf{U}, \mathbf{U}^\imath)$ of a quantized enveloping algebra $\mathbf{U}$ and its coideal subalgebra $\mathbf{U}^\imath$. The $\mathbf{U}^\imath$ itself is referred to as an $\imath$quantum group. It is defined once we are given a Satake diagram (in the sense of [20, Definition 2.7]) of symmetrizable Kac-Moody type. Consider the Dynkin diagram of a symmetrizable Kac-Moody Lie algebra and let I be the set of its nodes. A Satake diagram is a triple $(I, I_\bullet, \tau)$, where $I_\bullet \subset I$ is a subdiagram of finite type and τ is a diagram automorphism of order at most 2 satisfying certain conditions. From a Satake diagram of finite type (i.e., the Dynkin diagram I is of finite type), Letzter constructed a right coideal subalgebra $\mathbf{U}^\imath$ depending on parameters [17]. Later, Kolb generalized her method to a wider class including arbitrary symmetrizable Kac-Moody types [14]. Recently, new developments have been made. Universal K-matrix was obtained in [2] (see also [1]), and the theory of canonical bases for symmetric pairs, also known as $\imath$canonical bases, was initiated in [2] and developed in [3, 4].

Let us go back to the usual quantum groups. We consider a quantum group $\mathbf{U}$ associated to an affine Lie algebra. It is known [9, 8, 19] that there exists a distinguished family of finite-dimensional $\mathbf{U}$-module, called Kirillov-Reshetikhin modules, that have crystal bases in the sense of Kashiwara [11]. For such Kirillov-Reshetikhin crystals, a combinatorial version of the Yang-Baxter equation is satisfied by the combinatorial R-matrix, and it is applied to the analysis of box-ball systems, a kind of discrete integrable dynamical systems [7, 10]. We can also consider a box-ball system with boundary [15]. In that case, together with the Yang-Baxter equation, a combinatorial version of the reflection equation is also needed for integrability. It should be satisfied by the crystal limit of a quantum K-matrix. However, we cannot take such crystal limit directly from the universal K-matrix. In the case of the quantum R-matrix, the combinatorial R-matrix was determined by using the crystal theory. Very recently, the theory of $\imath$crystal bases was initiated by the last author in [22], although it is restricted to quasi-split types $I_\bullet = \emptyset$. We would like to apply it to determine combinatorial K-matrices for Kirillov-Reshetikhin crystals that satisfy the combinatorial reflection equation.

In this paper, we construct a quantum K-matrix for arbitrary Kirillov-Reshetikhin modules from that of fundamental representations. This method is well known as fusion construction. However, we could only find [5] for the case related to the quantum K-matrix, although it is not easy to see how their construction and ours are related. So we decided to explain it in rather detail in this paper. We then use the theory of $\imath$crystals in [22] and obtain explicitly combinatorial K-matrices of the Kirillov-Reshetikhin crystals associated to the first fundamental representation, or symmetric tensor representations of arbitrary level, for all Satake diagrams of quasi-split type based on the Dynkin diagram of affine type A. Satake diagrams of affine type are classified in [20, Table 16]. According to it, all quasi-split types are A.1, A.3 with no $\bullet$ and A.4. To be more precise, we also exclude A.3c to restrict our cases when $\tau(0) = 0$ where 0 is the affine node. All the combinatorial K-matrices in this paper are new, although the A.1 case was also treated in [16]. This is due to the fact that the choices of parameters appearing in $\mathbf{U}^\imath$ are different. The parameters in this paper are chosen in a way such that the theory of $\imath$crystal is applicable, and the $\imath$crystal graph of the Kirillov-Reshetikhin crystals under consideration are connected.

Throughout the paper, we use the following notations: $\chi(st) = 1\,(st \text{ is true}), = 0\,(st \text{ is false})$. For an integer m, we set $(m)_+ = \max(m, 0), \theta(m) = \chi(m \text{ is odd})$.

2. Kirillov-Reshetikhin modules and fusion construction of quantum K-matrices

2.1. **Quantum group.** We recall the definition of the quantum group associated with a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$. There exists a diagonal matrix $D = \mathrm{diag}(d_i)_{i\in I}$ such that DA is symmetric. We take d_i's to be pairwise coprime positive integers. Let $\{\alpha_i \mid i \in I\}$ and $\{h_i \mid i \in I\}$ be the sets of simple roots and simple coroots. Then we have $\langle h_i, \alpha_j\rangle = a_{ij}$ for $i, j \in I$. The quantum group $\mathbf{U}$ associated to A is defined to be an associative algebra over $\mathbb{C}(q)$ generated by $E_i, F_i, K_i^{\pm 1}$ $(i \in I)$ subject to the relations

$$K_iK_j = K_jK_i, \quad K_iE_j = q_i^{a_{ij}}E_jK_i, \quad K_iF_j = q_i^{-a_{ij}}F_jK_i, \quad E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{r=0}^{1-a_{ij}}(-1)^rE_i^{(r)}E_jE_i^{(1-a_{ij}-r)} = \sum_{r=0}^{1-a_{ij}}(-1)^rF_i^{(r)}F_jF_i^{(1-a_{ij}-r)} = 0 \quad \text{if } i \neq j$$

for $i, j \in I$, where $q_i = q^{d_i}, [m]_i = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}, [m]_i! = \prod_{k=1}^m [k]_i, E_i^{(m)} = E_i^m/[m]_i!, F_i^{(m)} = F_i^m/[m]_i!$. $\mathbf{U}$ has the structure of a Hopf algebra with coproduct

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i. \tag{1}$$

This coproduct is sometimes denoted by Δ_+ (see [11] for instance).

2.2. $\imath$**Quantum group.** We introduce an $\imath$quantum group $\mathbf{U}^\imath$. To do this, we first recall a Satake diagram in the sense of [20, Definition 2.7]. A Satake diagram is a pair of a Dynkin diagram whose vertices are painted white or black and an involutive permutation τ on it satisfying $a_{\tau(i)\tau(j)} = a_{ij}$ for all $i, j \in I$. It is represented by a triple $(I, I_\bullet, \tau)$ where $I_\bullet (\subset I)$ is the set of black vertices. We also set $I_\circ = I \setminus I_\bullet$. Let $w_\bullet$ be the longest element of the Weyl group $W_\bullet$ associated to $I_\bullet$ and $\rho_\bullet^\vee$ the half sum of the positive coroots of $I_\bullet$. Then $(I, I_\bullet, \tau)$ should satisfy

$$w_\bullet(\alpha_i) = -\alpha_{\tau(i)} \quad \text{for } i \in I_\bullet, \tag{2}$$
$$\langle \rho_\bullet^\vee, \alpha_i\rangle \in \mathbb{Z} \quad \text{for } i \in I_\circ \text{ such that } \tau(i) = i. \tag{3}$$

Next we recall Lusztig's braid group action $T_i = T''_{i,1}$ on $\mathbf{U}$ ([18, 37.1.3]). It is defined by

$$T_i(E_j) = \begin{cases} -F_iK_i & \text{if } i = j, \\ \sum_{r+s=-a_{ij}}(-1)^rq_i^{-r}E_i^{(s)}E_jE_i^{(r)} & \text{if } i \neq j, \end{cases}$$
$$T_i(F_j) = \begin{cases} -K_i^{-1}E_i & \text{if } i = j, \\ \sum_{r+s=-a_{ij}}(-1)^rq_i^{r}F_i^{(s)}F_jF_i^{(r)} & \text{if } i \neq j, \end{cases}$$
$$T_i(K_j) = K_jK_i^{-a_{ij}}.$$

Set

$$T_{w_\bullet} = T_{i_1}\cdots T_{i_l}$$

when $w_\bullet = s_{i_1}\cdots s_{i_l}$ is a reduced expression.

To define a generator B_i, we introduce two sets of parameters $(\varsigma_i)_{i\in I_\circ} \in (\mathbb{C}(q)^\times)^{I_\circ}, (\kappa_i)_{i\in I_\circ} \in \mathbb{C}(q)^{I_\circ}$. ς_i should satisfy $\varsigma_i = \varsigma_{\tau(i)}$ for $i \in I_\circ$. We set

$$B_i = F_i + \varsigma_iT_{w_\bullet}(E_{\tau(i)})K_i^{-1} + \kappa_iK_i^{-1}. \tag{4}$$

The $\imath$quantum group $\mathbf{U}^\imath$ defined as a subalgebra of $\mathbf{U}$ generated by $\mathbf{U}(I_\bullet)$ and $\{B_i, K_iK_{\tau(i)}^{-1}\}_{i\in I_\circ}$, where $\mathbf{U}(I_\bullet)$ is a usual quantized enveloping algebra associated to $I_\bullet$. $\mathbf{U}^\imath$ is a right coideal with respect to the coproduct given in the previous subsection.

2.3. **Kirillov-Reshetikhin modules.** In this subsection, we assume A is of affine type and $0 \in I$ to be the affine node. Let $\{\Lambda_i \mid i \in I\}$ be the set of fundamental weights and set $P = \bigoplus_{i\in I}\mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ where δ is the standard null root. Since we are interested in finite-dimensional $\mathbf{U}$-modules, our weight lattice to consider should be $P_{\mathrm{cl}} = P/\mathbb{Z}\delta$. For a finite-dimensional $\mathbf{U}$-module U, let U_{aff} denote the $\mathbf{U}$-module $\mathbb{C}(q)[z, z^{-1}] \otimes U$ with the actions of E_i and F_i by $z^{\delta_{i0}} \otimes E_i$ and $z^{-\delta_{i0}} \otimes F_i$. For $x \in \mathbb{C}(q)$, we define the $\mathbf{U}$-module $U(x)$ by $U_{\mathrm{aff}}/(z - x)U_{\mathrm{aff}}$. For finite-dimensional modules U, V and $x, y \in \mathbb{C}(q)$, we introduce the quantum R-matrix $R_{U,V}(x, y) : U(x) \otimes V(y) \to V(y) \otimes U(x)$ as a $\mathbf{U}$-linear operator. If $U(x) \otimes V(y)$

is irreducible, $R_{U,V}(x,y)$ is an isomorphism and is determined up to scalar multiple. But it happens for some special elements x,y of $\mathbb{C}(q)$ that $R_{U,V}(x,y)$ is not an isomorphism. Quantum R-matrices satisfy the Yang-Baxter equation.

$$R_{V,W}(y,z)R_{U,W}(x,z)R_{U,V}(x,y) = R_{U,V}(x,y)R_{U,W}(x,z)R_{V,W}(y,z). \tag{5}$$

We next explain Kirillov-Reshetikhin modules, KR modules for short, following [19, §3]. Set $I_0 = I \setminus \{0\}$. For $r \in I_0$, we define $\varpi_r = \Lambda_r - \langle h_r, c\rangle \Lambda_0$ and call it a level 0 fundamental weight. Here c is the canonical central element. In [13], a finite-dimensional $\mathbf{U}$-module $W(\varpi_r)$ is constructed as a quotient of the extremal weight module $V(\varpi_r)$. It is called a (level 0) fundamental module. Let $\mathcal{W}$ be a fundamental module. From $\mathcal{W}$, we can construct a KR module $\mathcal{W}_s$ ($s \in \mathbb{Z}_{>0}$) by fusion construction. For $s \geq 2$, let $\mathfrak{S}_s$ denote the group of permutations on s letters generated by $\sigma_i = (i\ i+1)$ for $1 \leq i \leq s-1$. We have $\mathbf{U}$-linear maps

$$R_w(x_1,\dots,x_s) : \mathcal{W}(x_1)\otimes\cdots\otimes\mathcal{W}(x_s) \longrightarrow \mathcal{W}(x_{w(1)})\otimes\cdots\otimes\mathcal{W}(x_{w(s)}), \tag{6}$$

for $w \in \mathfrak{S}_s$ and $x_1,\dots,x_s \in \mathbb{C}(q)$ satisfying

$$\begin{aligned} &R_1(x_1,\dots,x_s) = \mathrm{id}_{\mathcal{W}(x_1)\otimes\cdots\otimes\mathcal{W}(x_s)},\\ &R_{\sigma_i}(x_1,\dots,x_s) = \left(\otimes_{j<i}\mathrm{id}_{\mathcal{W}(x_j)}\right)\otimes R(x_i,x_{i+1})\otimes\left(\otimes_{j>i+1}\mathrm{id}_{\mathcal{W}(x_j)}\right),\\ &R_{ww'}(x_1,\dots,x_s) = R_{w'}(x_{w(1)},\dots,x_{w(s)})R_w(x_1,\dots,x_s), \end{aligned}$$

for $w,w' \in \mathfrak{S}_s$ with $\ell(ww') = \ell(w)+\ell(w')$ where $\ell(w)$ denotes the length of w and $R(x,y) = R_{\mathcal{W},\mathcal{W}}(x,y)$. To construct a KR module $\mathcal{W}_s$ for $s \geq 2$, we need to set $x_i = q^{d_r(s-2i+1)} = q_r^{s-2i+1}$, where d_r is determined in the beginning of §2.1. In particular, $d_r = 1$ for all $r \in I_0$ for untwisted ADE cases. Hence we have a $\mathbf{U}$-linear map $R_s = R_{\sigma_0}(x_1,\dots,x_s)$:

$$R_s : \mathcal{W}(q_r^{s-1})\otimes\ldots\otimes\mathcal{W}(q_r^{1-s}) \longrightarrow \mathcal{W}(q_r^{1-s})\otimes\ldots\otimes\mathcal{W}(q_r^{s-1}).$$

Here σ_0 is the longest element in $\mathfrak{S}_s$. Now we define a KR module corresponding to $\mathcal{W}$ and s by

$$\mathcal{W}_s = \operatorname{Im} R_s. \tag{7}$$

For a KR module $\mathcal{W}_s$ corresponding to ϖ_r ($r \in I_0$), we also define the dual KR module $\mathcal{W}_s^\vee$ as follows. Let W be the affine Weyl group and W_0 its subgroup generated by $\{s_i \mid i \in I_0\}$ where s_i stands for the simple reflection for α_i. Both act on P_{cl}. Let w_0 be the longest element of W_0. For $r \in I_0$, we define $r^\vee \in I_0$ by $-w_0\varpi_r = \varpi_{r^\vee}$. From this fixed $r^\vee$, we set $\mathcal{W}^\vee = W(\varpi_{r^\vee})$. $\mathcal{W}_s^\vee$ is constructed similarly from $\mathcal{W}^\vee$ by the fusion construction.

