

Approximation by interval-decomposables and interval resolutions of persistence modules^{*,**}

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Abstract

In topological data analysis, two-parameter persistence can be studied using the representation theory of the 2d commutative grid, the tensor product of two Dynkin quivers of type A. In a previous work, we defined interval approximations using restrictions to essential vertices of intervals together with Mobius inversion. In this work, we consider homological approximations using interval resolutions, and show that the interval resolution global dimension is finite for finite posets, and that it is equal to the maximum of the interval dimensions of the Auslander-Reiten translates of the interval representations. Furthermore, in the commutative ladder case, by a suitable modification of our interval approximation, we provide a formula linking the two conceptions of approximation.

Keywords: Representation theory, Relative homological algebra, Multidimensional persistence

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1. Introduction

The field of topological data analysis is a rapidly growing field applying the ideas of algebraic topology for data analysis. In particular, one of its main tools, persistent homology [16, 15], is able to express the “birth” and “death” of topological features in data with respect to some parameter. The so-called interval modules play a central role in persistent homology, as they are used to express the “birth” and “death” of topological features. Algebraically, this follows from the fact that every pointwise finite dimensional module of a totally ordered poset decomposes as a direct sum of interval modules [7, 12]. For details concerning one-parameter persistent homology, we refer the reader to [16, 15].

However, when dealing with multiple parameters, as in multiparameter persistent homology [11], there are difficulties with adopting exactly the same techniques. In particular, there are many indecomposable modules that are not interval modules (see [9, 10] for concrete examples). In terms of representation theory, this means that for a large enough underlying grid of parameters (an n -dimensional commutative grid for n

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parameters), the corresponding algebra is of wild representation type ([5, Theorem 1.3], [21, Theorem 2.5], [22, Theorem 5]).

Thus there are many attempts to construct new invariants to capture persistent topological information even in the multiparameter setting. One idea is to use the intervals, as they have served very well in the one-parameter case. In particular, one recent direction is to study generalized persistence diagrams [20, 2, 8, 13] which associates “multiplicities” to intervals or subclasses of intervals, much like how the persistence diagram d_M of a persistence module M associates to each interval I its multiplicity $d_M(I)$ as a direct summand in M .

Recently, the works [8, 6] introduced the explicit use of relative homological algebra in persistence theory. With this, a new perspective on several existing invariants in multiparameter persistence was provided. In particular, one of the contributions of [8] is the study of rank decompositions of the rank invariant, and their relationships with the generalized persistence diagrams and with projective resolutions relative to the so-called rank-exact structure. The work [6] provides a systematic use of relative homological algebra in the study of invariants via their framework of “homological invariants”.

In this work, we study the relative homological algebra with respect to the interval modules. In particular, we show that for an arbitrary finite poset, the interval resolution global dimension of its incidence algebra is finite (Proposition 4.5). This gives an affirmative answer to Question 6.5 (whether or not the set of all interval modules \mathbb{I} gives a homological invariant) of [6].

We then provide a formula to compute the interval resolution global dimension as the maximum of a finite number of terms (Proposition 4.6), which we use in computational experiments. In particular, we obtain some conjectures about the value of the interval resolution global dimension of the 2D commutative grids (Conjectures 4.8 and 4.9).

We previously proposed a notion of compressed multiplicities relative to some set of essential vertices of intervals [2]. These compressed multiplicities are related to (and can be seen as variants of) the generalized rank invariant proposed in [20] (see [2] for a careful comparison). Applying a Möbius inversion, one respectively gets a generalized persistence diagram. However, [6] showed that the generalized rank invariant (and thus the corresponding generalized persistence diagram) is not a dim-hom invariant relative to \mathbb{I} , and conjectures that it is not a homological invariant.

Here, in the case of the commutative ladder (an $m \times 2$ commutative grid), we provide a modification (Definition 5.1) of our previous notion of compressed multiplicity. This gives a new invariant that is intimately related to an alternating sum of the terms appearing in the interval resolution (Theorem 5.4) and whose Möbius inversion is a homological invariant relative to \mathbb{I} (Corollary 5.8). That is, this gives an alternative definition for the homological invariant relative to \mathbb{I} .

We note that this paper is heavily grounded in the representation theory of algebras and uses its methods freely. For example, Proposition 4.5 is proved by using a Theorem [23, Theorem in §5] on quasi-hereditary algebras due to Ringel (see also Iyama’s [19, Lemma 2.2]). The proofs of Proposition 4.6 and Theorem 5.4 essentially uses the theory of almost split sequences due to Auslander and Reiten [4], and one of the key points for the latter is in the fact that the left adjoint functor to the restriction functor (compression functor) sends all modules (involved in an almost split sequence) to interval decomposable modules. Thus, we assume background knowledge in the representation theory of algebras, for which we refer the reader to [3, 4].

2. Background

Throughout this paper \mathbb{k} is a field, and \mathcal{P} always denotes a finite poset. We use the following definition concerning subsets of a finite poset \mathcal{P} .

Definition 2.1. Let \mathcal{P} be a finite poset.

- (1) For $p, q \in \mathcal{P}$, the *segment* from p to q is $[p, q] := \{x \in \mathcal{P} \mid p \leq x \leq q\}$
- (2) A subset $S \subseteq \mathcal{P}$ is said to be *connected* if it is connected as a subgraph of the Hasse diagram of \mathcal{P}
- (3) A subset $S \subseteq \mathcal{P}$ is said to be *convex* if for any $p, q \in \mathcal{P}$, $[p, q] \subseteq S$.
- (4) A subset $S \subseteq \mathcal{P}$ is said to be an *interval* if it is connected and convex.

We use the notation $\mathbb{I}(\mathcal{P})$ for the set of all intervals in \mathcal{P} . Where the poset is clear, we simplify this to just \mathbb{I} . Note that in the literature, it is standard to call the subsets defined in Definition 2.1(1) as “intervals”. However, following the persistence literature, we use the word “interval” for Definition 2.1(4). Note that however, to avoid this potential source of confusion, [6] calls them “spreads”. We do not adopt this terminology.

Let $\text{Seg}(\mathcal{P})$ be the set of segments in \mathcal{P} . The *incidence algebra* of \mathcal{P} over \mathbb{k} is the \mathbb{k} -vector space of functions from $\text{Seg}(\mathcal{P})$ to \mathbb{k} , with “pointwise” $+$ operation and \mathbb{k} -multiplication, together with the following convolution $*$ as the multiplication operation. More precisely, for $f, g : \text{Seg}(\mathcal{P}) \rightarrow \mathbb{k}$, define $f * g : \text{Seg}(\mathcal{P}) \rightarrow \mathbb{k}$ by

$$(f * g)([x, y]) := \sum_{x \leq z \leq y} g([x, z])f([z, y]).^1$$

We denote by $\mathbb{k}\mathcal{P}$ the incidence algebra of \mathcal{P} over \mathbb{k} .

The incidence algebra can be identified with the path algebra of a bound quiver (Q, R) , where Q is the Hasse diagram of \mathcal{P} regarded as a quiver² and R is the two-sided ideal of the path-algebra of Q generated by all the commutativity relations.

Let $\text{mod } \mathbb{k}\mathcal{P}$ be the category of finitely generated left $\mathbb{k}\mathcal{P}$ modules. We also consider \mathcal{P} as a category with a unique morphism $p \rightarrow q$ whenever $p \leq q$. A pointwise finite dimensional (pfd) persistence module over \mathcal{P} is a functor $M : \mathcal{P} \rightarrow \text{vect}_{\mathbb{k}}$, where $\text{vect}_{\mathbb{k}}$ is the category of finite dimensional \mathbb{k} -vector spaces. Note that $\text{mod } \mathbb{k}\mathcal{P}$ can be identified with the category pfd persistence modules over \mathcal{P} , which, of course, can be identified with the category of pfd representations of the bound quiver (Q, R) .

Of great importance in persistence are the interval persistence modules.

