$W^{1,p}$ APPROXIMATION OF THE MOSER–TRUDINGER INEQUALITY

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Abstract. We propose a power type approximation of the Moser–Trudinger functional and show that its concentration level converges to the Carleson–Chang limit.

1. Introduction

Let $N \geq 2$, $1 < p < N$, B be the unit ball in $\mathbb{R}^N$, and $W_0^{1,p}(B)$ be the completion with respect to the Sobolev norm $\|u\|_{W^{1,p}(B)} = (\|u\|_{L^p(B)}^p + \|\nabla u\|_{L^p(B)}^p)^{1/p}$ of smooth compactly supported functions in B. Then, the Sobolev inequality in B states that, there exists a constant $C > 0$ such that

$$\|u\|_{L^{p^*}(B)} \leq C\|\nabla u\|_{L^p(B)} \tag{1.1}$$

for every $u \in W_0^{1,p}(B)$, where $p^* = Np/(N-p)$ is the critical Sobolev exponent.

In the borderline situation where $p = N$, the inequality (1.1) is known to hold when p^* is replaced by any number greater than or equal to 1. However, a stronger result, proved by Trudinger [24] (see also [19, 26]) is available. This is

$$\sup_{\substack{u \in W_0^{1,N}(B) \\ \|\nabla u\|_{L^N(B)} \leq 1}} \int_B e^{\alpha |u|^{\frac{N}{N-1}}} dx \leq C|B| \tag{1.2}$$

and is true for some constants α and C, depending only on N. Moser [18] sharpened the inequality to

$$\sup_{\substack{u \in W_0^{1,N}(B) \\ \|\nabla u\|_{L^N(B)} \leq 1}} \int_B e^{\alpha |u|^{\frac{N}{N-1}}} dx \begin{cases} \leq C|B| & \text{if} \quad \alpha \leq \alpha_N, \\ = +\infty & \text{if} \quad \alpha > \alpha_N, \end{cases} \tag{1.3}$$

where $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and ω_{N-1} is the surface measure of the unit sphere in $\mathbb{R}^N$.

The Trudinger inequality (1.2) is considered to be a limiting case of the Sobolev inequality in the framework of Orlicz spaces. After the contribution of Adams [1], Cianchi [7] established the optimal extension of inequalities (1.1) and (1.2) to the case where Lebesgue norms are replaced by any Orlicz norm. This extension coincides with (1.1) for $W_0^{1,p}(B)$ and (1.2) for $W_0^{1,N}(B)$, respectively. Even though the Sobolev inequality (1.1) and its limiting case (1.2) were unified in [7], the latter is not obtained via a direct limiting procedure in the former as $p \to N$.

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In this paper, we focus on this discontinuity and propose an equivalent form of the L^{p^*} norm that converges to the Moser–Trudinger functional in (1.3) as $p \to N$. To state our result, we define a function $F_p : \mathbb{R} \to \mathbb{R}_+$ for $p \in (1, N)$ by

$$\begin{cases} F_p(s) := \left[1 + \dfrac{N-p}{N(p-1)}\alpha_p |s|^{\frac{p}{p-1}}\right]^{\frac{N(p-1)}{N-p}}, \\ \alpha_p := \left(\alpha_N^{\frac{N-1}{N}} |B|^{\frac{1}{p}-\frac{1}{N}}\right)^{\frac{p}{p-1}}. \end{cases} \tag{1.4}$$

Proposition 1.1. *Let u be a smooth compactly supported function in B. Then there holds*

$$c_1 \|u\|_{L^{p^*}(B)}^{p^*} \le \int_B [F_p(u) - 1]\, dx \le c_2 \|u\|_{L^{p^*}(B)}^{p^*}$$

for some $c_1, c_2 > 0$ depending on p, N, and $|B|$. Furthermore, there holds

$$\int_B F_p(u) \to \int_B e^{\alpha_N |u|^{\frac{N}{N-1}}}\, dx \quad (p \to N).$$

Proposition 1.1 clearly follows from $\lim_{s\to\infty} F_p(s)/s^{p^*} = c$ for some constant $c > 0$. Furthermore, as we will see, the proposed function F_p yields new insight into the concentration level of the Moser–Trudinger functional in (1.3).