2.4. **Quantum R-matrix.** Fusion construction is used not only for defining KR modules but also for giving a quantum R-matrix for two KR modules. For two KR modules $\mathcal{W}_s, \mathcal{W}'_{s'}$, define a linear map $R_{\mathcal{W}_s,\mathcal{W}'_{s'}}(x,y)$ from

$$\mathcal{W}_s(x)\otimes\mathcal{W}'_{s'}(y) \subset (\mathcal{W}(q_r^{1-s}x)\otimes\ldots\otimes\mathcal{W}(q_r^{s-1}x))\otimes(\mathcal{W}'(q_{r'}^{1-s'}y)\otimes\ldots\otimes\mathcal{W}'(q_{r'}^{s'-1}y)) \tag{8}$$

by

$$R_{\mathcal{W}_s,\mathcal{W}'_{s'}}(x,y) = (R_{s,s+1}\cdots R_{2,3}R_{1,2})\cdots(R_{s+s'-2,s+s'-1}\cdots R_{s,s+1}R_{s-1,s})(R_{s+s'-1,s+s'}\cdots R_{s+1,s+2}R_{s,s+1}),$$

where $R_{i,i+1}$ stands for the quantum R-matrix acting only on the i-th and $i+1$-th components of the right hand side of (8). Hence, it maps to

$$(\mathcal{W}'(q_{r'}^{1-s'}y)\otimes\ldots\otimes\mathcal{W}'(q_{r'}^{s'-1}y))\otimes(\mathcal{W}(q_r^{1-s}x)\otimes\ldots\otimes\mathcal{W}(q_r^{s-1}x)),$$

namely, it interchanges $\mathcal{W}(q_r^{1-s}x)\otimes\ldots\otimes\mathcal{W}(q_r^{s-1}x)$ and $\mathcal{W}'(q_{r'}^{1-s'}y)\otimes\ldots\otimes\mathcal{W}'(q_{r'}^{s'-1}y)$.

Proposition 1. (i) *The image of $R_{\mathcal{W}_s,\mathcal{W}'_{s'}}(x,y)$ belongs to $\mathcal{W}'_{s'}(y)\otimes\mathcal{W}_s(x)$.*

(ii) *They satisfy the Yang-Baxter equation:*

$$R_{\mathcal{W}'_{s'},\mathcal{W}''_{s''}}(y,z)R_{\mathcal{W}_s,\mathcal{W}''_{s''}}(x,z)R_{\mathcal{W}_s,\mathcal{W}'_{s'}}(x,y) = R_{\mathcal{W}_s,\mathcal{W}'_{s'}}(x,y)R_{\mathcal{W}_s,\mathcal{W}''_{s''}}(x,z)R_{\mathcal{W}'_{s'},\mathcal{W}''_{s''}}(y,z).$$

Proof. To prove (i), we slightly generalize the construction of $R_w(x_1,\dots,x_s)$ in (6). This time, for two fundamental modules $\mathcal{W},\mathcal{W}'$ and $x_1,\dots,x_s,y_1,\dots,y_{s'}\in\mathbb{C}(q)$, we consider the $\mathbf{U}$-linear map

$$R'_{\sigma_0^{(s+s')}}(x_1,\dots,x_s,y_1,\dots,y_{s'}):\mathcal{W}(x_1)\otimes\cdots\otimes\mathcal{W}(x_s)\otimes\mathcal{W}'(y_1)\otimes\cdots\otimes\mathcal{W}'(y_{s'})$$
$$\longrightarrow\mathcal{W}'(y_{s'})\otimes\cdots\otimes\mathcal{W}'(y_1)\otimes\mathcal{W}(x_s)\otimes\cdots\otimes\mathcal{W}(x_1).$$

Here $\sigma_0^{(s+s')}$ is the longest element of $\mathfrak{S}_{s+s'}$. Let $\sigma_0^{(s)}$ (resp. $\sigma_0^{(s')}$) be the longest element of the subgroup $\mathfrak{S}_s$ of the former s letters $\{1,\dots,s\}$ (resp. $\mathfrak{S}_{s'}$ of the latter s' letters $\{s+1,\dots,s+s'\}$) of $\mathfrak{S}_{s+s'}$. Then $\sigma_0^{(s)}$ and $\sigma_0^{(s')}$ commute with each other. Let $\bar\sigma_0^{(s)}$ (resp. $\bar\sigma_0^{(s')}$) again be the longest element of $\mathfrak{S}_s$ but of the latter s letters $\{s'+1,\dots,s+s'\}$ (resp. of the former s' letters $\{1,\dots,s'\}$). $\bar\sigma_0^{(s)}$ and $\bar\sigma_0^{(s')}$ also commute with each other. Set

$$\tau=\begin{pmatrix}1&\cdots&s&s+1&\cdots&s+s'\\ s'+1&\cdots&s+s'&1&\cdots&s'\end{pmatrix}.$$

Then we have $\tau\sigma_0^{(s')}\sigma_0^{(s)}=\bar\sigma_0^{(s')}\bar\sigma_0^{(s)}\tau=\sigma_0^{(s+s')}$. The corresponding relation for the quantum R-matrices R_w's verifies the assertion.

(ii) is shown by successive uses of the Yang-Baxter equations among quantum R-matrices $R_{\mathcal{W},\mathcal{W}'}$, $R_{\mathcal{W},\mathcal{W}''},R_{\mathcal{W}',\mathcal{W}''}$ with various parameters. □

2.5. **Quantum K-matrix.** Quantum K-matrix is a solution to the reflection equation. Let $\mathcal{W}$ be a fundamental module. Following [20], we consider two cases: Untwisted case

$$K_{\mathcal{W}}(x):\mathcal{W}(x)\to\mathcal{W}(x^{-1}),\tag{9}$$

and twisted case

$$K_{\mathcal{W}}(x):\mathcal{W}(x)\to\mathcal{W}^\vee(x^{-1}).\tag{10}$$

To deal with two cases together, we introduce symbols $*=\emptyset,\vee$, so that one can write uniformly as $K_{\mathcal{W}}(x):\mathcal{W}(x)\to\mathcal{W}^*(x^{-1})$. Under this notation, for two fundamental modules $\mathcal{W},\mathcal{W}'$, the reflection equation reads as

$$R_{\mathcal{W}'^*,\mathcal{W}^*}(y^{-1},x^{-1})K_{\mathcal{W}'}(y)R_{\mathcal{W}^*,\mathcal{W}'}(x^{-1},y)K_{\mathcal{W}}(x)=K_{\mathcal{W}}(x)R_{\mathcal{W}'^*,\mathcal{W}}(y^{-1},x)K_{\mathcal{W}'}(y)R_{\mathcal{W},\mathcal{W}'}(x,y)\tag{11}$$

as a map from $\mathcal{W}(x)\otimes\mathcal{W}'(y)$ to $\mathcal{W}^*(x^{-1})\otimes\mathcal{W}'^*(y^{-1})$.

We define a quantum K-matrix for the KR module $\mathcal{W}_s$ by composing the ones and the quantum R matrices for fundamental modules by

$$K_{\mathcal{W}_s}(x)=K^{(s)}(x)\cdots K^{(2)}(x)K^{(1)}(x),$$
$$K^{(j)}(x)=K_1(q_r^{-s+2j-1}x)R^*_{1,2}(q_r^{s-2j+3}x^{-1},q_r^{-s+2j-1}x)R^*_{2,3}(q_r^{s-2j+5}x^{-1},q_r^{-s+2j-1}x)\cdots$$
$$\cdots R^*_{j-1,j}(q_r^{s-1}x^{-1},q_r^{-s+2j-1}x).$$

Here $R^*_{i,i+1}(x,y)$ is the quantum R-matrix $R^*_{\mathcal{W},\mathcal{W}}(x,y)$ acting on the i-th and $i+1$-th components. $K_{\mathcal{W}_s}(x)$ is a map from $\mathcal{W}(q_r^{1-s}x)\otimes\cdots\otimes\mathcal{W}(q_r^{s-1}x)$ to $\mathcal{W}^*(q_r^{1-s}x^{-1})\otimes\cdots\otimes\mathcal{W}^*(q_r^{s-1}x^{-1})$. We can also write inductively as

$$K_{\mathcal{W}_s}(x)=K_1(q_r^{s-1}x)R^*_{1,2}(q_r^{-s+3}x^{-1},q_r^{s-1}x)R^*_{2,3}(q_r^{-s+5}x^{-1},q_r^{s-1}x)\cdots$$
$$\cdots R^*_{s-1,s}(q_r^{s-1}x^{-1},q_r^{s-1}x)(K_{\mathcal{W}_{s-1}}(q_r^{-1}x)\otimes 1).\tag{12}$$

Proposition 2. (i) *The image of $K_{\mathcal{W}_s}(x)$ belongs to $\mathcal{W}_s^*(x^{-1})$.*

(ii) *The quantum K-matrices satisfy the reflection equation:*

$$R_{\mathcal{W}'^*_{s'},\mathcal{W}^*_s}(y^{-1},x^{-1})K_{\mathcal{W}'_{s'}}(y)R_{\mathcal{W}^*_s,\mathcal{W}'_{s'}}(x^{-1},y)K_{\mathcal{W}_s}(x)=K_{\mathcal{W}_s}(x)R_{\mathcal{W}'^*_{s'},\mathcal{W}_s}(y^{-1},x)K_{\mathcal{W}'_{s'}}(y)R_{\mathcal{W}_s,\mathcal{W}'_{s'}}(x,y).$$

Proof. To prove (i), we note that $K_{\mathcal{W}_s}(x)R_s=R_s\bar K_{\mathcal{W}_s}(x)$ holds, where $\bar K_{\mathcal{W}_s}(x)$ is defined by replacing q_r with q_r^{-1} in $K_{\mathcal{W}_s}(x)$. It can be proven by successive uses of the Yang-Baxter equation and the reflection equation (11). (ii) is also shown by successive uses of (11). □

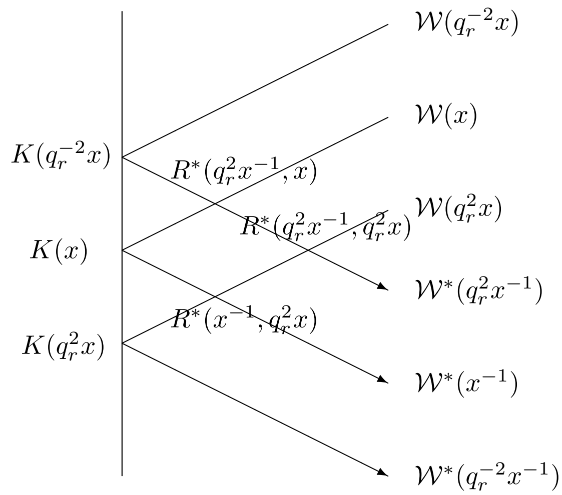


FIGURE 1. Graphical representation for $K_{\mathcal{W}_3}(x)$

3. Crystals and ıcrystals

3.1. **Crystals.** A crystal is a set $\mathcal{B}$ equipped with maps $\tilde{E}_i, \tilde{F}_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ for $i \in I$, where 0 is a formal symbol. By depicting $b \xrightarrow{i} b'$ when $\tilde{F}_i b = b'$ for $b, b' \in \mathcal{B}$, $\mathcal{B}$ becomes a colored oriented graph called crystal graph. We set $\varepsilon_i(b) = \max\{m \geq 0 \mid \tilde{E}_i^m b \neq 0\}, \varphi_i(b) = \max\{m \geq 0 \mid \tilde{F}_i^m b \neq 0\}$ for $b \in \mathcal{B}$.

Let $\mathcal{B}$ and $\mathcal{B}'$ be crystals. Then $\mathcal{B} \otimes \mathcal{B}'$ also has the structure of crystal by

$$\begin{aligned}
&\varepsilon_i(b_1 \otimes b_2) = \varepsilon_i(b_2) + (\varepsilon_i(b_1) - \varphi_i(b_2))_+,\\
&\varphi_i(b_1 \otimes b_2) = \varphi_i(b_1) + (\varphi_i(b_2) - \varepsilon_i(b_1))_+,\\
&\tilde{E}_i(b_1 \otimes b_2) = \begin{cases} \tilde{E}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) > \varphi_i(b_2),\\ b_1 \otimes \tilde{E}_i b_2 & \text{if } \varepsilon_i(b_1) \leq \varphi_i(b_2),\end{cases}\\
&\tilde{F}_i(b_1 \otimes b_2) = \begin{cases} \tilde{F}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) \geq \varphi_i(b_2),\\ b_1 \otimes \tilde{F}_i b_2 & \text{if } \varepsilon_i(b_1) < \varphi_i(b_2).\end{cases}
\end{aligned}$$

It is called the tensor product of crystals, though it is the Cartesian product of two sets. Note that this convention is opposite to [11].