Definition 2.2 (Interval persistence modules). Let \mathcal{P} be a poset. A persistence module M over \mathcal{P} is said to be an *interval persistence module* (*interval module*, for short) if

- $\dim M(x) \leq 1$ for each $x \in \mathcal{P}$,
- its “support” $\text{supp}(M) := \{x \in \mathcal{P} \mid M(x) \neq 0\}$ is an interval of \mathcal{P} , and
- for all $x \leq y$ with $x, y \in \text{supp}(M)$, $M(x \rightarrow y)$ is an identity map.

For each $I \in \mathbb{I}(\mathcal{P})$, the interval module with support I is uniquely determined up to isomorphism. We denote that interval module by V_I .

Definition 2.3 (2D commutative grids and commutative ladders). Let $m, n \in \mathbb{N} = \{1, 2, 3, \dots\}$.

- The *commutative 2D grid of size $m \times n$* is bound quiver defined by the following quiver Q together with all possible commutativity relations:

$$\begin{array}{ccccccc}
 (1, n) & \xrightarrow{(\alpha_1, n)} & (2, n) & \xrightarrow{(\alpha_2, n)} & \cdots & \xrightarrow{(\alpha_{m-1}, n)} & (m, n) \\
 \uparrow (1, \beta_{n-1}) & & \uparrow (2, \beta_{n-1}) & \cdots & & & \uparrow (m, \beta_{n-1}) \\
 \vdots & & \vdots & & \vdots & & \vdots \\
 \uparrow (1, \beta_2) & & \uparrow (2, \beta_2) & \cdots & & & \uparrow (m, \beta_2) \\
 (1, 2) & \xrightarrow{(\alpha_1, 2)} & (2, 2) & \xrightarrow{(\alpha_2, 2)} & \cdots & \xrightarrow{(\alpha_{m-1}, 2)} & (m, 2) \\
 \uparrow (1, \beta_1) & & \uparrow (2, \beta_1) & \cdots & & & \uparrow (m, \beta_1) \\
 (1, 1) & \xrightarrow{(\alpha_1, 1)} & (2, 1) & \xrightarrow{(\alpha_2, 1)} & \cdots & \xrightarrow{(\alpha_{m-1}, 1)} & (m, 1)
 \end{array} \tag{2.1}$$

¹Note that this definition is opposite to the usual definition of the incidence algebra [24]. We adopt this multiplication to make it match up to the usual interpretation of composition of functions from the left.

²For any $x, y \in \mathcal{P}$, the quiver has a (unique) arrow $x \rightarrow y$ iff there is an edge between x and y in H with $x \leq y$. In the path algebra, we compose arrows from the left. That is, for $\alpha : x \rightarrow y$ and $\beta : y \rightarrow z$, $\beta\alpha$ is the path from x to z going through α and then β , consistent with the convolution in the incidence algebra.

We denote this bound quiver by $\vec{G}_{m,n} = (Q, \rho)$.

- From the discussion above, pfd representations of the bound quiver $\vec{G}_{m,n}$ can be identified with the pfd persistence modules over the poset given by $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ with partial order defined by $(i, j) \leq (k, \ell)$ if and only if $i \leq k$ and $j \leq \ell$. By abuse of notation, we also denote this poset by $\vec{G}_{m,n}$.
- When $n = 2$ (or symmetrically $m = 2$), it is called the *commutative ladder* of length m (or n), which was studied in the context of persistence in [18].

By the discussion in Section 4.1 of [1], each interval I in $\mathbb{I}(\vec{G}_{m,n})$ has a “staircase” form, which was denoted by:

$$I = \bigsqcup_{i=s}^t [b_i, d_i]_i$$

for some integers $1 \leq s \leq t \leq m$ and some integers $1 \leq b_i \leq d_i \leq n$ for each $s \leq i \leq t$ such that

$$b_{i+1} \leq b_i \leq d_{i+1} \leq d_i \text{ for all } i \in \{s, \dots, t-1\}.$$

In this notation, each $[b_i, d_i]_i$ is the “slice” of the staircase at height i .

We use the following notation.

Notation 2.4. For a finite-dimensional algebra A over \mathbb{k} , fix a complete set \mathcal{L} of representatives of isoclasses of indecomposable modules in $\text{mod } A$. For X in $\text{mod } A$, by the Krull–Schmidt theorem, there exists a unique function $d_X: \mathcal{L} \rightarrow \mathbb{Z}_{\geq 0}$ such that $X \cong \bigoplus_{L \in \mathcal{L}} L^{(d_X(L))}$.

3. Right approximations and resolutions

Throughout this section A is a finite-dimensional algebra over a field \mathbb{k} . We here prepare necessary facts for right minimal approximations and resolutions with respect to the subcategory $\mathcal{S} = \text{add } G$ for a generator G in the category $\text{mod } A$ of finitely generated left A -modules. Note that all of the material here is not new, as it is standard in relative homological algebra (see e.g., [17], [14], or [6]).

We start with the following definition where \mathcal{S} is not necessarily given by $\text{add } G$ for some generator G .

Definition 3.1. Let M be in $\text{mod } A$, and \mathcal{S} an additive subcategory of $\text{mod } A$.

- (1) A right \mathcal{S} -approximation of M is a morphism $f \in \text{Hom}_A(X, M)$ with $X \in \mathcal{S}$ satisfying the condition that for any $g \in \text{Hom}_A(Y, M)$ with $Y \in \mathcal{S}$, there exists some $h \in \text{Hom}_A(Y, X)$ such that $g = fh$. This condition is equivalent to saying that f induces an epimorphism

$$\text{Hom}_A(-, f): \text{Hom}_A(-, X)|_{\mathcal{S}} \rightarrow \text{Hom}_A(-, M)|_{\mathcal{S}}.$$

- (2) An \mathcal{S} -resolution of M is a sequence

$$\dots \rightarrow X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M$$

such that f_0 is a right \mathcal{S} -approximation of M , and for each $i \geq 1$, f_i is a right \mathcal{S} -approximation of $\text{Ker } f_{i-1}$. Therefore, this sequence yields an exact sequence

$$\begin{aligned} \dots \rightarrow \text{Hom}_A(-, X_2)|_{\mathcal{S}} &\xrightarrow{\text{Hom}_A(-, f_2)} \text{Hom}_A(-, X_1)|_{\mathcal{S}} \\ &\xrightarrow{\text{Hom}_A(-, f_1)} \text{Hom}_A(-, X_0)|_{\mathcal{S}} \xrightarrow{\text{Hom}_A(-, f_0)} \text{Hom}_A(-, M)|_{\mathcal{S}} \rightarrow 0. \end{aligned} \tag{3.2}$$

Dually we define a *left \mathcal{S} -approximation* of M , and an *\mathcal{S} -coresolution* of M .

Remark 3.2. Let M and \mathcal{S} be as in the definition above. Assume that \mathcal{S} contains A , or equivalently that \mathcal{S} contains all finitely generated projective A -modules (equivalently, \mathcal{S} is a generator). Then any right \mathcal{S} -approximation $f: X \rightarrow M$ of M is actually an epimorphism. Indeed, let $g: P \rightarrow M$ be an epimorphism from a finitely generated projective module P (e.g., a projective cover of M). Then since $P \in \mathcal{S}$, we have $g = fh$ for some $h: P \rightarrow X$, which shows that f is an epimorphism. Therefore, any \mathcal{S} -resolution of M in (2) above turns out to be an exact sequence

$$\cdots \rightarrow X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \rightarrow 0.$$

The dual remark holds for left \mathcal{S} -approximations and \mathcal{S} -coresolutions. Namely, if \mathcal{S} contains all indecomposable injective A -modules (in other words, if \mathcal{S} is a cogenerator), then any left \mathcal{S} -approximation is a monomorphism, and any \mathcal{S} -coresolution is an exact sequence.

In the following, we only explain right \mathcal{S} -approximations and \mathcal{S} -resolutions, and omit the dual version.