The concentration level of the Moser–Trudinger functional (1.3) for $\alpha = \alpha_N$ was first investigated by Carleson–Chang [6]. Let B_ε be the ball centered at the origin with radius $\varepsilon > 0$ and

$$G(u) := \int_B e^{\alpha_N |u|^{\frac{N}{N-1}}}\, dx.$$

It is revealed in [6] that

$$\limsup_{n\to\infty} G(u_n) \le |B|(1 + e^{\sum_{k=1}^{N-1} \frac{1}{k}}) \tag{1.5}$$

if $\{u_n\}$ is a concentrating sequence, that is $\lim_{n\to\infty} \|\nabla u_n\|_{L^N(B_\varepsilon)} = 1$ holds for every $\varepsilon > 0$. The maximal limit in (1.5) on concentrating sequences is called the Carleson–Chang limit. Later, de Figueiredo–do Ó–Ruf [9] constructed a concentrating sequence $\{y_n\}$ such that $\lim_{n\to\infty} G(y_n) = |B|(1 + e^{\sum_{k=1}^{N-1}\frac{1}{k}})$, which means the value of the Carleson–Chang limit is the right hand side of (1.5). It should also be mentioned that Carleson–Chang [6] considered (1.5) to prove the existence of a function which attains the supremum in (1.3) for $\alpha = \alpha_N$. Specifically, they described a function u^* such that $G(u^*) > |B|(1 + e^{\sum_{k=1}^{N-1}\frac{1}{k}})$, and combined this fact, the concentration compactness argument, and (1.5) to show that all maximizing sequences of the supremum in (1.3) with $\alpha = \alpha_N$ are precompact. This method has been extended to more general cases, as shown in the works of Struwe [22], Flucher [10] and Li [15]. See also [20, 14, 12, 17, 11] and references therein for other discussion of maximizing problems related to the Moser–Trudinger functional.

We define $\mathscr{B}_p$ and X_p by

$$\mathscr{B}_p := \left\{ u \in W_{0,rad}^{1,p}(B) \;\middle|\; \|\nabla u\|_p \le 1 \right\}, \tag{1.6}$$

where $W_{0,rad}^{1,p}(B)$ denotes the set of radially symmetric functions belonging to $W_0^{1,p}(B)$, and

$$X_p := \left\{ \{u_n\} \subset \mathscr{B}_p \;\middle|\; \lim_{n\to\infty} \|\nabla u_n\|_{L^p(B_\varepsilon)} = 1 \text{ for any } \varepsilon > 0 \right\}. \tag{1.7}$$

The main result of the presented paper shows that the concentration level associated with F_p converges to the Carleson–Chang limit.

Theorem 1.2. *It holds that*

$$\sup_{\{u_n\}\in X_p}\left(\limsup_{n\to\infty}\int_B F_p(u_n)dx\right)\to |B|(1+e^{\sum_{k=1}^{N-1}\frac{1}{k}})\quad (p\to N).$$

Remark 1.3. It is important to notice the derivation of F_p. In the proof of (1.5), the following inequality, termed Alvino's inequality or the radial lemma, plays an essential role:

$$|u(x)|\le \alpha_N^{-\frac{N-1}{N}}\left(N\log\frac{1}{|x|}\right)^{\frac{N-1}{N}}\|\nabla u\|_{L^N(B)} \tag{1.8}$$

for every $x\in B$ and $u\in W^{1,N}_{0,rad}(B)$. It is easy to check that (1.8) is equivalent to

$$e^{\alpha_N\left(|u(x)|/\|\nabla u\|_{L^N(B)}\right)^{\frac{N}{N-1}}}\le\frac{1}{|x|^N}.$$