The notion of crystal was abstracted from that of crystal basis. We again use a different convention from [11], namely, we replace q with q^{-1}. Set $\mathbf{A} = \{f(q) \in \mathbb{C}(q) \mid f(q) \text{ has no pole at } q = \infty\}$ and let M be a $\mathbf{U}$-module. Then a crystal basis of M is a pair $(\mathcal{L}, \mathcal{B})$ of a $\mathbf{A}$-lattice $\mathcal{L}$ and a $\mathbb{C}$-basis $\mathcal{B}$ of $\mathcal{L}/q^{-1}\mathcal{L}$. Moreover, operators $\tilde{E}_i, \tilde{F}_i$ have concrete meanings on M and we have $\tilde{E}_i\mathcal{L} \subset \mathcal{L}, \tilde{F}_i\mathcal{L} \subset \mathcal{L}$, so $\tilde{E}_i\mathcal{B} \subset \mathcal{B} \sqcup \{0\}, \tilde{F}_i\mathcal{B} \subset \mathcal{B} \sqcup \{0\}$ where $0 \in M$.

It is known that a KR module $\mathcal{W}_s$ introduced in section 2.3 has a crystal basis when our Cartan matrix A is of nonexceptional affine type [19]. We call it a KR crystal and denote it by $\mathcal{B}_s$. We remark that because of the convention of crystal basis with q replaced by q^{-1}, our tensor product rule of crystals is consistent with our coproduct (1). Let $\mathcal{B}_s$ a KR crystal. One can consider its affinization $\mathcal{B}_s(x) = \{b \otimes x^d \mid b \in \mathcal{B}_s, d \in \mathbb{Z}\}$. We often write $x^d b$ instead of $b \otimes x^d$. Crystal operators $\tilde{E}_i, \tilde{F}_i$ act on $\mathcal{B}_s(x)$ as $\tilde{E}_i(x^d b) = x^{d+\delta_{i0}}\tilde{E}_i b, \tilde{F}_i(x^d b) = x^{d-\delta_{i0}}\tilde{F}_i b$. Let $\mathcal{B}_s, \mathcal{B}'_{s'}$ be KR crystals. Then, there exists an isomorphism of crystals $\mathcal{R}_{\mathcal{B}_s,\mathcal{B}'_{s'}} : \mathcal{B}_s(x) \otimes \mathcal{B}'_{s'}(y) \to \mathcal{B}'_{s'}(y) \otimes \mathcal{B}_s(x)$, namely, a map that commutes with the action of crystal operators. By taking a suitable limit of q, $q \to \infty$ in our case, we obtain a set-theoretical Yang-Baxter equation

$$\mathcal{R}_{\mathcal{B}'_{s'},\mathcal{B}''_{s''}}(y,z)\mathcal{R}_{\mathcal{B}_s,\mathcal{B}''_{s''}}(x,z)\mathcal{R}_{\mathcal{B}_s,\mathcal{B}'_{s'}}(x,y) = \mathcal{R}_{\mathcal{B}_s,\mathcal{B}'_{s'}}(x,y)\mathcal{R}_{\mathcal{B}_s,\mathcal{B}''_{s''}}(x,z)\mathcal{R}_{\mathcal{B}'_{s'},\mathcal{B}''_{s''}}(y,z).$$

from (5).

For a crystal $\mathcal{B}$ there is a notion of its dual crystal $\mathcal{B}^\vee$ [12]. It is defined by $\mathcal{B}^\vee = \{b^\vee \mid b \in \mathcal{B}\}$ with

$$\tilde{E}_i b^\vee = (\tilde{F}_i b)^\vee, \qquad \tilde{F}_i b^\vee = (\tilde{E}_i b)^\vee. \tag{13}$$

Although it is not written in the literature except in simpler cases, the dual crystal of the KR crystal $\mathcal{B}_s$ is given by the KR crystal of the KR module $\mathcal{W}_s^\vee$ introduced in section 2.3. We will see examples in type A in section 5.

3.2. **$\imath$Crystals.** In this subsection, we briefly recall the notion of $\imath$crystals, which is an analogous notion to crystal for the $\imath$quantum group. It was introduced in [22] under the following assumptions:

(A1) $I_\bullet = \emptyset$.
(A2) $a_{i,\tau(i)} \in \{2, 0, -1\}$ for all $i \in I$.
(A3) If $a_{i,\tau(i)} = 2$, then $\varsigma_i = q_i^{-1}$ and $\kappa_i = \frac{q_i^{s_i} - q_i^{-s_i}}{q_i - q_i^{-1}}$ for some $s_i \in \mathbb{Z}$.
(A4) If $a_{i,\tau(i)} = 0$, then $\varsigma_i = 1$ and $\kappa_i = 0$.
(A5) If $a_{i,\tau(i)} = -1$, then $\varsigma_i = q_i^{s_i}$, $\varsigma_i\varsigma_{\tau(i)} = q_i$, and $\kappa_i = 0$ for some $s_i \in \mathbb{Z}$.

Therefore, we always keep this assumption whenever we consider $\imath$crystals.

An $\imath$crystal is a set $\mathcal{B}$ equipped with structure maps $\mathrm{wt}^\imath$, β_i, and $\tilde{B}_i$ for $i \in I$. As a special case of [22, Corollary 5.2.2], we obtain the following.

Proposition 3. *Let $\mathcal{B}$ be a* KR *crystal. Then, it has an $\imath$crystal structure as follows: Let $b \in \mathcal{B}$ and $i \in I$.*

- $\mathrm{wt}^\imath(b) = \overline{\mathrm{wt}(b)}$, *where $\overline{\lambda}$ denote the image of $\lambda \in P_{cl}$ in $P_{cl}^\imath := P_{cl}/\{\lambda + \tau(\lambda) \mid \lambda \in P_{cl}\}$.*
- *If $a_{i,\tau(i)} = 2$, then*

$$\beta_i(b) = \begin{cases} \varepsilon_i(b) + 1 & \text{if } |s_i| \le \varphi_i(b) \text{ and } s_i - \varphi_i(b) \text{ is odd}, \\ |s_i| - \langle h_i, \mathrm{wt}(b) \rangle & \text{if } |s_i| > \varphi_i(b), \\ \varepsilon_i(b) & \text{if } |s_i| \le \varphi_i(b) \text{ and } s_i - \varphi_i(b) \text{ is even}, \end{cases}$$

$$\tilde{B}_i b = \begin{cases} \tilde{F}_i b & \text{if } |s_i| \le \varphi_i(b) \text{ and } s_i - \varphi_i(b) \text{ is odd}, \\ \mathrm{sgn}(s_i) b & \text{if } |s_i| > \varphi_i(b), \\ \tilde{E}_i b & \text{if } |s_i| \le \varphi_i(b) \text{ and } s_i - \varphi_i(b) \text{ is even}, \end{cases}$$

  where $\mathrm{sgn}(n)$ denotes the sign of $n \in \mathbb{Z} \setminus \{0\}$.
- *If $a_{i,\tau(i)} = 0$, then*

$$\beta_i(b) = \max(\varphi_i(b), \varphi_{\tau(i)}(b)) - \langle h_{\tau(i)}, \mathrm{wt}(b) \rangle,$$

$$\tilde{B}_i b = \begin{cases} \tilde{F}_i b & \text{if } \varphi_i(b) > \varphi_{\tau(i)}(b), \\ \tilde{E}_{\tau(i)} b & \text{if } \varphi_i(b) \le \varphi_{\tau(i)}(b). \end{cases}$$

- *If $a_{i,\tau(i)} = -1$, then*

$$\beta_i(b) = \max(\varphi_i(b), \varphi_{\tau(i)}(b) + s_i) - s_i - \langle h_{\tau(i)}, \mathrm{wt}(b) \rangle.$$

  – *When $\varphi_i(b) > \varphi_{\tau(i)}(b) + s_i$,*

$$\tilde{B}_i b = \begin{cases} \frac{1}{\sqrt{2}} \tilde{F}_i b & \text{if } \varphi_i(b) = \varphi_{\tau(i)}(b) + s_i + 1 \text{ and } \varphi_{\tau(i)}(\tilde{F}_i b) = \varphi_{\tau(i)}(b) + 1, \\ \tilde{F}_i b & \text{otherwise}. \end{cases}$$

  – *When $\varphi_i(b) \le \varphi_{\tau(i)}(b) + s_i$,*

$$\tilde{B}_i b = \begin{cases} \frac{1}{\sqrt{2}} \tilde{E}_{\tau(i)} b & \text{if } \varphi_i(b) = \varphi_{\tau(i)}(b) + s_i \text{ and } \varphi_i(\tilde{E}_{\tau(i)} b) = \varphi_i(b), \\ \frac{1}{\sqrt{2}} (\tilde{E}_{\tau(i)} b + \tilde{F}_i b) & \text{if } \varphi_i(b) = \varphi_{\tau(i)}(b) + s_i > (-s_{\tau(i)})_+ \text{ and } \varphi_i(\tilde{E}_{\tau(i)} b) = \varphi_i(b) - 1, \\ \tilde{E}_{\tau(i)} b & \text{otherwise}. \end{cases}$$

4. Existence of the combinatorial K-matrix

In this section, we define the notion of combinatorial K-matrix in a similar way to that of combinatorial R-matrix. Then, we show that each combinatorial K-matrix satisfies the set-theoretical reflection equation. After that, we give an interpretation of the combinatorial K-matrix from a viewpoint of $\imath$crystals.

4.1. **Definition.** Let $\mathcal{W}_s$ be a KR module, x a formal spectral parameter, and $K_{\mathcal{W}_s}(x) : \mathcal{W}_s(x) \to \mathcal{W}_s^*(x^{-1})$ the quantum K-matrix. Recall that $* = \emptyset, \vee$ depending on the Satake diagram we consider. Let $(\mathcal{L}_s, \mathcal{B}_s)$ denote the crystal basis of $\mathcal{W}_s$. Set $\mathcal{L}_s(x) := \mathcal{L}_s \otimes_{\mathbb{C}} \mathbb{C}(x)$ and $\mathcal{B}_s(x) := \{b \otimes x^d \mid b \in \mathcal{B}_s,\ d \in \mathbb{Z}\}$. We often write $x^d b$ instead of $b \otimes x^d$. As seen in section 2.5, matrix coefficients of our quantum K-matrix $K_{\mathcal{W}_s}(x)$ are rational functions in q, x. Hence, by multiplying a suitable power of q, one can normalize it in a way such that $K_{\mathcal{W}_s}(x)(\mathcal{L}_s(x)) \subset \mathcal{L}_s^*(x^{-1})$ and the induced $\mathbb{C}(x)$-linear map $\bar{K}_{\mathcal{W}_s}(x) : \mathcal{L}_s(x)/q^{-1}\mathcal{L}_s(x) \to \mathcal{L}_s^*(x^{-1})/q^{-1}\mathcal{L}_s^*(x^{-1})$ is not zero. Set $\mathcal{K}_{\mathcal{B}_s}(x) = \bar{K}_{\mathcal{W}_s}(x)|_{\mathcal{B}_s(x)}$. If $\mathcal{K}_{\mathcal{B}_s}(x)$ gives rise to a bijection $\mathcal{B}_s(x) \to \mathcal{B}_x^*(x^{-1})$, then we call it a *combinatorial K-matrix* on $\mathcal{B}_s$.

Lemma 4. *If a combinatorial K-matrix exists, then it is unique up to multiplication of x^d for some $d \in \mathbb{Z}$.*

Proof. Let K_1, K_2 be normalizations of $K_{\mathcal{W}_s}$ which give rise to combinatorial K-matrices $\mathcal{K}_1, \mathcal{K}_2$. Then, there exists a unique $f \in \mathbb{C}[\![q^{-1}]\!](x) \cap \mathbb{C}(q, x)$ such that $K_2 = fK_1$. Hence, we have

$$\mathcal{K}_2 = f_0 \mathcal{K}_1, \tag{14}$$

where $f_0 \in \mathbb{C}(x)$ denotes the constant term of f. In order to prove the assertion, we need to show that $f_0 = x^d$ for some $d \in \mathbb{Z}$.

Let $b \in \mathcal{B}_s$. Then, we have $\mathcal{K}_1(b), \mathcal{K}_2(b) \in \mathcal{B}_s^*(x^{-1})$. On the other hand, by equation (14), we have

$$\mathcal{K}_2(b) = f_0 \mathcal{K}_1(b).$$

Since $\mathcal{B}_s^*$ is a basis of the $\mathbb{C}$-vector space $\mathcal{L}(\mathcal{W}_s^*)/q^{-1}\mathcal{L}(\mathcal{W}_s^*)$, the fact that both $\mathcal{K}_1(b)$ and $f_0\mathcal{K}_1(b)$ belong to $\mathcal{B}_s^*(x^{-1})$ implies that $f_0 = x^d$ for some $d \in \mathbb{Z}$. Thus, the proof completes. □

4.2. **Set-theoretical reflection equation.** Let $\mathcal{W}_s, \mathcal{W}'_{s'}$ be KR modules, and $\mathcal{B}_s, \mathcal{B}'_{s'}$ their crystal bases. Assume that the combinatorial K-matrices $\mathcal{K}_{\mathcal{B}_s}(x), \mathcal{K}_{\mathcal{B}'_{s'}}(y)$ exist.