By considering the case of $\mathcal{S} = \text{add } A$, it is easy to see that the concept of a \mathcal{S} -resolution of M is a generalization of the idea of a (not necessarily minimal) projective resolution of M . For minimality, we need the following definitions.

Definition 3.3. Let $M \in \text{mod } A$.

- (1) The comma category $(\text{mod } A) \downarrow M$ from $\text{mod } A$ to M is defined as follows. Its objects are the morphisms $f: X \rightarrow M$ in $\text{mod } A$, and for any objects $f: X \rightarrow M, f': X' \rightarrow M$, the morphism set $\text{Hom}(f, f')$ is given as the set of morphisms $g: X \rightarrow X'$ in $\text{mod } A$ satisfying $f = f'g$. The composition is given by that of morphisms in $\text{mod } A$.
Note that for each $g \in \text{Hom}(f, f')$, g is an isomorphism in this category if and only if it is an isomorphism in $\text{mod } A$.
- (2) A morphism $f: X \rightarrow M$ in $\text{mod } A$ is called *right minimal* if every morphism $g: X \rightarrow X$ with $f = fg$ (i.e., every $g \in \text{Hom}(f, f)$) is an automorphism.
- (3) Let f, f' be morphisms in $(\text{mod } A) \downarrow M$. They are said to be *equivalent* in $(\text{mod } A) \downarrow M$ if $\text{Hom}(f, f') \neq \emptyset$ and $\text{Hom}(f', f) \neq \emptyset$. This is an equivalence relation on the set of morphisms in $(\text{mod } A) \downarrow M$. If f' is right minimal and is equivalent to f , then f' is called a *right minimal version* of f .

We give the following two properties of right minimal morphisms, which we will use later.

Theorem 3.4 ([4, Theorem 2.2]). *Let $f: X \rightarrow M$ in $\text{mod } A$. Then there exists a decomposition $X = X' \oplus X''$ such that $f|_{X'}$ is a right minimal version of f and $f|_{X''} = 0$. Moreover, $f|_{X'}$ is uniquely determined by f up to equivalences.*

The following is immediate from the theorem above.

Corollary 3.5. *Let $f: X \rightarrow M$ be a morphism in $\text{mod } A$. Then the following are equivalent.*

- (1) f is right minimal.
- (2) For any section $s: S \rightarrow X$, $fs = 0$ implies $s = 0$.

The following can be shown using the corollary above, but here we give a simple alternative proof.

Lemma 3.6. *Let $f: X \rightarrow M$ and $f': X' \rightarrow M'$ be morphisms in $\text{mod } A$. If both of them are right minimal, then so is $f \oplus f'$.*

Proof. Let $T := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{End}(X \oplus X')$, $F := f \oplus f' = \begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}$, and assume $FT = F$. This immediately yields $fa = f, f'c = 0$, and $f'd = f'$, and we have $a \in \text{Aut } X$ because f is right minimal. Since $T = \begin{pmatrix} \mathbf{1} & 0 \\ ca^{-1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d-ca^{-1}b \end{pmatrix}$, it is sufficient to show that $u := d - ca^{-1}b \in \text{Aut } X'$. But it is clear from the right minimality of f' and the fact that $f'u = f'd - f'ca^{-1}b = f' - 0 = f'$. \square

Throughout the rest of this section, we let G be a generator in $\text{mod } A$, and set $\mathcal{S} := \text{add } G$, the full subcategory of $\text{mod } A$ consisting of all direct summands of finite direct sums of copies of G . Then we have $A \in \mathcal{S}$, and Remark 3.2 is applicable in this setting.

Now we are ready to give the definition of minimal \mathcal{S} -resolutions, which generalizes the definition of minimal projective resolutions.

Definition 3.7. Let M be in $\text{mod } A$.

- (1) A *right minimal \mathcal{S} -approximation* of M is a right \mathcal{S} -approximation $f: X \rightarrow M$ that is right minimal.
- (2) A *minimal \mathcal{S} -resolution* of M is a sequence

$$\cdots \rightarrow X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \rightarrow 0$$

such that f_0 is a right minimal \mathcal{S} -approximation of M , and for each $i \geq 1$, f_i is a right minimal \mathcal{S} -approximation of $\text{Ker } f_{i-1}$. By Remark 3.2, this is automatically an exact sequence.

Theorem 3.4 gives the following link between minimal right \mathcal{S} -approximations and not-necessarily-minimal ones.

Lemma 3.8. Let M be in $\text{mod } A$, $f: I \rightarrow M$ a minimal right \mathcal{S} -approximation, and $g: J \rightarrow M$ a right \mathcal{S} -approximation. Then there exist an $I' \in \mathcal{S}$ and an isomorphism $h: J \rightarrow I \oplus I'$ such that $g = (f, 0)h$, namely we have the commutative diagram

$$\begin{array}{ccc} J & \xrightarrow{g} & M \\ \downarrow h & & \parallel \\ I \oplus I' & \xrightarrow{(f, 0)} & M \end{array} .$$

Using Lemmas 3.6 and 3.8, we obtain the following that links minimal resolutions and not-necessarily-minimal ones.

Proposition 3.9. Let M be in $\text{mod } A$, $I_* := (\cdots \rightarrow I_1 \xrightarrow{f_1} I_0 \xrightarrow{f_0} M \rightarrow 0)$ a minimal \mathcal{S} -resolution, and $J_* := (\cdots \rightarrow J_1 \xrightarrow{g_1} J_0 \xrightarrow{g_0} M \rightarrow 0)$ an \mathcal{S} -resolution. Set $K_i := \text{Im } f_i$ and $L_i := \text{Im } g_i$ for all $i \geq 1$.

Then for each $i \geq 0$, there exists $I'_i \in \mathcal{S}$ such that $J_i \cong I_i \oplus \text{cone}(\mathbb{1}_{I'_0}) \oplus \text{cone}(\mathbb{1}_{I'_1})[1] \oplus \cdots \oplus \text{cone}(\mathbb{1}_{I'_i})[i] \oplus \cdots$ and $L_i \cong K_i \oplus I'_{i-1}$ for all $i \geq 1$. In particular, a minimal \mathcal{S} -resolution is unique up to isomorphism.

Definition 3.10. Let M be in $\text{mod } A$.

- (1) If there exists an \mathcal{S} -resolution J_* of M of the form $J_* = (0 \rightarrow J_n \xrightarrow{g_n} \cdots \rightarrow J_1 \xrightarrow{g_1} J_0 \xrightarrow{g_0} M \rightarrow 0)$ for some $n \geq 0$ (noting that we placed no restrictions on J_n, \dots, J_1 being nonzero), then we say that *\mathcal{S} -resolution dimension* of M is at most n , and write $\mathcal{S}\text{-res-dim } M \leq n$. Otherwise we say that *\mathcal{S} -resolution dimension* of M is infinity.
- (2) If $\mathcal{S}\text{-res-dim } M \leq n$ and $\mathcal{S}\text{-res-dim } M \not\leq n-1$, then we say that *\mathcal{S} -resolution dimension* of M is equal to n , and denote it by $\mathcal{S}\text{-res-dim } M = n$.
- (3) Finally, we set $\mathcal{S}\text{-res-gldim } A := \sup\{\mathcal{S}\text{-res-dim } M \mid M \in \text{mod } A\}$, and call it the *\mathcal{S} -resolution global dimension* of A .

Lemma 3.11. Let M be in $\text{mod } A$, $0 \leq n$ an integer, $I_* = (I_i, f_i)_i$ a minimal \mathcal{S} -resolution of M , and $J_* = (J_i, g_i)_i$ an \mathcal{S} -resolution of M . Then the following are equivalent:

- (1) $\mathcal{S}\text{-res-dim } M \leq n$.
- (2) $\text{Im } g_n \in \mathcal{S}$.
- (3) I_* has the form $0 \rightarrow I_n \xrightarrow{f_n} \cdots \rightarrow I_1 \xrightarrow{f_1} I_0 \xrightarrow{f_0} M \rightarrow 0$.