Similarly, for the case of $1<p<N$, it holds that

$$|u(x)|\le\left(|B|^{\frac{1}{p}-\frac{1}{N}}\alpha_N^{\frac{N-1}{N}}\right)^{-1}\left\{N\frac{p-1}{N-p}\left(|x|^{-\frac{N-p}{p-1}}-1\right)\right\}^{\frac{p-1}{p}}\|\nabla u\|_{L^p(B)} \tag{1.9}$$

for every $x\in B$ and $u\in W^{1,p}_{0,rad}(B)$. The function F_p is defined so that (1.9) is equivalent to

$$F_p\big(u(x)/\|\nabla u\|_{L^p(B)}\big)\le\frac{1}{|x|^N}.$$

We prove (1.9) at the end of Section 2 for the convenience of readers.

Remark 1.4. Theorem 1.2 can be rewritten by using q-exponential function and q-logarithmic function, which are defined by

$$\exp_q(r):=[1+(1-q)r]^{\frac{1}{1-q}},\quad \ln_q r:=\frac{r^{1-q}-1}{1-q},$$

for $q>0$, $q\neq 1$, and $r>0$. It is easy to verify that

$$\lim_{q\to 1}\ln_q r=\log r \text{ and } \lim_{q\to 1}\exp_q r=e^r \tag{1.10}$$

for every $r>0$ and

$$\exp_q(\ln_q r)=\ln_q(\exp_q r)=r \tag{1.11}$$

for every $q>0$, $q\neq 1$, and $r>0$. These modified functions were originally introduced by Tsallis [25] to study nonextensive statistics. Then, it clearly holds that

$$F_p(u)=\exp_{\frac{Np+p-2N}{N(p-1)}}\left(\alpha_p|u|^{\frac{p}{p-1}}\right). \tag{1.12}$$

These relations in (1.10)–(1.12) make the proof clearer.

2. Proofs

In this section we prove Theorem 1.2. The best possible constant in (1.1) obtained by Aubin [4] and Talenti [23] is

(2.1)
$$S_p := \inf_{u\in W_0^{1,p}(B)} \frac{\|\nabla u\|_{L^p(B)}}{\|u\|_{L^{p^*}(B)}} = \sqrt{\pi}N^{\frac{1}{p}}\left(\frac{N-p}{p-1}\right)^{\frac{p-1}{p}}\left[\frac{\Gamma\left(\frac{N}{p}\right)\Gamma\left(N+1-\frac{N}{p}\right)}{\Gamma(N)\Gamma\left(1+\frac{N}{2}\right)}\right]^{\frac{1}{N}},$$

This plays a crucial role. First we define $\mathscr{B}_p$, X_p by (1.6), (1.7), and then we set a constant M_p by

$$M_p := \sup_{\{u_n\}\in X_p}\left(\limsup_{n\to\infty}\int_B F_p(u_n)dx\right).$$

With this setting, we divide the rest of the proof into two steps.

Proposition 2.1. *If $p > 2N/(N+1)$ then*

$$M_p = |B| + \left[\frac{N-p}{N(p-1)}\alpha_p\right]^{\frac{N(p-1)}{N-p}} S_p^{-p^*}.$$

Proposition 2.2. *It holds that*

$$\lim_{p\to N} M_p = |B|(1+e^{\sum_{k=1}^{N-1}\frac{1}{k}}).$$

Theorem 1.2 follows from Propositions 2.1 and 2.2.

Proof of Proposition 2.1. For any positive constants a, b and $\gamma > 1$, it holds that

$$a^\gamma + b^\gamma \le (a+b)^\gamma \le a^\gamma + b^\gamma + \gamma 2^{\gamma-1}\left(ab^{\gamma-1}+a^{\gamma-1}b\right).$$

Thus, for $p > 2N/(N+1)$ we have that

$$1+\left[\frac{N-p}{N(p-1)}\alpha_p\right]^{\frac{N(p-1)}{N-p}}|s|^{p^*} \le F_p(s) \tag{2.2}$$

and

$$F_p(s) \le 1+\left[\frac{N-p}{N(p-1)}\alpha_p\right]^{\frac{N(p-1)}{N-p}}|s|^{p^*} + H(s), \tag{2.3}$$

where $H(s) = C_1|s|^{\frac{p}{p-1}} + C_2|s|^{p^*-\frac{p}{p-1}}$ with positive constants C_1, C_2.