Proposition 5. *The combinatorial K-matrices $\mathcal{K}_{\mathcal{B}_s}(x)$ and $\mathcal{K}_{\mathcal{B}'_{s'}}(y)$ satisfy the set-theoretical reflection equation*

$$\mathcal{R}_{\mathcal{B}'^*_{s'}, \mathcal{B}^*_s}(y^{-1}, x^{-1}) \mathcal{K}_{\mathcal{B}'_{s'}}(y) \mathcal{R}_{\mathcal{B}^*_s, \mathcal{B}'_{s'}}(x^{-1}, y) \mathcal{K}_{\mathcal{B}_s}(x) = \mathcal{K}_{\mathcal{B}_s}(x) \mathcal{R}_{\mathcal{B}'^*_{s'}, \mathcal{B}_s}(y^{-1}, x) \mathcal{K}_{\mathcal{B}'_{s'}}(y) \mathcal{R}_{\mathcal{B}_s, \mathcal{B}'_{s'}}(x, y). \tag{15}$$

Proof. The assertion is clear from Proposition 2. □

4.3. **ıCrystal theoretical viewpoint.** Let $\mathcal{W}_s$ be a KR module and $\mathcal{B}_s$ its crystal basis. Assume that the combinatorial K-matrix $\mathcal{K}_{\mathcal{B}_s}(x)$ exists.

Proposition 6. *The combinatorial K-matrix is an isomorphism of ıcrystals.*

Proof. Immediate from the definitions. □

By Propositions 5 and 6, we see that when finding the combinatorial K-matrix, it would be meaningful to find an ıcrystal isomorphism from $\mathcal{B}_s(x)$ to $\mathcal{B}_s^*(x^{-1})$. Indeed, as we will see below, in some cases, we can obtain the combinatorial K-matrix explicitly in this way without knowing the quantum K-matrix. This can be seen as a natural generalization of the fact that the combinatorial R-matrix can be constructed from the tensor product of two KR crystals, without knowing the quantum R-matrix. This fact suggests that there is a general theory which ensures the existence of combinatorial K-matrix by means of ıcrystals.

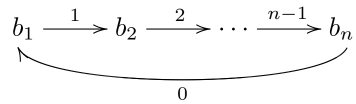
5. Type A case

In this section, we consider the quantum affine algebra of type $A_{n-1}^{(1)}$ with $n \geq 3$, the simplest family of KR modules $\mathcal{W}_s$ and its dual $\mathcal{W}_s^\vee$, and investigate quantum/combinatorial K-matrices for all quasi-split cases.

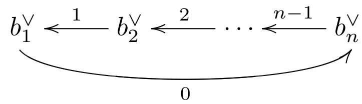
5.1. **The simplest KR modules and their R-matrices.** The fundamental module $\mathcal{W}$ associated to the first level 0 fundamental weight ϖ_1 is the vector representation $\mathcal{W} = \bigoplus_{i=1}^n \mathbb{C}(q)v_i$. Since $-w_0\varpi_1 = \varpi_{n-1}$, its dual representation $\mathcal{W}^\vee = \bigoplus_{i=1}^n \mathbb{C}(q)v_i^\vee$ has a highest weight ϖ_{n-1}. The actions of Chevalley generators on $\mathcal{W}$ and $\mathcal{W}^\vee$ are given as follows.

$$E_i v_j = \delta_{i+1,j} v_i, \quad F_i v_j = \delta_{ij} v_{i+1}, \quad K_i v_j = q^{\delta_{ij} - \delta_{i+1,j}} v_j,$$
$$E_i v_j^\vee = \delta_{ij} v_{i+1}^\vee, \quad F_i v_j^\vee = \delta_{i+1,j} v_i^\vee, \quad K_i v_j^\vee = q^{-\delta_{ij} + \delta_{i+1,j}} v_j^\vee.$$

In this section, indices i, j like in the above should be considered in $\mathbb{Z}/n\mathbb{Z}$. The crystal basis of $\mathcal{W}$ is $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ and its crystal graph is given as follows:



The crystal basis of $\mathcal{W}^\vee$ is $\mathcal{B}^\vee := \{b_1^\vee, b_2^\vee, \dots, b_n^\vee\}$ and its crystal graph is obtained from that of $\mathcal{B}$ by reversing the arrows:



The quantum R-matrices among $\mathcal{W}$ and $\mathcal{W}^\vee$ are given as follows [6].

$$R_{\mathcal{W},\mathcal{W}}(x,y) = \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} \frac{1-q^2}{1-q^2 z} z^{\chi(i<j)} E_{ii} \otimes E_{jj} + \sum_{i \neq j} q \frac{1-z}{1-q^2 z} E_{ij} \otimes E_{ji},$$

$$R_{\mathcal{W}^\vee,\mathcal{W}}(x,y) = \sum_{i \neq j} (-q)^{i-j-1} ((-q)^n z)^{\chi(i<j)} E_{ij} \otimes E_{ij} + \sum_{i,j} \frac{q^{\delta_{ij}} - q^{-\delta_{ij}}(-q)^n z}{1-q^2} E_{ij} \otimes E_{ji},$$

$$R_{\mathcal{W}^\vee,\mathcal{W}^\vee}(x,y) = \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} \frac{1-q^2}{1-q^2 z} z^{\chi(i>j)} E_{ii} \otimes E_{jj} + \sum_{i \neq j} q \frac{1-z}{1-q^2 z} E_{ij} \otimes E_{ji}.$$

Here $z = x/y$ and E_{ij} is the linear operator such that $E_{ij} v_j^* = v_i^{*'}$ when it belongs to $\mathrm{Hom}(\mathcal{W}^*, \mathcal{W}^{*'})$ $(*, *' = \emptyset, \vee)$.

Next we describe $\mathcal{W}_s$ obtained from $\mathcal{W}$ by fusion construction. The linear space $\mathcal{W}^{\otimes s}$ has a standard basis $\{v_{\mathbf{i}} \mid \mathbf{i} \in \{1, \dots, n\}^s\}$. Here, for $\mathbf{i} = (i_1, \dots, i_s)$, we set $v_{\mathbf{i}} = v_{i_1} \otimes \cdots \otimes v_{i_s}$. Define the weight of $\mathbf{i}$ $\mathrm{wt}(\mathbf{i})$ by $\alpha = (\alpha_1, \dots, \alpha_n)$ where α_i is the number of i's in $\mathbf{i}$. For $\alpha \in \{(\alpha_1, \dots, \alpha_n) \mid \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = s\}$, define

$$v_\alpha = \sum_{\mathrm{wt}(v_{\mathbf{i}})=\alpha} q^{-\tau(\mathbf{i})} v_{\mathbf{i}}, \qquad \tau(\mathbf{i}) = \sharp\{(p,q) \mid 1 \leq p < q \leq n, i_p > i_q\}.$$

Then $\{v_\alpha \mid \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = s\}$ turns out a basis of the KR module $\mathcal{W}_s$ defined in (7). The actions of Chevalley generators are given by

$$E_i v_\alpha = [\alpha_i + 1] v_{\alpha + \mathbf{e}_i - \mathbf{e}_{i+1}}, \quad F_i v_\alpha = [\alpha_{i+1} + 1] v_{\alpha - \mathbf{e}_i + \mathbf{e}_{i+1}}, \quad K_i v_\alpha = q^{\alpha_i - \alpha_{i+1}} v_\alpha.$$

Here $\mathbf{e}_i$ is the standard basis vector and subscripts of α should be considered modulo n. If α contains a negative integer upon application, then v_α should be considered as 0.

Similarly, one can realize a basis of $\mathcal{W}_s^\vee$ as a linear combination of $v_{\mathbf{i}}^\vee = v_{i_1}^\vee \otimes \cdots \otimes v_{i_s}^\vee \in \mathcal{W}^{\vee \otimes s}$. Set

$$v_\alpha^\vee = \sum_{\mathrm{wt}(\mathbf{i})=\alpha} q^{-\bar{\tau}(\mathbf{i})} v_{\mathbf{i}}^\vee, \qquad \bar{\tau}(\mathbf{i}) = \sharp\{(p,q) \mid 1 \leq p < q \leq n, i_p < i_q\}.$$

$\{v_\alpha^\vee \mid \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = s\}$ turns out a basis of $\mathcal{W}_s^\vee$. The actions of Chevalley generators are given by

$$E_i v_\alpha^\vee = [\alpha_{i+1} + 1] v_{\alpha - \mathbf{e}_i + \mathbf{e}_{i+1}}^\vee, \quad F_i v_\alpha^\vee = [\alpha_i + 1] v_{\alpha + \mathbf{e}_i - \mathbf{e}_{i+1}}^\vee, \quad K_i v_\alpha^\vee = q^{-\alpha_i + \alpha_{i+1}} v_\alpha^\vee.$$

We also review the piece-wise linear formulas for the combinatorial R-matrices. See e.g. [15], but note that the tensor product rule in this paper is opposite to it.

$$\mathcal{R}_{\mathcal{B}_s,\mathcal{B}_{s'}}(x,y)(x^d b_\alpha \otimes y^e b_\beta) = y^{e+Q_0(\alpha,\beta)} b_{\beta'} \otimes x^{d-Q_0(\alpha,\beta)} b_{\alpha'}$$
$$\beta'_i = \beta_i + Q_i(\alpha,\beta) - Q_{i-1}(\alpha,\beta), \quad \alpha'_i = \alpha_i + Q_{i-1}(\alpha,\beta) - Q_i(\alpha,\beta),$$
$$\mathcal{R}_{\mathcal{B}_s^\vee,\mathcal{B}_{s'}}(x,y)(x^d b_\alpha^\vee \otimes y^e b_\beta) = y^{e+Q_0(\alpha,\beta)} b_{\beta'} \otimes x^{d-Q_0(\alpha,\beta)} b_{\alpha'}^\vee$$
$$\beta'_i = \beta_i + P_i(\alpha,\beta) - P_{i-1}(\alpha,\beta), \quad \alpha'_i = \alpha_i + P_i(\alpha,\beta) - P_{i-1}(\alpha,\beta),$$
$$\mathcal{R}_{\mathcal{B}_s^\vee,\mathcal{B}_{s'}^\vee}(x,y)(x^d b_\alpha^\vee \otimes y^e b_\beta^\vee) = y^{e+Q_0(\beta.\alpha)} b_{\beta'}^\vee \otimes x^{d-Q_0(\beta,\alpha)} b_{\alpha'}^\vee$$
$$\beta'_i = \beta_i + Q_{i-1}(\beta,\alpha) - Q_i(\beta,\alpha), \quad \alpha'_i = \alpha_i + Q_i(\alpha,\beta) - Q_{i-1}(\alpha,\beta),$$

where Q_i, P_i are given by

$$Q_i(\alpha,\beta) = \min_{1\le k\le n}\Big(\sum_{j=k+1}^{n} \alpha_{i+j} + \sum_{k=1}^{k-1} \beta_{i+j}\Big), \qquad P_i(\alpha,\beta) = \min(\alpha_{i+1}, \beta_{i+1}).$$

Let $\mathcal{L}_s^*$ ($* \in \{\emptyset, \vee\}$) denote the crystal lattice of $\mathcal{W}_s$ spanned by v_α's. Set $b_\alpha^* := v_\alpha + q^{-1}\mathcal{L}_s^*$. Then, $\mathcal{B}_s^* := \{b_\alpha^*\}$ forms the crystal basis. We write α instead of b_α if no confusion can occur. We summarize the crystal and $\imath$crystal structure below. Note that for the parameters s_i in (A3) in section 3.2, we set $s_i = 0$. The crystal structure of $\mathcal{B}_s(x) = \{x^d b_\alpha \mid d \in \mathbb{Z},\ b_\alpha \in \mathcal{B}_s\}$ is as follows:

$$\begin{aligned}
&\langle h_i, \mathrm{wt}(x^d b_\alpha)\rangle = \alpha_i - \alpha_{i+1},\\
&\varepsilon_i(x^d b_\alpha) = \alpha_{i+1},\\
&\varphi_i(x^d b_\alpha) = \alpha_i,\\
&\tilde{E}_i(x^d b_\alpha) = x^{d+\delta_{i,0}} b_{\alpha+\mathbf{e}_i-\mathbf{e}_{i+1}},\\
&\tilde{F}_i(x^d b_\alpha) = x^{d-\delta_{i,0}} b_{\alpha-\mathbf{e}_i+\mathbf{e}_{i+1}}.
\end{aligned}$$

Then, the $\imath$crystal structure of $\mathcal{B}_s(x)$ is described as follows (below, $a_{i,\tau(i)} = -1$ occurs only when $n = 2n'+1$ for some $n' \in \mathbb{Z}_{>0}$ and $i = n', n'+1$):

- When $a_{i,\tau(i)} = 2$.

$$\begin{aligned}
&\beta_i(x^d b_\alpha) = \alpha_{i+1} + \theta(\alpha_i),\\
&\tilde{B}_i(x^d b_\alpha) = x^{d+(-1)^{\theta(\alpha_i)}} b_{\alpha+(-1)^{\theta(\alpha_i)}(\mathbf{e}_i-\mathbf{e}_{i+1})}.
\end{aligned}$$

- When $a_{i,\tau(i)} = 0$.

$$\beta_i(x^d b_\alpha) = \begin{cases} \alpha_i - \alpha_{\tau(i)} + \alpha_{\tau(i)+1} & \text{if } \alpha_i > \alpha_{\tau(i)},\\ \alpha_{\tau(i)+1} & \text{if } \alpha_i \le \alpha_{\tau(i)},\end{cases}$$

$$\tilde{B}_i(x^d b_\alpha) = \begin{cases} x^{d-\delta_{i,0}} b_{\alpha-\mathbf{e}_i+\mathbf{e}_{i+1}} & \text{if } \alpha_i > \alpha_{\tau(i)},\\ x^{d+\delta_{\tau(i),0}} b_{\alpha+\mathbf{e}_{\tau(i)}-\mathbf{e}_{\tau(i)+1}} & \text{if } \alpha_i \le \alpha_{\tau(i)}.\end{cases}$$

- When $a_{i,\tau(i)} = -1$.