Set $\Lambda := \text{End}_A(G)$, and regard G as an A - Λ -bimodule. Then we can consider the functor $\text{Hom}_A(G, -): \text{mod } A \rightarrow \text{mod } \Lambda$. We denote by $\text{pd}_\Lambda Y$ the projective dimension of a left Λ -module Y . Now we give a way to compute the \mathcal{I} -resolution dimension of an A -module by the projective dimension of a Λ -module.

The following is well-known (for instance, see [4, Proposition II.2.1] or [6, Proposition 4.16]).

Lemma 3.12. *Let $X, Y \in \text{mod } A$ and $F: \text{Hom}_A(G, X) \rightarrow \text{Hom}_A(G, Y)$ be in $\text{mod } \Lambda$. If $X \in \text{add } G$, then there exists a unique $f \in \text{Hom}_A(X, Y)$ such that $F = \text{Hom}_A(G, f)$.*

Using the above, the next Proposition follows. (cf. [6, Proposition 4.17(1)(2)])

Proposition 3.13. *Let M be in $\text{mod } A$, G a generator, and $\mathcal{I} := \text{add } G$. Then*

$$\mathcal{I}\text{-res-dim } M = \text{pd}_\Lambda \text{Hom}_A(G, M).$$

The following is immediate from Proposition 3.13 (cf. [6, Proposition 4.17(3)]).

Corollary 3.14. *Let G be a generator, and $\mathcal{I} := \text{add } G$. Then*

$$\mathcal{I}\text{-res-gldim } A = \sup\{\text{pd}_\Lambda \text{Hom}_A(G, M) \mid M \in \text{mod } A\} \leq \text{gldim } \Lambda.$$

In the case that G is a generator-cogenerator, we get not only an upper bound for \mathcal{I} -res-gldim A as above, but the following equality, which is immediate from Erdmann–Holm–Iyama–Schröer [17, Lemma 2.1]. This will be useful for our setting.

Proposition 3.15. *Assume further that G is also a cogenerator, i.e., all indecomposable injective A -modules are contained in \mathcal{I} . Then*

$$\mathcal{I}\text{-res-gldim } A = \text{gldim } \Lambda - 2$$

where $\mathcal{I} := \text{add } G$.

Finally, we recall the following definitions from [6], in order to put our results in the context of their framework of homological invariants. For this part, we let \mathcal{X} be a finite set of indecomposable modules of A , $\mathcal{I} = \text{add } \mathcal{X}$, and let

$$\mathcal{E}_{\mathcal{X}} = \left\{ 0 \rightarrow L \xrightarrow{f_2} M \xrightarrow{f_1} N \rightarrow 0 \text{ short exact} \mid f_1 \text{ is a right } \mathcal{I}\text{-approximation} \right\}.$$

Definition 3.16 (Relative Grothendieck groups).

- (1) Let F be the free abelian group generated by the symbols $[M]$ for each isomorphism class of $M \in \text{mod } A$. Let $H_{\mathcal{X}}$ be the subgroup generated by $[M] - [L] - [N]$ for $(0 \rightarrow L \xrightarrow{f_2} M \xrightarrow{f_1} N \rightarrow 0) \in \mathcal{E}_{\mathcal{X}}$. The *Grothendieck group of A relative to \mathcal{X}* is

$$K_0(A, \mathcal{X}) := F/H_{\mathcal{X}}.$$

- (2) Letting \mathcal{E}_{\min} be the class of all split short exact sequences, the *split Grothendieck group of A* , denoted $K_0^{\text{split}}(A)$, is defined as above but replacing $\mathcal{E}_{\mathcal{X}}$ with \mathcal{E}_{\min} .

Definition 3.17 (Homological invariants [6]). Let \mathcal{X} be a finite set of indecomposable modules of A and $\mathcal{I} = \text{add } \mathcal{X}$.

- (1) An *invariant* is a surjective group homomorphism $p: K_0^{\text{split}}(A) \rightarrow \mathbb{Z}^n$ for some $n \in \mathbb{N}$.
- (2) Two invariants p and q are said to be equivalent if and only if $\ker p = \ker q$.
- (3) An invariant p is said to be a *homological invariant relative to \mathcal{X}* if all of the following hold:
 - (a) All of the indecomposable projectives are in \mathcal{I} .
 - (b) For each $M \in \text{mod } A$, $\mathcal{I}\text{-res-dim } M < \infty$.
 - (c) p is equivalent to the invariant given by the canonical quotient map

$$K_0^{\text{split}}(A) \xrightarrow{\pi} K_0(A, \mathcal{X}) \cong \mathbb{Z}^{|\mathcal{X}|}$$

where the last isomorphism follows from [6, Proposition 4.9].

4. Finiteness and computation of the interval resolution global dimension

Throughout this section, we set $A := \mathbb{k}\mathcal{P}$ to be the incidence algebra of a poset \mathcal{P} , and $\mathcal{I} = \text{add } G$ with $G := \bigoplus_{I \in \mathbb{I}} V_I$, where \mathbb{I} is the set of all intervals in \mathcal{P} . Where convenient, we abuse the notation and write \mathbb{I} to also mean the set of modules $\{V_I \mid I \in \mathbb{I}\}$. With this abuse of notation, $\mathcal{I} = \text{add } \mathbb{I}$. Modules in \mathcal{I} are called *interval-decomposable* modules.

Since all projective indecomposable modules and all injective indecomposable modules are isomorphic to interval modules, the module G is indeed a generator and a cogenerator.

We specialize Definitions 3.1 and 3.10 to this setting.

Definition 4.1. Let M be in $\text{mod } \mathbb{k}\mathcal{P}$ and $\mathcal{I} = \text{add } G$ with $G := \bigoplus_{I \in \mathbb{I}(\mathcal{P})} V_I$.

- (1) A *right interval-approximation* of M is simply a right \mathcal{I} -approximation of M .
- (2) An *interval resolution* of M is simply an \mathcal{I} -resolution of M .
- (3) The *interval resolution dimension* of M is simply the \mathcal{I} -resolution dimension of M , and is denoted by $\text{int-res-dim } M$.
- (4) The *interval resolution global dimension* of A is simply the \mathcal{I} -resolution global dimension, and is denoted by int-res-gldim .

In this section, we will prove that the global dimension of $\Lambda = \text{End}_A(G)$ is finite. Then by Proposition 3.15, we see that the interval resolution global dimension of A is finite. We use a Theorem in Ringel's work [23] for this. Ringel [23] gave a concise proof of Iyama's finiteness theorem [19] for the representation dimension by using the concept of a *left quasi-hereditary algebra*. More precisely, he made use of a special type of that algebra with the additional property that all of its standard left modules have projective dimension at most 1, which is called a *left strongly quasi-hereditary algebra*. Note that any left quasi-hereditary algebra has finite global dimension.

We cite the following from [23, Theorem in §5] (cf. [19, Lemma 2.2]).

Theorem 4.2. *Let B be an artin algebra, and X a finitely generated B -module. Then there exists a B -module Y such that $C := \text{End}_B(X \oplus Y)$ is left strongly quasi-hereditary. In particular, the global dimension of C is finite. Moreover, we can take this Y as a module with the property that any indecomposable direct summand of Y is a submodule of an indecomposable direct summand of X .*

For our purposes, we extract the following Corollary.