We take any sequence $\{u_n\}\in X_p$. We prove that

$$\int_B H(u_n)dx = o(1) \tag{2.4}$$

as $n\to\infty$. In order to prove this, we set

$$\tau_1 := \int_B |u_n|^{\frac{p}{p-1}}dx, \quad \tau_2 := \int_B |u_n|^{p^*-\frac{p}{p-1}}dx \quad \text{and} \quad \kappa_{n,\varepsilon} = u_n|_{\partial B_\varepsilon}.$$

Recall that the embedding

$$W^{1,p}_{rad}(B\setminus\overline{B}_\varepsilon) \hookrightarrow C^0\left(\overline{B\setminus B_\varepsilon}\right) \tag{2.5}$$

holds for every $\varepsilon > 0$, and for every $q\in[1,p^*]$ there is a constant S_q such that

$$S_q\|u\|_{L^q(B\setminus\overline{B}_\varepsilon)} \le \|\nabla u\|_{L^p(B\setminus\overline{B}_\varepsilon)} \tag{2.6}$$

for every $u \in W^{1,p}_{rad}\left(B \setminus \overline{B_\varepsilon}\right)$ with $u = 0$ on ∂B. By the definitions of $\mathscr{B}_p$ and X_p, we obtain $\|\nabla u_n\|_{L^p(B\setminus\overline{B}_\varepsilon)} = o(1)$, and then $\|u_n\|_{L^p(B\setminus\overline{B}_\varepsilon)} \to 0$ as $n \to \infty$ for any $\varepsilon > 0$ by (2.6). It holds from the embedding (2.5) that $\kappa_{n,\varepsilon} \to 0$ as $n \to \infty$ for any $\varepsilon > 0$. Using this fact and (2.6) again, we observe that

$$\begin{aligned}
\tau_1 &= \int_{B_\varepsilon} |u_n|^{\frac{p}{p-1}} dx + \int_{B\setminus B_\varepsilon} |u_n|^{\frac{p}{p-1}} dx \\
&\leq \left(\|u_n - \kappa_{n,\varepsilon}\|_{L^{\frac{p}{p-1}}(B_\varepsilon)} + \|\kappa_{n,\varepsilon}\|_{L^{\frac{p}{p-1}}(B_\varepsilon)}\right)^{\frac{p}{p-1}} + \int_{B\setminus B_\varepsilon} |u_n|^{\frac{p}{p-1}} dx \\
&= \left(\|u_n - \kappa_{n,\varepsilon}\|_{L^{p^*}(B_\varepsilon)} |B_\varepsilon|^{\frac{(p-1)p^*}{(p-1)p^*-p}} + o_n(1)\right)^{\frac{p}{p-1}} + o_n(1) \\
&\leq \left(S_p^{-1} \|\nabla u_n\|_{L^p(B_\varepsilon)} |B_\varepsilon|^{\frac{(p-1)p^*}{(p-1)p^*-p}} + o_n(1)\right)^{\frac{p}{p-1}} + o_n(1) \\
&= o_\varepsilon(1) + o_n(1),
\end{aligned}$$

where $o_n(1) \to 0$ as $n \to \infty$ and $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. Letting $\varepsilon \to 0$ after $n \to \infty$, we obtain $\tau_1 = o(1)$ as $n \to \infty$. Similarly, we deduce that $\tau_2 = o(1)$ as $n \to \infty$. Thus, we obtain (2.4).