$$\beta_i(x^d b_\alpha) = \begin{cases} \alpha_i - \alpha_{\tau(i)} + \alpha_{\tau(i)+1} - s_i, & \text{if } \alpha_i > \alpha_{\tau(i)} + s_i\\ \alpha_{\tau(i)+1} & \text{if } \alpha_i \le \alpha_{\tau(i)} + s_i,\end{cases},$$

$$\tilde{B}_{n'}(x^d b_\alpha) = \begin{cases} \frac{1}{\sqrt{2}} x^d b_{\alpha-\mathbf{e}_{n'}+\mathbf{e}_{n'+1}} & \text{if } \alpha_{n'} = \alpha_{n'+1} + s_{n'} + 1,\\ x^d b_{\alpha-\mathbf{e}_{n'}+\mathbf{e}_{n'+1}} & \text{if } \alpha_{n'} > \alpha_{n'+1} + s_{n'} + 1,\\ \frac{1}{\sqrt{2}} x^d b_{\alpha+\mathbf{e}_{n'+1}-\mathbf{e}_{n'+2}} & \text{if } \alpha_{n'} = \alpha_{n'+1} + s_{n'},\\ x^d b_{\alpha+\mathbf{e}_{n'+1}-\mathbf{e}_{n'+2}} & \text{if } \alpha_{n'} < \alpha_{n'+1} + s_{n'}\end{cases},$$

$$\tilde{B}_{n'+1}(x^d b_\alpha) = \begin{cases} x^d b_{\alpha-\mathbf{e}_{n'+1}+\mathbf{e}_{n'+2}} & \text{if } \alpha_{n'+1} > \alpha_{n'} + s_{n'+1},\\ \frac{1}{\sqrt{2}} x^d (b_{\alpha+\mathbf{e}_{n'}-\mathbf{e}_{n'+1}} + b_{\alpha-\mathbf{e}_{n'+1}+\mathbf{e}_{n'+2}}) & \text{if } \alpha_{n'+1} = \alpha_{n'} + s_{n'+1} > (-s_{n'})_+,\\ x^d b_{\alpha+\mathbf{e}_{n'}-\mathbf{e}_{n'+1}} & \text{otherwise.}\end{cases}$$

Also, the crystal and $\imath$crystal structure of $\mathcal{B}_s^\vee(x^{-1}) = \{x^d b_\alpha^\vee \mid b_\alpha \in \mathcal{B}_s\}$ is as follows (below, we will not consider $\mathcal{B}_s^\vee(x^{-1})$ when we have $a_{i,\tau(i)} = -1$ for some i):

$$\begin{aligned}
&\langle h_i, \mathrm{wt}(x^d b_\alpha^\vee)\rangle = -\alpha_i + \alpha_{i+1},\\
&\varepsilon_i(x^d b_\alpha^\vee) = \alpha_i,\\
&\varphi_i(x^d b_\alpha^\vee) = \alpha_{i+1},\\
&\tilde{E}_i(x^d b_\alpha^\vee) = x^{d-\delta_{i,0}} b^\vee_{\alpha-\mathbf{e}_i+\mathbf{e}_{i+1}},\\
&\tilde{F}_i(x^d b_\alpha^\vee) = x^{d+\delta_{i,0}} b^\vee_{\alpha+\mathbf{e}_i-\mathbf{e}_{i+1}}.
\end{aligned}$$

- When $a_{i,\tau(i)} = 2$.

$$\beta_i(x^d b^\vee_\alpha) = \alpha_i + \theta(\alpha_{i+1}),$$
$$\tilde{B}_i(x^d b^\vee_\alpha) = x^{d-(-1)^{\theta(\alpha_{i+1})}} b^\vee_{\alpha-(-1)^{\theta(\alpha_{i+1})}(\mathbf{e}_i-\mathbf{e}_{i+1})}.$$

- When $a_{i,\tau(i)} = 0$.

$$\beta_i(x^d b^\vee_\alpha) = \begin{cases} \alpha_{i+1} + \alpha_{\tau(i)} - \alpha_{\tau(i)+1} & \text{if } \alpha_{i+1} > \alpha_{\tau(i)+1}, \\ \alpha_{\tau(i)} & \text{if } \alpha_{i+1} \leq \alpha_{\tau(i)+1}, \end{cases}$$
$$\tilde{B}_i(x^d b^\vee_\alpha) = \begin{cases} x^{d+\delta_{i,0}} b^\vee_{\alpha+\mathbf{e}_i-\mathbf{e}_{i+1}} & \text{if } \alpha_{i+1} > \alpha_{\tau(i)+1}, \\ x^{d-\delta_{\tau(i),0}} b^\vee_{\alpha-\mathbf{e}_{\tau(i)}+\mathbf{e}_{\tau(i)+1}} & \text{if } \alpha_{i+1} \leq \alpha_{\tau(i)+1}. \end{cases}$$

5.2. **Type A.1.** We consider the $\imath$quantum group $\mathbf{U}^\imath$ of type A.1. In this case, τ is the identity. We take parameters ς_i, κ_i in (4) as $\varsigma_i = q^{-1}, \kappa_i = 0$ for any i. The $\imath$crystal graph of $\mathcal{B}$ is given as follows:

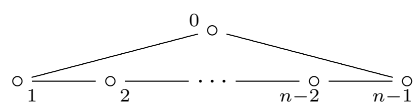
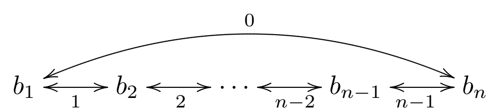


Figure 2. Satake diagram of type A.1.



Here $b_i = v_i + q^{-1}\mathcal{L}$. This type is twisted (10), and the quantum K-matrix for $\mathcal{W}$ is given by

$$K_{\mathcal{W}}(x) = \sum_j E_{n+1-j,j},$$

which is a $\mathbf{U}^\imath$-linear map from $\mathcal{W}(x)$ to $\mathcal{W}^\vee(x^{-1})$.

Let $s \geq 1$, and consider KR crystals $\mathcal{B}_s(x) = \{x^d b_\alpha\}$ and $\mathcal{B}^\vee_s(x^{-1}) = \{x^d b^\vee_\alpha\}$. We shall define a map

$$\mathcal{K} : \mathcal{B}_s(x) \to \mathcal{B}^\vee_s(x^{-1}), \quad x^d b_\alpha \mapsto x^{d+I(\alpha)} b^\vee_{\alpha'}, \tag{16}$$

where

$$I(\alpha) = 2\left\lfloor \frac{\alpha_1}{2} \right\rfloor = \alpha_1 - \theta(\alpha_1),$$
$$\alpha' = (\alpha'_1, \dots, \alpha'_n), \quad \alpha'_i = \alpha_{i+1} + \theta(\alpha_i) - \theta(\alpha_{i+1}).$$

Below, we shall show that the map $\mathcal{K}$ is the combinatorial K-matrix. First of all, let us observe that $\mathcal{K}$ is a bijection. Actually, the assignment

$$x^d b^\vee_\alpha \mapsto x^{d+J(\alpha)} b_{\alpha''},$$

where

$$J(\alpha) = -\alpha_n + \theta(\alpha_n),$$
$$\alpha''_i = \alpha_{i-1} + \theta(\alpha_i) - \theta(\alpha_{i-1}),$$

gives rise to the inverse map.

Next, let us fix a normalized K-matrix as follows. It is easy to see that

$$\{\alpha \mid \beta_1(b_\alpha) = s\} = \{\mathbf{e}_1 + (s-1)\mathbf{e}_2,\ s\mathbf{e}_2\},$$

and

$$\tilde{B}_1 b_{\mathbf{e}_1+(s-1)\mathbf{e}_2} = b_{s\mathbf{e}_2},\ \tilde{B}_1 b_{s\mathbf{e}_2} = b_{\mathbf{e}_1+(s-1)\mathbf{e}_2}.$$

Then, by [22, Section 4.1], there exists a unique $u \in \mathcal{L}_s$ such that

$$u + q^{-1}\mathcal{L}_s = b_{\mathbf{e}_1+(s-1)\mathbf{e}_2} + b_{s\mathbf{e}_2}$$

and that

$$\{w \in \mathcal{W}_s \mid B_1 w = [s]_1 w\} = \mathbb{C}(q)u.$$

Similarly, there exists a unique $u^\vee \in \mathcal{L}_s^\vee$ such that

$$u^\vee + q^{-1}\mathcal{L}_s^\vee = b^\vee_{(s-1)\mathbf{e}_1+\mathbf{e}_2} + b^\vee_{s\mathbf{e}_1},$$

and that

$$\{w \in \mathcal{W}_s^\vee \mid B_1 w = [s]_1 w\} = \mathbb{C}(q)u^\vee.$$

The above observation enables us to normalize the quantum K-matrix $K_{\mathcal{W}_s}(x)$ as

$$K_{\mathcal{W}_s}(x)(u) = u^\vee.$$

Lemma 7. *The ıcrystal $\mathcal{B}_s(x)$ is connected. Consequently, we have $K_{\mathcal{W}_s}(x)(\mathcal{L}_s(x)) \subset \mathcal{L}_s^\vee(x^{-1})$.*

Proof. By [21, Theorems 3.3.6 and 4.3.1], each b_α is connected to $b_{2\lfloor \frac{\alpha_1}{2} \rfloor \mathbf{e}_1 + (s-2\lfloor \frac{\alpha_1}{2} \rfloor)\mathbf{e}_2}$. Also, when $\alpha_1 > 2$, we have

$$\tilde{B}_1 \tilde{B}_0 b_{2\lfloor \frac{\alpha_1}{2} \rfloor \mathbf{e}_1 + (s-2\lfloor \frac{\alpha_1}{2} \rfloor)\mathbf{e}_2} = b_{(2\lfloor \frac{\alpha_1}{2} \rfloor - 2)\mathbf{e}_1 + (s-2\lfloor \frac{\alpha_1}{2} \rfloor + 1)\mathbf{e}_2 + \mathbf{e}_n}.$$

Since the right-hand side is connected to $b_{(2\lfloor \frac{\alpha_1}{2} \rfloor - 2)\mathbf{e}_1 + (s-2\lfloor \frac{\alpha_1}{2} \rfloor + 2)\mathbf{e}_2}$, so is α. Therefore, α is connected to $b_{s\mathbf{e}_2}$. This proves the assertion. □

By Lemma 7, the quantum K-matrix $K_{\mathcal{W}_s}(x)$ induces a $\mathbb{C}$-linear map

$$\bar{K}_{\mathcal{W}_s}(x) : \mathcal{L}_s(x)/q^{-1}\mathcal{L}_s(x) \to \mathcal{L}_s^\vee(x^{-1})/q^{-1}\mathcal{L}_s^\vee(x^{-1}).$$

As before, we set $\mathcal{K}_{\mathcal{B}_s}(x) := \bar{K}_{\mathcal{W}_s}(x)|_{\mathcal{B}_s(x)}$.

Lemma 8. *Let $\alpha = \mathbf{e}_1 + (s-1)\mathbf{e}_2$. Then, we have*

$$\mathcal{K}_{\mathcal{B}_s}(x)(b_\alpha) = \mathcal{K}(b_\alpha)$$

Proof. Since

$$\tilde{B}_0^2 b_{\mathbf{e}_1+(s-1)\mathbf{e}_2} = b_{\mathbf{e}_1+(s-1)\mathbf{e}_2},\ \tilde{B}_0 b_{s\mathbf{e}_2} = 0,$$

we see that

$$\tilde{B}_0^2 u + q^{-1}\mathcal{L}_s = b_{\mathbf{e}_1+(s-1)\mathbf{e}_2} = b_\alpha.$$

Similarly, we have

$$\begin{aligned} \tilde{B}_0^2 u^\vee + q^{-1}\mathcal{L}_s^\vee &= \begin{cases} b^\vee_{s\mathbf{e}_1} & \text{if } s \text{ is odd,} \\ b^\vee_{(s-1)\mathbf{e}_1+\mathbf{e}_2} & \text{if } s \text{ is even.} \end{cases} \\ &= \mathcal{K}(b_\alpha) \end{aligned}$$

Since we normalized the quantum K-matrix as $K_{\mathcal{W}_s}(x)(u) = u^\vee$, we obtain

$$\begin{aligned} \mathcal{K}_{\mathcal{B}_s}(x)(b_\alpha) &= \mathcal{K}_{\mathcal{B}_s}(x)(\tilde{B}_0^2(u + q^{-1}\mathcal{L}_s(x))) \\ &= \tilde{B}_0^2(u^\vee + q^{-1}\mathcal{L}_s^\vee(x^{-1})) \\ &= \mathcal{K}(b_\alpha), \end{aligned}$$

as desired. This proves the assertion. □

Lemma 9. *Let $x^d b_\alpha \in \mathcal{B}_s(x)$, and write $\mathcal{K}(x^d b_\alpha) = x^{d'} b^\vee_{\alpha'}$. Then, for each $i \in I$, we have*

$$\theta(\alpha'_i) = \theta(\alpha_i).$$

Proof. Noting that $n - \theta(n)$ is even for all $n \in \mathbb{Z}$, we see that the parity of $\alpha'_i = \alpha_{i+1} + \theta(\alpha_i) - \theta(\alpha_{i+1})$ coincides with that of $\theta(\alpha_i)$. Hence, the assertion follows. □

Proposition 10. *The map $\mathcal{K}$ given in (16) is the combinatorial K-matrix.*

Proof. We only need to show that

$$\mathcal{K}(x^d b_\alpha) = \mathcal{K}_{\mathcal{B}_s}(x)(x^d b_\alpha)$$

for all $x^d b_\alpha \in \mathcal{B}_s(x)$. To this end, by Lemmas 7 and 8, it suffices to prove that the map $\mathcal{K}$ commutes with $\tilde{B}_i$ for all $i \in I$. Below, we often use Lemma 9 without stating explicitly at each time.