Corollary 4.3. *Let B be an artin algebra, and X a finitely generated B -module. Assume that for each indecomposable direct summand X' of X , all submodules of X' are in $\text{add } X$, then $\text{End}_B(X)$ is left strongly quasi-hereditary, and its global dimension is finite.*

Proof. By Theorem 4.2, there exists a B -module Y such that $Y = \bigoplus_{i=1}^n Y_i$ with $Y_i \leq X_i$ for some indecomposable direct summand X_i of X for all $i = 1, \dots, n$ and that $\text{End}_B(X \oplus Y)$ is left strongly quasi-hereditary. By assumption, all Y_i are in $\text{add } X$, and hence $Y \in \text{add } X$. Therefore, $\text{End}_B(X \oplus Y)$ is Morita equivalent to $\text{End}_B(X)$. Thus $\text{End}_B(X)$ is also left strongly quasi-hereditary. \square

Lemma 4.4. *Let M be an interval module of $A = \mathbb{k}\mathcal{P}$, and N a submodule of M . Then, N is interval-decomposable.*

Proof. We set Q to be the Hasse diagram of \mathcal{P} regarded as a quiver. Denote by $1_x \in \mathbb{k}$ the basis of $e_x N$ for all $x \in \mathcal{P}$, where e_x is the idempotent in A associated to x .

We first show that the support $\text{supp } N := \{x \in \mathcal{P} \mid e_x N \neq 0\}$ of N is convex. Let $x, y \in \text{supp } N$, p a path from x to y in Q , and z any vertex occurring in p . It is enough to show that $e_z N \neq 0$. Let q be the subpath of p from x to z . Since M is an interval module, and $x, y \in \text{supp } N \subseteq \text{supp } M$, all the vertices occurring in p (and hence in q) are contained in $\text{supp } M$. Then since N is a submodule of M , we have

$$1_z = e_z q e_x 1_x \in e_z A e_x 1_x \leq e_z N.$$

Hence $e_z N \neq 0$, as desired.

Let S_1, \dots, S_n be the connected components of $\text{supp } N$. Then for each $i = 1, \dots, n$, S_i is convex and connected, thus an interval. Therefore, N is the direct sum of the interval modules defined by the S_i 's. \square

Corollary 4.3 and Lemma 4.4 immediately imply our desired result:

Proposition 4.5. *Let $G := \bigoplus_{I \in \mathbb{I}(\mathcal{P})} V_I$. The global dimension of $\Lambda := \text{End}(G)$ is finite. Hence $\text{int-res-gldim } A$ is finite.*

We next show the following, which enables us to compute $\text{int-res-gldim } A$ by a computer because \mathbb{I} is a finite set.

Proposition 4.6. *Let $A = \mathbb{k}\mathcal{P}$. The following holds.*

$$\text{int-res-gldim } A = \max_{I \in \mathbb{I}} \text{int-res-dim } \tau V_I.$$

Proof. Let $I \in \mathbb{I}$, and assume that V_I is not projective. Then there exists an almost split sequence of the form

$$0 \rightarrow \tau V_I \rightarrow E_I \rightarrow V_I \rightarrow 0,$$

which yields an exact sequence

$$0 \rightarrow \text{Hom}_A(G, \tau V_I) \rightarrow \text{Hom}_A(G, E_I) \rightarrow \text{rad}(G, V_I) \rightarrow 0$$

of left Λ -modules. This shows

$$\text{pd } \text{rad}(G, V_I) \leq \max\{\text{pd } \text{Hom}_A(G, E_I), \text{pd } \text{Hom}_A(G, \tau V_I) + 1\}.$$

Since Λ is not semisimple, we have

$$\begin{aligned} \text{int-res-gldim } A &= \text{gldim } \Lambda - 2 \\ &= \max\{\text{pd}(\text{Hom}_A(G, V_I)/\text{rad}(G, V_I) \mid I \in \mathbb{I}, \text{rad}(G, V_I)) \neq 0\} - 2 \\ &= \max\{\text{pd } \text{rad}(G, V_I) + 1 \mid I \in \mathbb{I}, \text{rad}(G, V_I) \neq 0\} - 2 \\ &= \max\{\text{pd } \text{rad}(G, V_I) - 1 \mid I \in \mathbb{I}\} \\ &\leq \max\{\text{pd } \text{Hom}_A(G, E_I) - 1, \text{pd } \text{Hom}_A(G, \tau V_I) \mid I \in \mathbb{I}\} \\ &= \max\{\text{int-res-dim } E_I - 1, \text{int-res-dim } \tau V_I \mid I \in \mathbb{I}\} \\ &\leq \text{int-res-gldim } A. \end{aligned}$$

Hence $\text{int-res-gldim } A = \max\{\text{int-res-dim } E_I - 1, \text{int-res-dim } \tau V_I \mid I \in \mathbb{I}\}$.

By Proposition 4.5, $\text{int-res-gldim } A = d$ for some positive integer d , and hence there exists some $I \in \mathbb{I}$ such that either $\text{int-res-dim } E_I - 1 = d$ or $\text{int-res-dim } \tau V_I = d$. But the former is impossible because if this is the case, then we have $\text{int-res-dim } E_I = d + 1 > \text{int-res-gldim } A = \max\{\text{int-res-dim } X \mid X \in \text{mod } A\}$, a contradiction. Therefore, d is in $\{\text{int-res-dim } \tau V_I \mid I \in \mathbb{I}\}$, which proves the assertion. \square

Using Proposition 4.6 we computed several cases using the excellent GAP [25] package QPA [26].

Example 4.7. Let $A = \mathbb{k}G_{m,n}$ ($m, n \geq 2$) and $\mathbb{k} = \mathbb{F}_2$, the finite field with 2 elements. Then the filled-in parts of the table below gives the value (or a lower bound) obtained by numerical computation of $\text{int-res-gldim } \mathbb{k}G_{m,n}$ in the row labelled m and column labelled n . By definition of $A = \mathbb{k}G_{m,n}$, the table is symmetric so that it suffices to compute only the upper or lower part.

	2	3	4	5	6	7	8	9	10
2	0	1	2	2	2	2	2	2	2
3	1	2	3	4	4	4			
4	2	3	4	5	≥ 6	≥ 6			
5	2	4	5						
6	2	4	≥ 6						
7	2	4	≥ 6						

These computations suggest some conjectures on the values of $\text{int-res-gldim } A$ as follows. The first conjecture is about the commutative ladders [18].

Conjecture 4.8. *For the path algebra $A = \mathbb{k}G_{m,n}$, $\text{int-res-gldim } A = 2$ if $n = 2$ and $m \geq 4$.*

If the next conjecture is correct, then the preceding conjecture also holds with $C(2) = 2$.

Conjecture 4.9. *There exists a function $C : \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$ such that for each $m \in \mathbb{N}$,*

$$\text{int-res-gldim } \mathbb{k}G_{m,n} = C(m)$$

for all $n \geq m + 2$. That is, considering $\text{int-res-gldim } \mathbb{k}G_{m,n}$ as a function of n for fixed m , this function stabilizes to some constant $C(m)$ starting from $m + 2$.

5. Relationship between interval-decomposable approximations and interval resolutions for commutative ladders

In this section, we relate the interval resolutions of persistence modules in the $2 \times n$ case with a modified version of the compressed multiplicity invariant [2]. Via Möbius inversion, we obtain an invariant that we show to be a homological invariant in Corollary 5.8.

To fix some notation, we let $\vec{G}_{m,2} = (Q, \rho)$ be a commutative ladder, where Q is presented as follows:

$$\begin{array}{ccccccc} y_1 & \xrightarrow{c_1} & y_2 & \xrightarrow{c_2} & \cdots & \xrightarrow{c_{n-1}} & y_n \\ b_1 \uparrow & & \uparrow b_2 & & \cdots & & \uparrow b_n \\ x_1 & \xrightarrow{a_1} & x_2 & \xrightarrow{a_2} & \cdots & \xrightarrow{a_{n-1}} & x_n \end{array}$$

and ρ the full commutativity relations. Note that compared to Diagram (2.1), we have set $x_i := (i, 1)$ and $y_i := (i, 2)$, $b_i := (1, \beta_i)$ for all $i = 1, \dots, n$, and $a_i := (\alpha_i, 1)$, $c_i := (\alpha_i, 2)$ for all for all $i = 1, \dots, n - 1$ to make the notation simpler. We regard the algebra $A := \mathbb{k}(Q, \rho)$ as a category defined by this quiver with the full commutativity relations ρ .