Applying (2.4) to (2.3) with the aid of the Sobolev inequality, we have

$$\begin{aligned}
\int_B F_p(u_n)dx &\leq \int_B \left\{1 + \left[\frac{N-p}{N(p-1)}\alpha_p\right]^{\frac{N(p-1)}{N-p}} |u_n|^{p^*} + H(u_n)\right\} dx \\
&\leq |B| + \left[\frac{N-p}{N(p-1)}\alpha_p\right]^{\frac{N(p-1)}{N-p}} S_p^{-p^*} \|\nabla u_n\|^{p^*}_{L^p(B)} + o(1) \\
&\leq |B| + \left[\frac{N-p}{N(p-1)}\alpha_p\right]^{\frac{N(p-1)}{N-p}} S_p^{-p^*} + o(1).
\end{aligned} \tag{2.7}$$

This proves the upper estimate of M_p.

Next, it remains to prove the lower estimate of M_p. We define

$$U(x) = (1 + |x|^{\frac{p}{p-1}})^{-\frac{N-p}{p}},$$

and then for $\varepsilon_n \to 0$ as $n \to \infty$, set

$$W_n(x) = K_n \left[{\varepsilon_n}^{-\frac{N-p}{p}} \left(U(x/\varepsilon_n) - U_{\varepsilon_n}\right)\right], \tag{2.8}$$

where $U_{\varepsilon_n} := (1 + {\varepsilon_n}^{-\frac{p}{p-1}})^{-\frac{N-p}{p}}$ and K_n is the constant satisfying $\|\nabla W_n\|_p = 1$. It is easy to see that $\{W_n\} \in X_p$. By direct computation, we have

$$\int_B |W_n|^{p^*} dx = S_p^{-p^*} + o(1)$$

as $n \to \infty$. Combining this with (2.2), we have

$$
\begin{aligned}
(2.9) \qquad & |B| + \left[\frac{N-p}{N(p-1)}\alpha_p\right]^{\frac{N(p-1)}{N-p}} S_p^{-p^*} \\
= \; & \lim_{n\to\infty} \int_B \left\{ 1 + \left[\frac{N-p}{N(p-1)}\alpha_p\right]^{\frac{N(p-1)}{N-p}} |W_n|^{p^*} \right\} dx \\
\leq \; & \sup_{\{u_n\}\in X_p} \limsup_{n\to\infty} \int_B F_p(u_n) dx.
\end{aligned}
$$

Hence, the lower estimate is obtained, and consequently (2.7) and (2.9) yield Proposition 2.1. □

Proof of Proposition 2.2. It follows from (1.4), (2.1), and Proposition 2.1 that

$$
\begin{aligned}
M_p = \; & |B| + \left[\frac{N-p}{N(p-1)} \left(|B|^{\frac{1}{p^*}} \frac{\sqrt{\pi}N}{\Gamma\left(1+\frac{N}{2}\right)^{\frac{1}{N}}} \right)^{\frac{p}{p-1}} \right]^{\frac{N(p-1)}{N-p}} \\
& \times \left\{ \sqrt{\pi} N^{\frac{1}{p}} \left(\frac{N-p}{p-1}\right)^{\frac{p-1}{p}} \left[\frac{\Gamma\left(\frac{N}{p}\right)\Gamma\left(N+1-\frac{N}{p}\right)}{\Gamma(N)\Gamma\left(1+\frac{N}{2}\right)} \right]^{\frac{1}{N}} \right\}^{-p^*} \\
= \; & |B| + |B| \left[\frac{\Gamma(N)}{\Gamma\left(\frac{N}{p}\right)\Gamma\left(N+1-\frac{N}{p}\right)} \right]^{\frac{p}{N-p}}.
\end{aligned}
$$

Let $q := \frac{Np-N+p}{Np}$. We observe that

$$
\left[\frac{\Gamma(N)}{\Gamma\left(\frac{N}{p}\right)\Gamma\left(N+1-\frac{N}{p}\right)} \right]^{\frac{p}{N-p}} = \exp_q \left\{ \ln_q \left[\frac{\Gamma(N)}{\Gamma\left(\frac{N}{p}\right)\Gamma\left(N+1-\frac{N}{p}\right)} \right]^{\frac{p}{N-p}} \right\}.
$$