Let $i \in I$, $x^d b_\alpha \in \mathcal{B}_s(x)$, and write $\mathcal{K}(x^d b_\alpha) = x^{d'} b^\vee_{\alpha'}$. We see that

$$\tilde{B}_i(x^{d'} b^\vee_{\alpha'}) = x^{d' - (-1)^{\theta(\alpha_{i+1})}\delta_{i,0}} b^\vee_{\alpha' - (-1)^{\theta(\alpha_{i+1})}(\mathbf{e}_i - \mathbf{e}_{i+1})}. \tag{17}$$

Setting $\alpha'' := \alpha' - (-1)^{\theta(\alpha_{i+1})}(\mathbf{e}_i - \mathbf{e}_{i+1})$, we have for each $j \in I$,

$$\begin{aligned} \alpha''_j &= \begin{cases} \alpha'_i - (-1)^{\theta(\alpha_{i+1})} & \text{if } j = i, \\ \alpha'_{i+1} + (-1)^{\theta(\alpha_{i+1})} & \text{if } j = i+1, \\ \alpha'_j & \text{otherwise,} \end{cases} \\ &= \begin{cases} \alpha_{i+1} + \theta(\alpha_i) - \theta(\alpha_{i+1}) - (-1)^{\theta(\alpha_{i+1})} & \text{if } j = i, \\ \alpha_{i+2} + \theta(\alpha_{i+1}) - \theta(\alpha_{i+2}) + (-1)^{\theta(\alpha_{i+1})} & \text{if } j = i+1, \\ \alpha_{j+1} + \theta(\alpha_j) - \theta(\alpha_{j+1}) & \text{otherwise,} \end{cases} \\ &= \begin{cases} \alpha_{i+1} + \theta(\alpha_i) + \theta(\alpha_{i+1}) - 1 & \text{if } j = i, \\ \alpha_{i+2} - \theta(\alpha_{i+1}) - \theta(\alpha_{i+2}) + 1 & \text{if } j = i+1, \\ \alpha_{j+1} + \theta(\alpha_j) - \theta(\alpha_{j+1}) & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand, we have

$$\tilde{B}_i(x^d b_\alpha) = x^{d + (-1)^{\theta(\alpha_i)}\delta_{i,0}} b_{\alpha + (-1)^{\theta(\alpha_i)}(\mathbf{e}_i - \mathbf{e}_{i+1})}. \tag{18}$$

Let us write

$$\mathcal{K}(\tilde{B}_i(x^d b_\alpha)) = x^{d'''} b^\vee_{\alpha'''}.$$

Then, we have

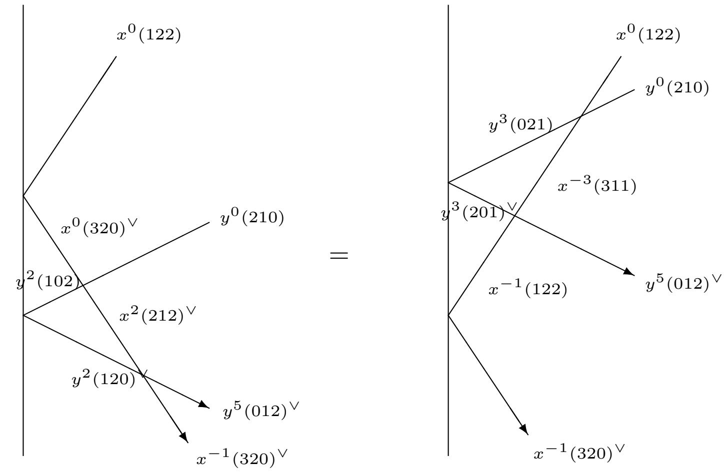
$$d''' = d + (-1)^{\theta(\alpha_i)}\delta_{i,0} + I(\alpha + (-1)^{\theta(\alpha_i)}(\mathbf{e}_i - \mathbf{e}_{i+1})),$$

and for each $j \in I$,

$$\alpha'''_j = \begin{cases} \alpha_i + (-1)^{\theta(\alpha_i)} + \theta(\alpha_{i-1}) - \theta(\alpha_i + (-1)^{\theta(\alpha_i)}) & \text{if } j = i-1, \\ \alpha_{i+1} - (-1)^{\theta(\alpha_i)} + \theta(\alpha_i + (-1)^{\theta(\alpha_i)}) - \theta(\alpha_{i+1} - (-1)^{\theta(\alpha_i)}) & \text{if } j = i, \\ \alpha_{i+2} + \theta(\alpha_{i+1} - (-1)^{\theta(\alpha_i)}) - \theta(\alpha_{i+2}) & \text{if } j = i+1, \\ \alpha_{j+1} + \theta(\alpha_j) - \theta(\alpha_{j+1}) & \text{otherwise.} \end{cases}$$

Now, it is an easy exercise to verify that $d''' = d' - (-1)^{\theta(\alpha_{i+1})}\delta_{i,0}$ and that $\alpha'''_j = \alpha''_j$ for all $j \in I$. Thus, we see that $\mathcal{K}$ commutes with $\tilde{B}_i$, and hence, the proof completes. □

Example 11. Set $n = 3$. Below is the graphical presentation when we apply the both hand sides of the combinatorial reflection equation (15) on $x^0 b_{(1,2,2)} \otimes y^0 b_{(2,1,0)} \in \mathcal{B}_5(x) \otimes \mathcal{B}_3(y)$.



5.3. **Type A.3.** We consider the $\imath$quantum group $\mathbf{U}^\imath$ of type A.3 in the quasi-split case. In this case, τ is given by $\tau(i) = -i \pmod{n}$. We set $n' = \lfloor \frac{n-1}{2} \rfloor$. We take parameters ς_i, κ_i in (4) as $\varsigma_0 = q^{-1}, \varsigma_{n'+1} = q^{-1}$ if n is even, $\varsigma_{n'+1} = q$ if n is odd, $\varsigma_i = 1$ otherwise, and $\kappa_i = 0$ for any i. The $\imath$crystal graph of $\mathcal{B}$ when n is even is given as follows:

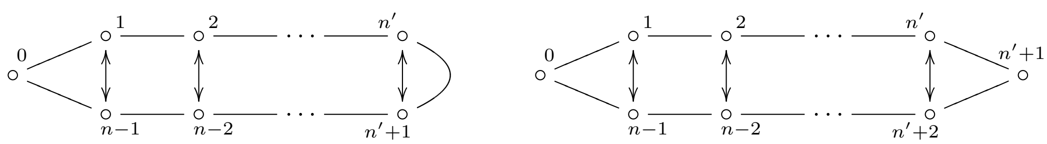
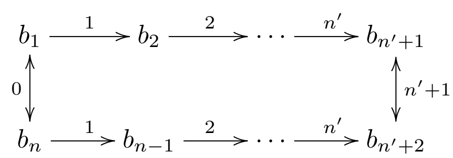


FIGURE 3. Satake diagrams of type A.3 when n is odd (left) and even (right).



Here $b_i = v_i + q^{-1}\mathcal{L}$. This type is untwisted (9), and the quantum K-matrix for $\mathcal{W}$ is given by

$$K_{\mathcal{W}}(x) = \sum_j E_{1-j,j},$$

which is a $\mathbf{U}^\imath$-linear map from $\mathcal{W}(x)$ to $\mathcal{W}(x^{-1})$.

Note that $\mathcal{B}_s = \{b_\alpha \,|\, \alpha = (\alpha_i)_{1\le i\le n}, \alpha_i \ge 0, \sum_i \alpha_i = s\}$. To describe the combinatorial K-matrix $\mathcal{K}_{\mathcal{B}_s}(x)$, we need to divide the set $\{\alpha = (\alpha_i)_{1\le i\le n} \mid b_\alpha \in \mathcal{B}_s\}$ into n subsets as follows. If (α_i) satisfies

$$\alpha_1 + \alpha_2 + \cdots + \alpha_i < \alpha_{n-i} + \cdots + \alpha_{n-1}, \alpha_2 + \cdots + \alpha_i < \alpha_{n-i} + \cdots + \alpha_{n-2}, \cdots, \alpha_i < \alpha_{n-i},$$
$$\text{and } \alpha_{i+1} \ge \alpha_{n-i-1}, \alpha_{i+1} + \alpha_{i+2} \ge \alpha_{n-i-2} + \alpha_{n-i-1}, \cdots, \alpha_{i+1} + \cdots + \alpha_{n'} \ge \alpha_{n-n'} + \cdots + \alpha_{n-i-1},$$
$$\text{and } \alpha_1 + \cdots + \alpha_i + \alpha_{n-i} + \cdots + \alpha_n : \text{odd}$$

for some i such that $0 \le i \le n'$, then say Case $(2i+1)$ holds. If (α_i) satisfies

$$\alpha_1 + \alpha_2 + \cdots + \alpha_i \le \alpha_{n-i} + \cdots + \alpha_{n-1}, \alpha_2 + \cdots + \alpha_i \le \alpha_{n-i} + \cdots + \alpha_{n-2}, \cdots, \alpha_i \le \alpha_{n-i},$$
$$\text{and } \alpha_{i+1} > \alpha_{n-i-1}, \alpha_{i+1} + \alpha_{i+2} > \alpha_{n-i-2} + \alpha_{n-i-1}, \cdots, \alpha_{i+1} + \cdots + \alpha_{n'} > \alpha_{n-n'} + \cdots + \alpha_{n-i-1},$$
$$\text{and } \alpha_1 + \cdots + \alpha_i + \alpha_{n-i} + \cdots + \alpha_n : \text{even}$$

for some i such that $0 \le i \le n'$, then say Case $(2i+2)$ holds. Note that when both n and s are odd, then Case $(n+1)$ does not occur. We use this case division only when s is odd.

Remark 12. We list the case divisions when n is small and s is odd.

$n = 3$

$$\begin{array}{lll} \text{Case (1)} & \alpha_1 \geq \alpha_2, & \alpha_3 \text{ is odd,} \\ \text{Case (2)} & \alpha_1 > \alpha_2, & \alpha_3 \text{ is even,} \\ \text{Case (3)} & \alpha_1 < \alpha_2. & \end{array}$$

$n = 4$

$$\begin{array}{lll} \text{Case (1)} & \alpha_1 \geq \alpha_3, & \alpha_4 \text{ is odd,} \\ \text{Case (2)} & \alpha_1 > \alpha_3, & \alpha_4 \text{ is even,} \\ \text{Case (3)} & \alpha_1 < \alpha_3, & \alpha_1+\alpha_3+\alpha_4 \text{ is odd,} \\ \text{Case (4)} & \alpha_1 \leq \alpha_3, & \alpha_1+\alpha_3+\alpha_4 \text{ is even.} \end{array}$$

$n = 5$

$$\begin{array}{llll} \text{Case (1)} & \alpha_1 \geq \alpha_4, & \alpha_1+\alpha_2 \geq \alpha_3+\alpha_4,, & \alpha_5 \text{ is odd,} \\ \text{Case (2)} & \alpha_1 > \alpha_4, & \alpha_1+\alpha_2 > \alpha_3+\alpha_4, & \alpha_5 \text{ is even,} \\ \text{Case (3)} & \alpha_1 < \alpha_4, & \alpha_2 \geq \alpha_3, & \alpha_1+\alpha_4+\alpha_5 \text{ is odd,} \\ \text{Case (4)} & \alpha_1 \leq \alpha_4, & \alpha_2 > \alpha_3, & \alpha_1+\alpha_4+\alpha_5 \text{ is even,} \\ \text{Case (5)} & \alpha_1+\alpha_2 < \alpha_3+\alpha_4, & \alpha_2 < \alpha_3. & \end{array}$$

Lemma 13. *Suppose s is odd. Then, for any α such that $b_\alpha \in \mathcal{B}_s$, it belongs to one and only one Case (i) for some $1 \leq i \leq n$.*

Proof. For $n = 3, 4$ cases, it is readily checked.

We show one can reduce the proof of the n case to the $n-2$ case by taking the $n = 5$ case for example. Suppose $\alpha_1 > \alpha_4$. Then, under this condition, only Case (1),(2),(5) can occur. Note that in Case (5), the second inequality automatically satisfied if the first one holds. Thus, the fact that one and only one case of Case (1),(2),(5) occurs is deduced from the $n = 3$ case by considering the case of $(\alpha_1+\alpha_2, \alpha_3+\alpha_4, \alpha_5)$. Next, suppose $\alpha_1 < \alpha_4$. Then, under this condition, only Case (3),(4),(5) can occur. Note that in Case (5), the first inequality automatically satisfied if the second one holds. Thus, the fact that one and only one case of Case (3),(4),(5) occurs is deduced again from the $n = 3$ case by considering the case of $(\alpha_2, \alpha_3, \alpha_1+\alpha_4+\alpha_5)$. Finally, suppose $\alpha_1 = \alpha_4$. Then, under this condition, only Case (1),(4),(5) can occur, and the fact that one and only one case occurs is deduced again from the $n = 3$ case by considering the case of $(\alpha_2, \alpha_3, \alpha_5)$. □

Proposition 14. *The $\imath$crystal $\mathcal{B}_s$ is connected.*

This proposition follows from the next three lemmas.