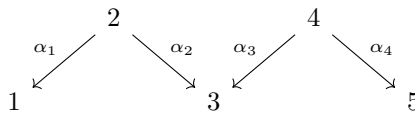
5.1. Compression

First, we introduce a modified version of compression of modules that was studied in [2].

Let us fix some notation. Let $I = [x_i, x_j]_1 \sqcup [y_k, y_l]_2$ be an interval of Q (recall the the discussion in Section 4.1 of [1] concerning the “staircase” shape of intervals). We have $1 \leq k \leq i \leq l \leq j \leq n$, and I is illustrated as follows:

$$\begin{array}{ccccccc} y_k & \xrightarrow{c_k} & \cdots & \xrightarrow{c_{i-1}} & y_i & \xrightarrow{c_i} & \cdots & \xrightarrow{c_{l-1}} & y_l \\ & & & & b_i \uparrow & & \cdots & & \uparrow b_l \\ & & & & x_i & \xrightarrow{a_i} & \cdots & \xrightarrow{a_{l-1}} & x_l & \xrightarrow{a_l} & \cdots & \xrightarrow{a_{j-1}} & x_j \end{array} \quad (5.3)$$

On the other hand, let Q' be the quiver:



and set $B := \mathbb{k}Q'$ to be the path category of the quiver Q' . For each vertex x of Q (resp. Q') we denote by e_x (resp. e'_x) the path of length 0 at x , and for each morphism f in $\mathbb{k}Q$, \bar{f} denotes the image of f under the canonical morphism $\mathbb{k}Q \rightarrow A$.

For the interval I , we define a quiver morphism $F := \xi_I: Q' \rightarrow A$ (depending on the form of I).

- If I is of the form (5.3), that is, if I spans two rows of the commutative ladder, define $F := \xi_I: Q' \rightarrow A$ by the following table:

$$\frac{x}{F(x)} \parallel \begin{array}{c|c|c|c|c|c} 1 & 2 & 3 & 4 & 5 & \\ \hline y_l & y_k & y_i & x_i & x_j & \end{array} \parallel \begin{array}{c|c|c|c} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \hline \overline{c_{l-1} \cdots c_k} & \overline{c_{i-1} \cdots c_k} & \overline{b_i} & \overline{a_{j-1} \cdots a_i} \end{array},$$

where if $k = i$, then $c_{i-1} \cdots c_k$ is replaced by e_i , (similar for the cases $i = l$ or $l = j$).

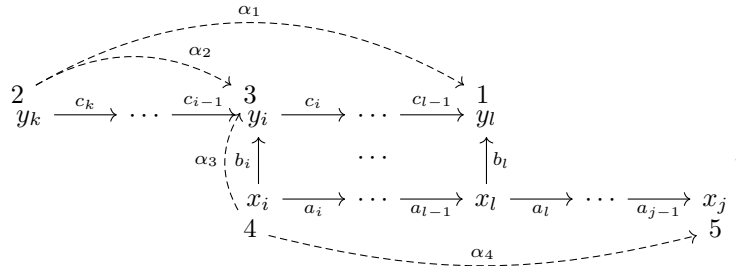
- When I is of the form $x_i \rightarrow \cdots \rightarrow x_j$ with $0 \leq i \leq j \leq n$ (I is contained in the lower row of the commutative ladder), we define $F = \xi_I$ by the following table:

$$\frac{x}{F(x)} \parallel \begin{array}{c|c|c|c|c|c} 1 & 2 & 3 & 4 & 5 & \\ \hline x_j & x_i & x_i & x_i & x_j & \end{array} \parallel \begin{array}{c|c|c|c} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \hline \overline{a_{j-1} \cdots a_i} & \overline{e_i} & \overline{e_i} & \overline{a_{j-1} \cdots a_i} \end{array}.$$

- We make a similar construction of $F = \xi_I$ when I is of the form $y_k \rightarrow \cdots \rightarrow y_l$ with $0 \leq k \leq l \leq n$ (I is contained in the upper row of the commutative ladder) by the following table:

$$\frac{x}{F(x)} \parallel \begin{array}{c|c|c|c|c|c} 1 & 2 & 3 & 4 & 5 & \\ \hline y_l & y_k & y_k & y_k & y_l & \end{array} \parallel \begin{array}{c|c|c|c} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \hline \overline{c_{l-1} \cdots c_k} & \overline{e_k} & \overline{e_k} & \overline{c_{l-1} \cdots c_k} \end{array}.$$

In the first case, F is visualized as follows:



where each broken arrow α represent an arrow in the quiver Q' , and the corresponding solid path represents its image $F(\alpha)$ in the path category A .

In any case, F uniquely extends to a linear functor $\hat{F}: B \rightarrow A$. By using \hat{F} , we regard A to be the A - B -bimodule ${}_A A_B = A(\hat{F}(\cdot), -)$, which gives us an adjoint pair

$$\text{mod } A \begin{array}{c} \xleftarrow{L} \\ \xrightarrow{R} \end{array} \text{mod } B,$$

where $\text{mod } A$ (resp. $\text{mod } B$) is the category of the finite-dimensional (left) A -modules (resp. B -modules), and $L = L_{\xi_I} := {}_A A_B \otimes_B -$ is a left adjoint to $R = R_{\xi_I} := \text{Hom}_A({}_A A_B, -)$. Recall that an A -module M is a functor $A \rightarrow \text{mod } \mathbb{k}$, and then $R(M) \cong M \circ \hat{F}$. For instance, $R(V_I) \cong V_{[1,5]}$.

Definition 5.1 (Compressed multiplicity). Let ξ be the function associating $I \in \mathbb{I}$ to $F = \xi_I$ as defined above. We define the *compressed multiplicity with respect to ξ* of V_I in M as

$$c_M^\xi(I) := d_{R_{\xi_I}(M)}(R_{\xi_I}(V_I)),$$

which is multiplicity of $R_{\xi_I}(V_I)$ as a direct summand of $R_{\xi_I}(M)$, for all $I \in \mathbb{I}$.

This is a modification of the compressed multiplicity introduced in [2]. In that previous work, for \hat{F} we used a functor defined by the inclusion of essential vertices, and so the corresponding $R(M) \cong M \circ \hat{F}$ is simply the compression functor of [2]. As also noted in [2], when \hat{F} is defined using the inclusion of I as is, the corresponding compressed multiplicity is equal to the generalized rank invariant of Kim and Memoli [20] (see [2] for a more detailed discussion). Here, we emphasize that instead of inclusion, \hat{F} is obtained from the quiver morphism F as defined above, thus yielding a new variant of the invariant.

Let us relate the above compressed multiplicities to the interval resolutions studied in the previous sections. First, we lay down some groundwork.

Lemma 5.2. *For each $x \in Q'_0$, we have $L(Be'_x) \cong Ae_{F(x)}$.*

Proof. Since the canonical morphism ${}_A A_B \otimes_B {}_B B_B \rightarrow {}_A A_B$ of A - B -bimodules defined by sending $a \otimes b$ to $a\hat{F}(b)$ for all morphisms a in A (resp. b in B) is an isomorphism, we have $A \otimes_B Be'_x \cong A\hat{F}(e'_x) = Ae_{F(x)}$. \square

The functor $L : \text{mod } B \rightarrow \text{mod } A$ satisfies the following nice property.