Due to the continuity of the q-exponential function, it suffices to prove that

$$
\lim_{p\to N} \ln_q \left[\frac{\Gamma(N)}{\Gamma\left(\frac{N}{p}\right)\Gamma\left(N+1-\frac{N}{p}\right)} \right]^{\frac{p}{N-p}} = \sum_{k=1}^{N-1} \frac{1}{k}.
$$

By the definition of the q-logarithmic function with $q = \frac{Np-N+p}{Np}$, we have

$$\begin{aligned}
&\ln_q \left[\frac{\Gamma(N)}{\Gamma\left(\frac{N}{p}\right)\Gamma\left(N+1-\frac{N}{p}\right)} \right]^{\frac{p}{N-p}} \\
=& \frac{\left[\frac{\Gamma(N)}{\Gamma\left(\frac{N}{p}\right)\Gamma\left(N+1-\frac{N}{p}\right)} \right]^{\frac{p}{N-p}\left(1-\frac{Np-N+p}{Np}\right)} - 1}{1-\frac{Np-N+p}{Np}} \\
=& \frac{N}{\Gamma\left(\frac{N}{p}\right)^{\frac{1}{N}}\Gamma\left(N+1-\frac{N}{p}\right)^{\frac{1}{N}}} \left[\frac{\Gamma(N)^{\frac{1}{N}} - \Gamma\left(\frac{N}{p}\right)^{\frac{1}{N}}\Gamma\left(N+1-\frac{N}{p}\right)^{\frac{1}{N}}}{\frac{N}{p}-1} \right].
\end{aligned}$$

Putting $t = \frac{N}{p} - 1$, we derive that

$$\begin{aligned}
&\lim_{p\to N} \ln_q \left[\frac{\Gamma(N)}{\Gamma\left(\frac{N}{p}\right)\Gamma\left(N+1-\frac{N}{p}\right)} \right]^{\frac{p}{N-p}} \\
=& \lim_{t\to 0} \frac{N}{\Gamma(t+1)^{\frac{1}{N}}\Gamma(N-t)^{\frac{1}{N}}} \left(\frac{\Gamma(N)^{\frac{1}{N}} - \Gamma(t+1)^{\frac{1}{N}}\Gamma(N-t)^{\frac{1}{N}}}{t} \right) \\
=& \frac{N}{\Gamma(N)^{\frac{1}{N}}} \frac{d}{dt} \left(-\Gamma(t+1)^{\frac{1}{N}}\Gamma(N-t)^{\frac{1}{N}} \right)\Big|_{t=0} \\
=& \frac{\Gamma'(N)}{\Gamma(N)} - \frac{\Gamma'(1)}{\Gamma(1)} \\
=& \frac{d}{dz}\log(\Gamma(z))\Big|_{z=N} + \frac{d}{dz}\log(\Gamma(z))\Big|_{z=1}.
\end{aligned}$$

Here $\frac{d}{dz}\log(\Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}$ is called the digamma function. It is known that the digamma function is written by (see for example Section 13.2 in [3])

$$\frac{d}{dz}\log(\Gamma(z)) = -\gamma + \sum_{j=1}^{\infty} \left(\frac{1}{j} - \frac{1}{z-1+j} \right),$$

where γ denotes Euler's constant. Thus, it holds

$$\lim_{p\to N} \ln_q \left[\frac{\Gamma(N)}{\Gamma\left(\frac{N}{p}\right)\Gamma\left(N+1-\frac{N}{p}\right)} \right]^{\frac{p}{N-p}} = \sum_{k=1}^{N-1} \frac{1}{k}.$$

This completes the proof of Proposition 2.2. □

We give a proof of (1.9) for the convenience of readers.