Lemma 15. *Suppose $i \neq 0$.*

(1) *If $\tilde{B}_i b_\alpha = b_{\alpha'}$, then $\tilde{B}_{n-i} b_{\alpha'} = b_\alpha$.*
(2) *If $\tilde{B}_i b_\alpha = \frac{1}{\sqrt{2}}(b_{\alpha'} + b_{\alpha''})$, then $\tilde{B}_{i-1} b_{\alpha'} = \frac{1}{\sqrt{2}} b_\alpha$ and $\tilde{B}_{i-1} b_{\alpha''} = \frac{1}{\sqrt{2}} b_\alpha$. It happens only when n is even and $i = n'+1$.*
(3) *If $\tilde{B}_i b_\alpha = \frac{1}{\sqrt{2}} b_{\alpha'}$, then $\tilde{B}_{i+1} b_{\alpha'} = \frac{1}{\sqrt{2}}(b_\alpha + b_{\alpha''})$ with some other α''. It happens only when n is even and $i = n'$.*

Proof. The assertions follow from [22, Sections 4.2 and 4.3]. □

Lemma 16. *Any element $b_\alpha \in \mathcal{B}_s$ is generated by $\tilde{B}_i$ $(i \neq 0)$ from $b_{\alpha'}$ where $\alpha' = (\sum_{j=1}^{n-1} \alpha_j - \gamma_1)\mathbf{e}_1 + (\alpha_n + \gamma_1)\mathbf{e}_n$. γ_1 is defined inductively by*

$$\gamma_{n'} = (\alpha_{n-n'} - \alpha_{n'})_+, \quad \gamma_j = (\gamma_{j+1} + \alpha_{n-j} - \alpha_j)_+ \text{ for } j = n'-1, n'-2, \dots, 1.$$

Proof. We give the proof for $n = 5$ only. Note that $n' = 2, n-n' = 3$ in this case. Applying $\tilde{B}_3$, we have

$$\tilde{B}_3^{\alpha_3} b_\alpha \propto b_{\alpha'} + \cdots, \quad \alpha' = (\alpha_1, \alpha_2+\alpha_3-\gamma_2, 0, \alpha_4+\gamma_2, \alpha_5).$$

We used the symbol $\propto$ since the factor $\frac{1}{\sqrt{2}}$ may occur upon application. Other terms may also appear. Similarly, we continue the calculation as

$$\tilde{B}_4^{\alpha_2+\alpha_3-\gamma_2+\gamma_1} b_{\alpha'} \propto b_{\alpha''} + \cdots, \quad \alpha'' = (\alpha_1+\alpha_2+\alpha_3-\gamma_2, 0, 0, \alpha_4+\gamma_2-\gamma_1, \alpha_5+\gamma_1),$$
$$(\tilde{B}_3\tilde{B}_2)^{\alpha_4+\gamma_2-\gamma_1} b_{\alpha''} \propto b_{\alpha'''} + \cdots, \quad \alpha''' = (\alpha_1+\alpha_2+\alpha_3-\gamma_2, \alpha_4+\gamma_2-\gamma_1, 0, 0, \alpha_5+\gamma_1),$$
$$\tilde{B}_4^{\alpha_4+\gamma_2-\gamma_1} b_{\alpha'''} \propto b_{\alpha''''} + \cdots, \quad \alpha'''' = (\alpha_1+\alpha_2+\alpha_3+\alpha_4-\gamma_1, 0, 0, 0, \alpha_5+\gamma_1).$$

In view of Lemma 15, we find that b_α is generated from $b_{\alpha''''}$. □

Lemma 17. (1) *If a is even, $b_{(s-a)\mathbf{e}_1+a\mathbf{e}_n}$ and $b_{(s-a-1)\mathbf{e}_1+(a+1)\mathbf{e}_n}$ are connected by $\tilde{B}_0$.*

(2) *If $n>3$ and a is even, $b_{(s-a)\mathbf{e}_1+a\mathbf{e}_n}$ and $b_{(s-a-2)\mathbf{e}_1+(a+2)\mathbf{e}_n}$ are connected by applications of $\tilde{B}_i$'s.*

(3) *If $n=3$ and a is odd, $b_{(s-a-1)\mathbf{e}_1+(a+1)\mathbf{e}_3}$ is represented as a linear combination of vectors which are generated from $b_{(s-a)\mathbf{e}_1+a\mathbf{e}_3}$ by applying $\tilde{B}_i$'s.*

Proof. (1) is immediate. For (2), consider two vectors $b_{\mathbf{e}_1+(s-a-2)\mathbf{e}_2+\mathbf{e}_{n-1}+a\mathbf{e}_n}$ and $b_{(s-a-2)\mathbf{e}_2+\mathbf{e}_{n-1}+(a+1)\mathbf{e}_n}$ which are connected by $\tilde{B}_0$. On the other hand, by Lemma 16, the former can be connected with $b_{(s-a)\mathbf{e}_1+a\mathbf{e}_n}$ and the latter $b_{(s-a-2)\mathbf{e}_1+(a+2)\mathbf{e}_n}$. Note that cases (2),(3) of Lemma 15 do not occur during the applications.

To prove (3), set $c=s-a$ and divide the cases whether c is even or odd. Suppose c is even. Then we have

$$\tilde{B}_2^2\tilde{B}_0\tilde{B}_1^{c/2+1}b_{c\mathbf{e}_1+a\mathbf{e}_3}=\frac{1}{\sqrt{2}}(b_{c/2\mathbf{e}_1+(c/2-1)\mathbf{e}_2+(a+1)\mathbf{e}_3}+b_{(c/2-1)\mathbf{e}_1+(c/2-1)\mathbf{e}_2+(a+2)\mathbf{e}_3}).$$

Applying $\tilde{B}_0$, we obtain a linear combination of the same two vectors, but with different coefficients. Hence, one obtains $b_{c/2\mathbf{e}_1+(c/2-1)\mathbf{e}_2+(a+1)\mathbf{e}_3}$ as a linear combination of vectors which are generated from $b_{c\mathbf{e}_1+a\mathbf{e}_3}$ by applying $\tilde{B}_i$'s. Applying $\tilde{B}_2^{c/2-1}$, one obtains $b_{(c-1)\mathbf{e}_1+(a+1)\mathbf{e}_3}$. Next suppose c is odd. We have

$$\tilde{B}_1^{(c-1)/2}b_{c\mathbf{e}_1+a\mathbf{e}_3}=b_{(c+1)/2\mathbf{e}_1+(c-1)/2\mathbf{e}_2+a\mathbf{e}_3},$$
$$\tilde{B}_2\tilde{B}_1^{(c+1)/2}b_{c\mathbf{e}_1+a\mathbf{e}_3}=\frac{1}{\sqrt{2}}(b_{(c+1)/2\mathbf{e}_1+(c-1)/2\mathbf{e}_2+a\mathbf{e}_3}+b_{(c-1)/2\mathbf{e}_1+(c-1)/2\mathbf{e}_2+(a+1)\mathbf{e}_3}).$$

Hence, one obtains $b_{(c-1)/2\mathbf{e}_1+(c-1)/2\mathbf{e}_2+(a+1)\mathbf{e}_3}$ as a linear combination of vectors which are generated from $b_{c\mathbf{e}_1+a\mathbf{e}_3}$ by applying $\tilde{B}_i$'s. Applying $\tilde{B}_2^{(c-1)/2}$ further, one obtains $b_{(c-1)\mathbf{e}_1+(a+1)\mathbf{e}_3}$. □

Lemma 18. *Under a suitable normalization, we have*

$$\mathcal{K}_{\mathcal{B}_s}(x)(b_{s\mathbf{e}_n})=\begin{cases}b_{\mathbf{e}_1+(s-1)\mathbf{e}_n} & \text{if } s \text{ is odd,}\\ b_{s\mathbf{e}_n} & \text{if } s \text{ is even.}\end{cases}$$

Proof. Let us first consider the case when n is odd. By weight consideration, we can normalize the quantum K-matrix as

$$K_{\mathcal{W}_s}(x)(v_{s\mathbf{e}_{n'+1}})=v_{s\mathbf{e}_{n'+1}}.$$

Hence, we have

$$\mathcal{K}_{\mathcal{B}_s}(x)(b_{s\mathbf{e}_{n'+1}})=b_{s\mathbf{e}_{n'+1}}.$$

Applying $\sqrt{2}\tilde{B}_{n-1}^s\tilde{B}_{n-2}^s\cdots\tilde{B}_{n'+2}^s\tilde{B}_{n'+1}^s$ on the both sides, we obtain

$$\mathcal{K}_{\mathcal{B}_s}(x)(b_{\mathbf{e}_1+(s-1)\mathbf{e}_n}+b_{s\mathbf{e}_n})=b_{\mathbf{e}_1+(s-1)\mathbf{e}_n}+b_{s\mathbf{e}_n}. \tag{19}$$

When s is odd, applying $\tilde{B}_0$ on the both sides of (19), we have

$$\mathcal{K}_{\mathcal{B}_s}(x)(xb_{s\mathbf{e}_1}+x^{-1}b_{\mathbf{e}_1+(s-1)\mathbf{e}_n})=x^{-1}b_{s\mathbf{e}_n}+xb_{\mathbf{e}_1+(s-1)\mathbf{e}_n}.$$

Comparing this with (19), the assertion follows.

Similarly, when s is even, applying $\tilde{B}_0^2$ on the both sides of (19), we have

$$\mathcal{K}_{\mathcal{B}_s}(x)(b_{\mathbf{e}_1+(s-1)\mathbf{e}_n})=b_{\mathbf{e}_1+(s-1)\mathbf{e}_n}.$$

Comparing this with identity (19), the assertion follows.

Next, let us consider the case when n is even. As in type A.1, from the consideration of $\beta_{n'+1}$, we can normalize the quantum K-matrix as

$$\mathcal{K}_{\mathcal{B}_s}(b_{\mathbf{e}_{n'+1}+(s-1)\mathbf{e}_{n'+2}}+b_{s\mathbf{e}_{n'+2}})=b_{\mathbf{e}_{n'+1}+(s-1)\mathbf{e}_{n'+2}}+b_{s\mathbf{e}_{n'+2}}.$$

Applying $\tilde{B}_{n-1}^s\tilde{B}_{n-2}^s\cdots\tilde{B}_{n'+2}^s$ on both sides, we obtain

$$\mathcal{K}_{\mathcal{B}_s}(x)(b_{\mathbf{e}_1+(s-1)\mathbf{e}_n}+b_{s\mathbf{e}_n})=b_{\mathbf{e}_1+(s-1)\mathbf{e}_n}+b_{s\mathbf{e}_n}.$$

Now, as in the n being odd case, applying $\tilde{B}_0$ when s is odd or $\tilde{B}_0^2$ when s is even, we obtain the desired identity. □

Proposition 19. *The combinatorial K-matrix $\mathcal{K}_{\mathcal{B}_s}(x)$ is given by*

$$\mathcal{K}_{\mathcal{B}_s}(x)(x^d b_\alpha) = x^{d+I(\alpha)} \begin{cases} b_\alpha & (s \textit{ is even}) \\ b_{\alpha+(-1)^{i-1}(\mathbf{e}_{i'}-\mathbf{e}_{n-i'+1})} & (s \textit{ is odd}) \end{cases}, \tag{20}$$

where we assume for α Case (i) $(1 \le i \le n)$ holds and $i' = \lfloor \frac{i+1}{2} \rfloor$. The energy function $I(\alpha)$ is given by

$$I(\alpha) = -2\lfloor \frac{\alpha_n + \gamma_1 + \chi(s \textit{ is even})}{2} \rfloor,$$

where γ_1 was defined in Lemma 16.

Proof. In view of Proposition 14 and Lemma 18, it suffices to show that the right hand side of (20) commutes with $\tilde{B}_i$ for any i, which is to be proved next. □

Lemma 20. *The right hand side of* (20) *commutes with $\tilde{B}_i$ for any i.*

Proof. We let $\mathcal{K}(x)$ denote the right hand side of (20), and show $\mathcal{K}(x)$ comments with $\tilde{B}_i$ for any i. If s is even, $\mathcal{K}(1)$ is the identity. Hence, the commutativity is trivial except the case of $\tilde{B}_0$ where there is a change in the power of x. Since this case is close to when s is odd, we omit its proof.

We assume s is odd below. To reduce the number of cases to handle, we first note the following facts that can be checked easily from previous results.

(i) If n is odd and Case (n) holds for α, then $\mathcal{K}(1)(b_\alpha) = b_\alpha$.
(ii) Suppose $i < n'$ if n is even. If Case $(2i+1)$ (resp. $(2i+2)$) holds for α, then for $\mathcal{K}(1)(b_\alpha)$ Case $(2i+2)$ (resp. $(2i+1)$) holds.
(iii) If Case $(2i+1)$ or $(2i+2)$ holds for α, then $I(\alpha) = -2\lfloor \frac{(\alpha_n+\cdots+\alpha_{n-i})-(\alpha_i+\cdots+\alpha_1)}{2} \rfloor$.
(iv) $\tilde{B}_i (i \neq 0)$ does not change the energy function, i.e., $I(\tilde{B}_i\alpha) = I(\alpha)$.

From these facts, one verifies that $\mathcal{K}(x^{-1})\mathcal{K}(x) = \mathrm{id}$. Regarding them and Lemma 15(1), one notices that we can restrict the cases for the proof of the commutativity of $\tilde{B}_i$ and $\mathcal{K}(x)$ to the following ones.

(a) $\tilde{B}_0$ and Case (1) holds.
(b) $\tilde{B}_0$ and Case $(2i+1)$ holds for $0 < i \le n'$.
(c) $\tilde{B}_{i+1}$ and Case $(2i+1)$ holds for $0 \le i < n'$.
(d) $\tilde{B}_j$ and Case $(2i+1)$ holds for $j \neq i+1, 1 \le j \le n', 0 \le i \le n'$.
(e) $\tilde{B}_{n'+1}$ and Case $(2n'+1)$ holds for n even.
(f) $\tilde{B}_{n'+1}$ and Case $(2i+1)$ holds for $0 \le i < n'$ and n even.
(g) $\tilde{B}_{n'+1}$ and Case $(2n'-1)$ holds for n odd.
(h) $\tilde{B}_{n'+1}$ and Case $(2i+1)$ holds for $0 \le i < n'-1$ and n odd.

We prove only in the cases of (a),(c) and (g).