Proposition 5.3. *For any indecomposable module X in $\text{mod } B$, $L(X)$ is isomorphic to an interval module in $\text{mod } A$, or is zero. Therefore, $L(Y)$ is interval decomposable or zero for all modules Y in $\text{mod } B$.*

Proof. B has 15 isoclasses of indecomposable modules, whose complete set of representatives is given by the interval modules $V_{[i,j]}$ for all i, j with $1 \leq i \leq j \leq n$, where we set $[i, j] := \{x \in Q'_0 \mid i \leq x \leq j\}$ in Q'_0 . The assertion is verified by explicit computation of $L(X)$ using a minimal projective presentation of X and the lemma above. For instance, we check the statement for $X = V_{[2,4]}$. This module has the following minimal projective presentation:

$$0 \rightarrow Be'_1 \oplus Be'_3 \oplus Be'_5 \xrightarrow{\begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ 0 & \alpha_3 & \alpha_4 \end{pmatrix}} Be'_2 \oplus Be'_4 \rightarrow X \rightarrow 0,$$

where α_i stands for the right multiplication by α_i for each $i = 1, \dots, 4$. Then since L is right exact, we have an exact sequence

$$Ae_{y_l} \oplus Ae_{y_i} \oplus Ae_{x_j} \xrightarrow{\begin{pmatrix} \alpha'_1 & \alpha'_2 & 0 \\ 0 & \alpha'_3 & \alpha'_4 \end{pmatrix}} Ae_{y_k} \oplus Ae_{x_i} \rightarrow L(X) \rightarrow 0,$$

where α'_i stands for the morphism $A \otimes_B \alpha_i$ for each $i = 1, \dots, 4$. By computing the cokernel of the morphism, we have

$$L(X) \cong V_{[x_i, x_{j-1}]_1 \sqcup [y_k, y_{l-1}]_2}.$$

(We set $V_\emptyset := 0$. Thus if $i = j$ and $k = l$, then $L(X) = 0$.)

Similarly, the other cases can be verified. \square

We then recall that for M in $\text{mod } A$, we have the following.

- (1) Since $A = \mathbb{k}\vec{G}_{m,2}$ has a finite interval resolution global dimension (by Proposition 4.5), say r , there exists an interval resolution of M in the form of the following exact sequence with some non-negative integers $d_J^{(i)}$ for all $i = 0, 1, \dots, r$ and $J \in \mathbb{I}$:

$$0 \rightarrow \bigoplus_{J \in \mathbb{I}} V_J^{d_J^{(r)}} \xrightarrow{f_r} \dots \xrightarrow{f_2} \bigoplus_{J \in \mathbb{I}} V_J^{d_J^{(1)}} \xrightarrow{f_1} \bigoplus_{J \in \mathbb{I}} V_J^{d_J^{(0)}} \xrightarrow{f_0} M \rightarrow 0. \quad (5.4)$$

- (2) If $X \in \mathcal{S}$, then the sequence (3.2) and the exactness of the sequence (5.4) yields an exact sequence

$$\begin{aligned} 0 \rightarrow [X, \bigoplus_{J \in \mathbb{I}} V_J^{d_J^{(r)}}] \xrightarrow{[X, f_r]} \dots \xrightarrow{[X, f_2]} [X, \bigoplus_{J \in \mathbb{I}} V_J^{d_J^{(1)}}] \\ \xrightarrow{[X, f_1]} [X, \bigoplus_{J \in \mathbb{I}} V_J^{d_J^{(0)}}] \xrightarrow{[X, f_0]} [X, M] \rightarrow 0. \end{aligned} \quad (5.5)$$

where $[\cdot, \cdot] := \text{Hom}_A(\cdot, \cdot)$ for short. Note that the requirement that $X \in \mathcal{S}$ is needed here.

With the preparations finished, we give the following Theorem 5.4 relating the interval resolution (5.4) with the compressed multiplicities c_M^ξ with respect to ξ of M .

Theorem 5.4. *Let M be in $\text{mod } A$ with an interval resolution (5.4). Then we have*

$$c_M^\xi(I) = \sum_{I \leq J \in \mathbb{I}} \sum_{i=0}^r (-1)^i d_J^{(i)},$$

where $I \leq J$ means that I is a subquiver of J .

Proof. Recall that $R(V_I) = V_{[1,5]}$. Then the almost split sequence starting from $R(V_I)$ has the following form:

$$0 \rightarrow R(V_I) \rightarrow V_{[2,5]} \oplus V_{[1,4]} \rightarrow V_{[2,4]} \rightarrow 0,$$

which yields a minimal projective resolution

$$0 \rightarrow \text{Hom}_B(V_{[2,4]}, -) \rightarrow \text{Hom}_B(V_{[2,5]} \oplus V_{[1,4]}, -) \rightarrow \text{Hom}_B(R(V_I), -) \rightarrow \mathcal{S}_{R(V_I)} \rightarrow 0,$$

of the simple functor $\mathcal{S}_{R(V_I)} := \text{Hom}_B(R(V_I), -) / \text{rad}_B(R(V_I), -)$ corresponding to $R(V_I)$ by Auslander-Reiten theory. This shows that

$$\begin{aligned} d_{R(M)}(R(V_I)) &= \dim \mathcal{S}_{R(V_I)}(R(M)) \\ &= \dim \text{Hom}_B(R(V_I), R(M)) - \dim \text{Hom}_B(V_{[2,5]} \oplus V_{[1,4]}, R(M)) \\ &\quad + \dim \text{Hom}_B(V_{[2,4]}, R(M)) \end{aligned} \quad (5.6)$$

Now since ${}_A A_B$ is projective as a left A -module, $R = \text{Hom}_A({}_A A_B, -)$ is an exact functor. Then the exact sequence (5.4) gives us an exact sequence

$$0 \rightarrow \bigoplus_{J \in \mathbb{I}} R(V_J)^{d_J^{(r)}} \xrightarrow{R(f_r)} \dots \xrightarrow{R(f_r)} \bigoplus_{J \in \mathbb{I}} R(V_J)^{d_J^{(1)}} \xrightarrow{R(f_1)} \bigoplus_{J \in \mathbb{I}} R(V_J)^{d_J^{(0)}} \xrightarrow{R(f_0)} R(M) \rightarrow 0 \quad (5.7)$$

Let Y be in $\text{mod } B$. Apply $\text{Hom}_B(Y, -)$ to the exact sequence (5.7), and consider the following commutative diagram obtained by the adjoint pair (L, R) :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (Y, \bigoplus_{J \in \mathbb{I}} R(V_J)^{d_J^{(r)}}) & \longrightarrow & \dots & \longrightarrow & (Y, \bigoplus_{J \in \mathbb{I}} R(V_J)^{d_J^{(1)}}) & \longrightarrow & (Y, \bigoplus_{J \in \mathbb{I}} R(V_J)^{d_J^{(0)}}) & \longrightarrow & (Y, R(M)) & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & \\ 0 & \longrightarrow & [L(Y), \bigoplus_{J \in \mathbb{I}} V_J^{d_J^{(r)}}] & \longrightarrow & \dots & \longrightarrow & [L(Y), \bigoplus_{J \in \mathbb{I}} V_J^{d_J^{(1)}}] & \longrightarrow & [L(Y), \bigoplus_{J \in \mathbb{I}} V_J^{d_J^{(0)}}] & \longrightarrow & [L(Y), M] & \longrightarrow & 0, \end{array}$$

where we set $(\cdot, \cdot) := \text{Hom}_B(\cdot, \cdot)$ and $[\cdot, \cdot] := \text{Hom}_A(\cdot, \cdot)$ for short. By Proposition 5.3, $L(Y) \in \mathcal{S}$ and thus the lower row is exact (replace X by $L(Y)$ in the exact sequence (5.5)). Since all the vertical maps are isomorphisms, the upper row is also exact.