Proof of (1.9). Let $1 < p < N$ and fix $u \in W^{1,p}_{0,rad}(B)$. Then there exists $v : [0,1) \to \mathbb{R}$ such that $u(x) = v(|x|)$. By the fundamental theorem of calculus and the Hölder inequality, we have

$$|v(r)| \leq \int_r^1 |v'(s)|ds \leq \left(\int_r^1 s^{N-1}|u'(s)|^p ds\right)^{\frac{1}{p}} \left(\int_r^1 s^{-\frac{N-1}{p-1}} ds\right)^{\frac{p-1}{p}}$$

$$\leq \omega_{N-1}^{-\frac{1}{p}} \|\nabla u\|_{L^p(B)} \left\{\frac{N-p}{p-1}\left(r^{-\frac{N-p}{p-1}}-1\right)\right\}^{\frac{p-1}{p}}.$$

The conclusion follows from $\omega_{N-1}^{-\frac{1}{p}} = N^{\frac{p-1}{p}}\left(|B|^{\frac{1}{p}-\frac{1}{N}}\alpha_N^{\frac{N-1}{N}}\right)^{-1}$. □

3. Additional remarks

In a final section, we state some remarks.

Remark 3.1. Theorem 1.2 still holds for more general settings. Indeed, Proposition 2.1, and hence Theorem 1.2, holds when $\mathscr{B}_p$ and X_p are replaced by the following sets, without assuming radially symmetric conditions:

$$\mathscr{C}_p := \left\{u \in W_0^{1,p}(B) \;\middle|\; \|\nabla u\|_{L^p(B)} \leq 1\right\},$$

$$\hat{X}_p := \left\{\{u_n\} \subset \mathscr{C}_p \;\middle|\; u_n \rightharpoonup 0 \text{ weakly in } W_0^{1,p}(B)\right\}.$$

We give a sketch of the proof. Since $\{W_n\}$ constructed in (2.8) belongs to $\hat{X}_p$, the lower estimate of Proposition 2.1 holds by the same argument as in the proof for X_p. For the upper estimate, it is enough to prove $\int_B H(u_n)dx = o(1)$ for any $\{u_n\} \in \hat{X}_p$, where H is defined in (2.3). This computation is a direct consequence of the definition of H and the compactness of subcritical Sobolev embeddings. Hence, Proposition 2.1 holds for $\hat{X}_p$.

Remark 3.2. By Lions [16], it has been proven that if a sequence $\{u_n\} \subset \mathscr{C}_N$ satisfies $u_n \rightharpoonup u_0$ weakly in $W_0^{1,N}(B)$ and

$$\liminf_{n\to\infty} \int_B e^{\alpha_N |u_n|^{\frac{N}{N-1}}} dx > \int_B e^{\alpha_N |u_0|^{\frac{N}{N-1}}} dx,$$

then $\{u_n\} \in \hat{X}_N$. This means that the Moser–Trudinger functional can be discontinuous only for some sequences in $\hat{X}_N$. However, the situation $p < N$ is different. Indeed, one can construct a sequence $\{u_n\} \subset \mathscr{C}_p$ such that $u_n \rightharpoonup u_0 \not\equiv 0$ weakly in $W_0^{1,p}(B)$ and $\liminf_{n\to\infty} \int_B F_p(u_n)dx > \int_B F_p(u_0)dx$ as follows:

We first consider

$$T_p(s) = F_p(s) - \left[\frac{N-p}{N(p-1)}\alpha_p\right]^{\frac{N(p-1)}{N-p}} |s|^{p^*}.$$

By (2.2) and (2.3), we observe that $1 \leq T_p(s) \leq 1 + H(s)$. Applying a variant of the dominated convergence theorem, we have

$$\int_B T_p(u_n)dx \to \int_B T_p(u_0)dx$$

for any $\{u_n\} \subset W_0^{1,p}(B)$ with $u_n \rightharpoonup u_0$ weakly in $W_0^{1,p}(B)$. Therefore, it suffices to identify a sequence $\{u_n\} \subset \mathscr{C}_p$ such that $u_n \rightharpoonup u_0 \not\equiv 0$ weakly in $W_0^{1,p}(B)$ and $\liminf_{n\to\infty} \int_B |u_n|^{p^*} dx > \int_B |u_0|^{p^*} dx$.