Let us show (a). Suppose α_n is odd. Then we have $\mathcal{K}(x)(b_\alpha) = x^{-\alpha_n+1} b_{\alpha'}, \tilde{B}_0 b_\alpha = x^{-1} b'_\alpha$, where $\alpha' = \alpha + \mathbf{e}_1 - \mathbf{e}_n$. Since Case (2) holds for α', $\mathcal{K}(x)(\tilde{B}_0 b_\alpha) = \tilde{B}_0 \mathcal{K}(x)(b_\alpha) = x^{-\alpha_n} b_\alpha$. Note that $\mathcal{K}(x)(b_\alpha)$ belongs to $\mathcal{B}_s(x^{-1})$. The case α_n is even is similar.

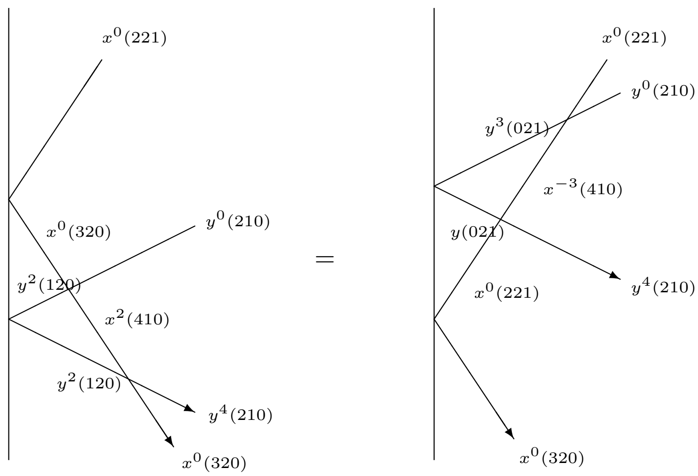
Next we show (c). Because of the fact (iv) above, one can ignore the dependence of x. We have $\mathcal{K}(1)(b_\alpha) = b_{\alpha+\mathbf{e}_{i+1}-\mathbf{e}_{n-i}}$, and

$$\tilde{B}_{i+1} b_\alpha = \begin{cases} b_{\alpha-\mathbf{e}_{i+1}+\mathbf{e}_{i+2}} & (\alpha_{i+1} > \alpha_{n-i-1}) \\ b_{\alpha+\mathbf{e}_{n-i-1}-\mathbf{e}_{n-i}} & (\alpha_{i+1} = \alpha_{n-i-1}). \end{cases}$$

For the former case, Case $(2i+1)$ holds, whereas for the latter, Case $(2i+3)$ holds. In either case, $\mathcal{K}(1)(\tilde{B}_{i+1} b_\alpha) = b_{\alpha+\mathbf{e}_{i+2}-\mathbf{e}_{n-i}}$, which agrees with $\tilde{B}_{i+1}\mathcal{K}(1)(b_\alpha)$.

Finally, we show (g). In this case, we have $\mathcal{K}(1)(b_\alpha) = b_{\alpha+\mathbf{e}_{n'}-\mathbf{e}_{n'+2}}$ and $\tilde{B}_{n'+1} b_\alpha = b_{\alpha+\mathbf{e}_{n'}-\mathbf{e}_{n'+1}}$ since Case $(2n'-1)$ for α implies $\alpha_{n'} \ge \alpha_{n'+1}$. Since Case $(2n'-1)$ holds also for $\tilde{B}_{n'+1} b_\alpha$, one finds $\tilde{B}_{n'+1}\mathcal{K}(1)(b_\alpha) = \mathcal{K}(1)(\tilde{B}_{n'+1} b_\alpha) = b_{\alpha+2\mathbf{e}_{n'}-\mathbf{e}_{n'+1}-\mathbf{e}_{n'+2}}$. □

Example 21. Set $n = 3$. Below is the graphical presentation when we apply the both hand sides of the combinatorial reflection equation (15) on $x^0 b_{(2,2,1)} \otimes y^0 b_{(2,1,0)} \in \mathcal{B}_5(x) \otimes \mathcal{B}_3(y)$.



5.4. **Type A.4.** We consider the $\imath$quantum group of type A.4. In this case, n should be even and we set $n' = n/2$. This type is twisted (10), and the quantum K-matrix for $\mathcal{W}$ is given by

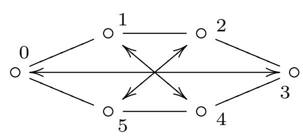


FIGURE 4. Satake diagram of type A.4 when $n = 6$.

$$K_{\mathcal{W}}(x) = \sum_j x^{\chi(j>n')} E_{j+n',j},$$

which is a $\mathbf{U}^\imath$-linear map from $\mathcal{W}(x)$ to $\mathcal{W}^\vee(x^{-1})$.

Proposition 22. *Define* $\mathcal{K} : \mathcal{B}_s(x) \to \mathcal{B}_s^\vee(x^{-1})$ *by* $\mathcal{K}(x^d b_\alpha) = x^{d+I(\alpha)} b^\vee_{\alpha'}$ *where*

$$I(\alpha) = \min(\alpha_1, \alpha_{n'+1}) - \sum_{j=n'+1}^{n} \alpha_j,$$
$$\alpha'_i = \alpha_{i+1} + \alpha_{i+n'+1} + \max(\alpha_i, \alpha_{i+n'}) - \alpha_i - \max(\alpha_{i+1}, \alpha_{i+n'+1}).$$

Then, it is the combinatorial K-matrix $\mathcal{K}_{\mathcal{B}_s}(x)$.

We list necessary lemmas to prove this proposition below.

Lemma 23. *We have*

$$\mathcal{K}(b_{s\mathbf{e}_1}) = \mathcal{K}_{\mathcal{B}_s}(x)(b_{s\mathbf{e}_1}) = b^\vee_{s\mathbf{e}_{n'+1}}.$$

Proof. The assertion follows from weight consideration and easy calculation. □

Lemma 24. $\mathcal{K}$ *commutes with* $\tilde{B}_i$ *for any* $i \in I$.

Proof. We first prove the commutativity when $x = 1$. Let $b_\alpha \in \mathcal{B}_s$. In the case of type A.4, the action of $\tilde{B}_i$ on $\mathcal{B}_s$ is given by the case of $a_{i,\tau(i)} = 0$. To describe it concretely, we assume $\varphi_i(b_\alpha) > \varphi_{i+n'}(b_\alpha)$. In this case, it satisfies $\tilde{B}_i b_\alpha = b_{\alpha - \mathbf{e}_i + \mathbf{e}_{i+1}}$. Then, the entries of $\mathcal{K}\tilde{B}_i b_\alpha$ related to the i-th and $(i+1)$-th ones of $\tilde{B}_i b_\alpha$ are described as follows:

$(i-1)$-th	$\alpha_{i+n'} - \alpha_{i-1} + \max(\alpha_{i-1}, \alpha_{i+n'-1})$
i-th	$\alpha_{i+1} + \alpha_{i+n'+1} - \max(\alpha_{i+1}+1, \alpha_{i+n'+1}) + 1$
$(i+1)$-th	$\alpha_{i+2} + \alpha_{i+n'+2} + \max(\alpha_{i+1}+1, \alpha_{i+n'+1}) - \alpha_{i+1} - \max(\alpha_{i+2}, \alpha_{i+n'+2}) - 1$
$(i+n'-1)$-th	$\alpha_{i+n'} - \alpha_{i+n'-1} + \max(\alpha_{i-1}, \alpha_{i+n'-1})$
$(i+n')$-th	$\alpha_i + \alpha_{i+1} - \alpha_{i+n'} + \alpha_{i+n'+1} - \max(\alpha_{i+1}+1, \alpha_{i+n'+1})$
$(i+n'+1)$-th	$\alpha_{i+2} + \alpha_{i+n'+2} + \max(\alpha_{i+1}+1, \alpha_{i+n'+1}) - \alpha_{i+n'+1} - \max(\alpha_{i+2}, \alpha_{i+n'+2})$

To see the action of $\tilde{B}_i$ on $\mathcal{K}(b_\alpha)$, we should compare $\varphi_i(\mathcal{K}(b_\alpha))$ and $\varphi_{i+n'}(\mathcal{K}(b_\alpha))$. Assume $\varphi_i(\mathcal{K}(b_\alpha)) > \varphi_{i+n'}(\mathcal{K}(b_\alpha))$. We can see immediately the condition is equivalent to $\alpha_{i+n'+1} > \alpha_{i+1}$. Then, it satisfies $\tilde{B}_i\mathcal{K}(b_\alpha) = \tilde{F}_i\mathcal{K}(b_\alpha)$. The entries of $\mathcal{K}(b_\alpha)$ except i-th and $(i+1)$-th ones are invariant by the action of $\tilde{B}_i$ and equivalent to the ones of $\mathcal{K}(\tilde{B}_i b_\alpha)$ in this case. The i-th entry is $\alpha_{i+1}+1$ and $(i+1)$-th one is $\alpha_{i+2} + \alpha_{i+n'+2} + \alpha_{i+n'+1} - \alpha_{i+1} - \alpha_{i+1} - \max(\alpha_{i+2}, \alpha_{i+n'+2}) - 1$, and they are equivalent to the i-th and $(i+1)$-th one of $\mathcal{K}(\tilde{B}_i b_\alpha)$ in the condition $\alpha_{i+n'+1} > \alpha_{i+1}$. Similarly, we can see the assertion on the other conditions.

To get the formula for $I(\alpha)$, we compare the dependence of x by applying $\tilde{B}_0$ on both sides of $\mathcal{K}(b_\alpha) = x^{I(\alpha)} b^\vee_{\alpha'}$. Noting that $\alpha'_i - \alpha'_{i+n'} = \alpha_{i+n'} - \alpha_i$, we obtain

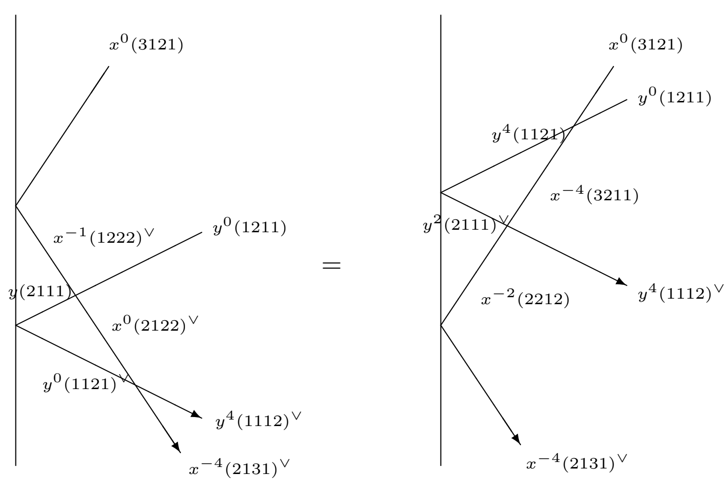
$$I(\tilde{F}_0\alpha) = \begin{cases} I(\alpha)+1 & (\alpha_n > \alpha_{n'}, \alpha_1 \geq \alpha_{n'+1}), \\ I(\alpha)+2 & (\alpha_n > \alpha_{n'}, \alpha_1 < \alpha_{n'+1}), \end{cases}$$
$$I(\tilde{E}_{n'}\alpha) = \begin{cases} I(\alpha) & (\alpha_n \leq \alpha_{n'}, \alpha_1 \geq \alpha_{n'+1}), \\ I(\alpha)+1 & (\alpha_n \leq \alpha_{n'}, \alpha_1 < \alpha_{n'+1}). \end{cases}$$

Here $\tilde{F}_0\alpha$ or $\tilde{E}_{n'}\alpha$ should be understood as an action on $\mathcal{B}_s(x=1)$. Noting the fact that $\tilde{B}_i$ with $i \neq 0, n'$ does not change the value of $I(\alpha)$, we obtain the desired formula. □

Lemma 25. *The ıcrystal $\mathcal{B}_s$ is connected.*

Proof. Let us introduce the notation $\tilde{B}_i^{\max}$ to mean $\tilde{B}_i^{\max} b = \tilde{B}_i^c b$ where $c = \max\{k \geq 0 \mid \tilde{B}_i^k b \neq 0\}$. Then, for any $b_\alpha \in \mathcal{B}_s$, we have $\tilde{B}_i^{\max} b_\alpha = b_{\alpha'}$ where $\alpha'_{i+n'+1} = 0$. Using this property of $\tilde{B}_i^{\max}$, apply to b_α $\tilde{B}^{\max}_{n'-1}, \tilde{B}^{\max}_{n'-2}, \dots, \tilde{B}^{\max}_1, \tilde{B}^{\max}_0, \tilde{B}^{\max}_{n-1}, \dots, \tilde{B}^{\max}_{n'+1}$, successively. Then, at each application, $\alpha_n, \alpha_{n-1}, \dots, \alpha_{n'+2}, \alpha_{n'+1}, \alpha_{n'}, \dots, \alpha_2$ turn out to be 0. Since $\tilde{B}_i\tilde{B}_{i+n'} = \mathrm{id}$, we are done. □

Example 26. Set $n = 4$. Below is the graphical presentation when we apply the both hand sides of the combinatorial reflection equation (15) on $x^0 b_{(3,1,2,1)} \otimes y^0 b_{(1,2,1,1)} \in \mathcal{B}_7(x) \otimes \mathcal{B}_5(y)$.



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Department of Mathematics, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan
Email address: m21sa009@st.osaka-cu.ac.jp

Osaka Central Advanced Mathematical Institute & Department of Mathematics, Osaka Metropolitan University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan
Email address: okado@omu.ac.jp

Osaka Central Advanced Mathematical Institute & Department of Mathematics, Osaka Metropolitan University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan
Email address: watanabehideya@gmail.com