This yields the following equality:

$$\dim \text{Hom}_B(Y, R(M)) = \sum_{J \in \mathbb{I}} \left(\sum_{i=0}^r (-1)^i d_J^{(i)} \right) \dim \text{Hom}_B(Y, R(V_J)).$$

By applying this formula to Equation (5.6) for $Y = R(V_I)$, $Y = V_{[2,5]} \oplus V_{[1,4]}$, and $Y = V_{[2,4]}$, we have

$$\begin{aligned} d_{R(M)}(R(V_I)) &= \sum_{J \in \mathbb{I}} \left(\sum_{i=0}^r (-1)^i d_J^{(i)} \right) (\dim \text{Hom}_B(R(V_I), R(V_J)) \\ &\quad - \dim \text{Hom}_B(V_{[2,5]} \oplus V_{[1,4]}, R(V_J)) + \dim \text{Hom}_B(V_{[2,4]}, R(V_J))) \\ &= \sum_{J \in \mathbb{I}} \left(\sum_{i=0}^r (-1)^i d_J^{(i)} \right) d_{R(V_J)}(R(V_I)), \end{aligned} \quad (5.8)$$

where the last equality follows from Equation (5.6) with $M = V_J$. Now since every entry of the dimension vector of $R(V_J)$ is at most 1, we have

$$d_{R(V_J)}(R(V_I)) = 0 \text{ or } 1.$$

If I is a subquiver of J , then it is obvious that $R(V_J) = R(V_I)$, and hence $d_{R(V_J)}(R(V_I)) = 1$. Conversely, if $d_{R(V_J)}(R(V_I)) = 1$, then $\dim R(V_J)(x) = \dim R(V_I)(x) \neq 0$ for all $x \in \{x_i, x_j, y_k, y_l\}$, and hence $\{x_i, x_j, y_k, y_l\} \subseteq J_0$. Therefore, $I_0 \subseteq J_0$, and I is a subquiver of J . As a consequence,

$$d_{R(V_J)}(R(V_I)) = \begin{cases} 1 & \text{if } I \leq J, \\ 0 & \text{otherwise.} \end{cases}$$

This together with Equation (5.8) proves the assertion. \square

That is, the compressed multiplicity c_M^ξ with respect to ξ of M can be expressed in terms of a formula involving only the multiplicities of the intervals in an interval resolution of M . Following the ideas in previous works [20, 2], we use Möbius inversion to obtain another invariant.

Definition 5.5 (Interval approximation). Recall that $c_M^\xi(I) := d_{R_{\xi_I}(M)}(R_{\xi_I}(V_I))$ for all $I \in \mathbb{I}$. We define *interval approximation* δ_M^ξ with respect to ξ to be the the Möbius inversion of c_M^ξ , which is defined by

$$\delta_M^\xi(J) := \sum_{S \subseteq \text{cov}(J)} (-1)^{\#S} c_M^\xi(\bigvee S)$$

for all $J \in \mathbb{I}$.

By the general theory of Möbius inversion, and since by Theorem 5.4

$$c_M^\xi(I) = \sum_{I \leq J \in \mathbb{I}} \left(\sum_{i=0}^r (-1)^i d_J^{(i)} \right),$$

we obtain the following.

Corollary 5.6. *Let M be in $\text{mod } A$ with an interval resolution (5.4). Then we have*

$$\delta_M^\xi(J) = \sum_{i=0}^r (-1)^i d_J^{(i)}$$

for all $J \in \mathbb{I}$.

Remark 5.7. Assume that M is interval-decomposable and that the interval resolution (5.4) is a minimal interval resolution. Then we have $d_J^{(i)} = 0$ for all $i \geq 1$ and all $J \in \mathbb{I}$, and $M \cong \bigoplus_{J \in \mathbb{I}} V_J^{d_J^{(0)}}$. Therefore, $\delta_M^\xi(J) = d_J^{(0)} = d_M(V_J)$ for all $J \in \mathbb{I}$.

That is, for M interval-decomposable, the value $\delta_M^\xi(J)$ of interval approximation at J is exactly equal to the multiplicity of the interval module V_J as a direct summand of M .

Furthermore, expressed in the framework of [6], our results Proposition 4.5 and Corollary 5.6 translates to the following Corollary 5.8.

Corollary 5.8. *Let $A = \mathbb{k}\vec{G}_{m,2}$, and let M be in $\text{mod } A$. Then, interval approximation with respect to ξ , which associates M to $\delta_M^\xi : \mathbb{I} \rightarrow \mathbb{Z}$, is the homological invariant relative to \mathbb{I} (and thus equivalent to the dim-hom invariant relative to \mathbb{I} by [6, Theorem 1.1]).*

Proof. Note first that \mathbb{I} contains all indecomposable projectives, and that $A = \mathbb{k}\vec{G}_{m,2}$ has a finite interval resolution global dimension by Proposition 4.5.

For $M \in \text{mod } A$ with the interval resolution

$$0 \rightarrow \bigoplus_{J \in \mathbb{I}} V_J^{d_J^{(r)}} \xrightarrow{f_r} \dots \xrightarrow{f_2} \bigoplus_{J \in \mathbb{I}} V_J^{d_J^{(1)}} \xrightarrow{f_1} \bigoplus_{J \in \mathbb{I}} V_J^{d_J^{(0)}} \xrightarrow{f_0} M \rightarrow 0,$$

the equivalence class of M in $K_0(A, \mathbb{I})$ is

$$[M] = \sum_{i=0}^r (-1)^i \left[\bigoplus_{J \in \mathbb{I}} V_J^{d_J^{(i)}} \right]$$

by the usual argument of breaking an exact sequence into short exact sequences. Thus,

$$[M] = \sum_{J \in \mathbb{I}} \sum_{i=0}^r (-1)^i d_J^{(i)} [V_J] = \sum_{J \in \mathbb{I}} \delta_M^\xi(J) [V_J]$$

by applying Corollary 5.6. Under the identifications $K_0(A, \mathbb{I}) \cong \mathbb{Z}^{|\mathbb{I}|} \cong \mathbb{Z}^{\mathbb{I}}$ (functions from \mathbb{I} to \mathbb{Z}) using [6, Proposition 4.9], this simply means that $[M] = \delta_M^\xi$. Thus, the canonical quotient map

$$\pi : K_0^{\text{split}}(A) \rightarrow K_0(A, \mathbb{I})$$

takes the equivalence class of $M \in \text{mod } A$ to δ_M^ξ . □

6. Discussion

We have shown that for an arbitrary finite poset the interval resolution global dimension (int-res-gldim) of its incidence algebra is finite (Proposition 4.5) and provided a formula to compute int-res-gldim as the maximum of a finite number of terms (Proposition 4.6). Beyond its applications in persistence, we hope that our results will be of independent interest in the representation theory of algebras.

Removing some of the finiteness conditions may provide interesting (and useful) settings for further investigation.

- (1) What can be said about the resolution global dimension relative to other classes of indecomposable modules \mathcal{X} with an infinite number of elements? In the setting of finite posets, there are only a finite number of interval modules (up to isomorphism). Our proof of the finiteness of int-res-gldim (Proposition 4.5) relies heavily on this finiteness (from which follows that $X = G = \bigoplus_{I \in \mathbb{I}(\mathcal{P})} V_I$ is finitely generated) in order to apply Theorem 4.2 (cited from [23, Theorem in §5]).
- (2) What can be said in the setting of locally finite posets, or posets in general? We have obtained some conjectures about the value of the interval resolution global dimension of the 2D commutative grids $\vec{G}_{m,n}$ (Conjectures 4.8 and 4.9). They also suggest that the interval resolution global dimension is not bounded as a function of (m, n) if both m and n are allowed to vary. This suggests that a commutative grid infinite in both axes will have infinite int-res-gldim.

In the $2 \times n$ commutative grid case, we provided a new invariant (the modified compressed multiplicity) that is intimately related to an alternating sum of the terms appearing in the interval resolution (Theorem 5.4). This is related to the result relating the rank invariant to the projective resolutions in the rank-exact structure [8, Section 4].

Furthermore, the invariant obtained via Möbius inversion of the modified compressed multiplicity was shown to be a homological invariant (and thus a dim-hom invariant) relative to the intervals (Corollary 5.8). Since the compressed multiplicity can be seen as a further modification of the generalized rank invariant [20], and since the generalized rank invariant was shown to not be a dim-hom invariant [6, Corollary 7.10], our results in the $2 \times n$ case may point to potential modifications to obtain a generalized rank invariant that is also a dim-hom invariant in the general case.

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