Take $\phi, \psi \in W_0^{1,p}(B)$ with $\|\nabla\phi\|_p^p = 1/2, \|\nabla\psi\|_p^p = 1/2$ and consider zero extension of ψ outside of B. Define a sequence by

$$u_n(x) = C_n \left(\phi(x) + n^{\frac{N-p}{p}}\psi(nx)\right),$$

where C_n is taken such that $\|\nabla u_n\|_p = 1$. Under the setting, we see that

$$C_n \to 1, \quad u_n \rightharpoonup \phi \text{ weakly in } W_0^{1,p}(B)$$

as $n \to \infty$. Therefore, $\{u_n\}$ does not belong to $\hat{X}_p$. Moreover, it follows from the theorem of Brezis and Lieb [5] that

$$\lim_{n\to\infty}\int_B |u_n|^{p^*}dx = \int_B |\phi|^{p^*}dx + \lim_{n\to\infty}\int_B |\psi_n|^{p^*}dx > \int_B |\phi|^{p^*}dx,$$

hence the sequence $\{u_n\}$ satisfies the desired condition.

Remark 3.3. Several inequalities for $W^{1,N}$ functions can be derived from $W^{1,p}$ cases using the direct limiting procedure as $p \to N$. For instance, $W^{1,p}$ approximation of the Alvino inequality (1.8) and the Hardy inequality in the half space was obtained in [13] and [21], respectively.

Remark 3.4. The optimal constant $\sup_{\|\nabla u\|_{L^p(B)}\le 1}\int_B F_p(u)dx$ is lower semicontinuous as $p \to N$, namely there holds

$$\liminf_{p\uparrow N}\left(\sup_{\|\nabla u\|_{L^p(B)}\le 1}\int_B F_p(u)dx\right) \ge \sup_{\|\nabla u\|_{L^N(B)}\le 1}\int_B e^{\alpha_N|u|^{\frac{N}{N-1}}}dx.$$

Indeed, it follows from $\|\nabla u\|_{L^p(B)} \le |B|^{\frac{1}{p}-\frac{1}{N}}\|\nabla u\|_{L^N(B)}$ that

$$\begin{aligned}
&\sup_{\|\nabla u\|_{L^p(B)}\le 1}\int_B F_p(u)dx \\
&= \sup_{\|\nabla u\|_{L^p(B)}\le |B|^{\frac{1}{p}-\frac{1}{N}}}\int_B \left[1 + \frac{N-p}{N(p-1)}\alpha_p\left(|B|^{\frac{1}{N}-\frac{1}{p}}|u|\right)^{\frac{p}{p-1}}\right]^{\frac{N(p-1)}{N-p}} dx \\
&= \sup_{\|\nabla u\|_{L^p(B)}\le |B|^{\frac{1}{p}-\frac{1}{N}}}\int_B \left[1 + \frac{N-p}{N(p-1)}\alpha_N^{\frac{p(N-1)}{N(p-1)}}|u|^{\frac{p}{p-1}}\right]^{\frac{N(p-1)}{N-p}} dx \\
&\ge \sup_{\|\nabla u\|_{L^N(B)}\le 1}\int_B \left[1 + \frac{N-p}{N(p-1)}\alpha_N^{\frac{p(N-1)}{N(p-1)}}|u|^{\frac{p}{p-1}}\right]^{\frac{N(p-1)}{N-p}} dx \\
&\to \sup_{\|\nabla u\|_{L^N(B)}\le 1}\int_B e^{\alpha_N|u|^{\frac{N}{N-1}}}dx \quad (p \to N).
\end{aligned}$$

The continuity of the optimal constant remains open.

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