Recent advances on Schrödinger equations with dissipative nonlinearities

Chunhua Li * Yoshinori Nishii† Yuji Sagawa ‡ Hideaki Sunagawa §

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Dedicated to Professor Tohru Ozawa on the occasion of his sixtieth birthday

Abstract: We give a survey on recent developments on nonlinear Schrödinger equations with dissipative structure based on the authors' recent works.

1 Introduction

This paper is intended to be a survey on recent advances on nonlinear Schrödinger equations with dissipative structure based on the authors' recent works [21], [22], [23], [24]. We refer the readers to these papers and the references cited therein for the detailed expositions.

This paper is organized as follows. In Section 2, we summarize typical previous results for nonlinear Schrödinger equations with the power-type nonlinearities. Section 3 is devoted to the case where the nonlinear term depends also on the derivative of the unknown function. Special attentions are paid to the weakly dissipative nonlinearities which never appear in the power-type nonlinearity situation. In Section 4, we focus our attentions on a two-component Schrödinger system which tells us that the system case is much more delicate than the single case. Finally, we enumerate the results obtained in [25] concerning general nonlinear Schrödinger systems of derivative type in the Appendix. Throughout this paper, we denotes by $\mathcal{L}$ the standard free Schrödinger operator $i\partial_t + \frac{1}{2}\partial_x^2$ for $(t,x) \in \mathbb{R}\times\mathbb{R}$ with $i=\sqrt{-1}$. The free evolution group $e^{\frac{it}{2}\partial_x^2}$ is written as $\mathcal{U}(t)$. The function space H^k stands for the L^2-based Sobolev space of order k equipped with the norm $\|\phi\|_{H^k} = \sum_{0\le j\le k} \|\partial_x^j\phi\|_{L^2}$, and the weighted Sobolev space $H^{k,m}$ is defined by $\{\phi \in L^2 \mid \langle\cdot\rangle^m\phi \in H^k\}$ with $\langle x\rangle = \sqrt{1+x^2}$. Several non-negative constants will be denoted by the same letter C, unless otherwise specified.

*Department of Mathematics, College of Science, Yanbian University. 977 Gongyuan Road, Yanji, Jilin 133002, China. (E-mail: sxlch@ybu.edu.cn)

†Department of Mathematics, Tokyo University of Science. 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan. (E-mail: yoshinori.nishii@rs.tus.ac.jp)

‡Mathematical Science Education Center, Kumamoto University. 2-40-1 Kurokami, Chuo-ku, Kumamoto 860-8555, Japan.

§Department of Mathematics, Graduate School of Science, Osaka Metropolitan University. 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan. (E-mail: sunagawa@omu.ac.jp)

2 Brief review on the basic facts

First of all, let us recall some of well-known results on large-time behavior of small data solutions to the cubic power-type nonlinear Schrödinger equation in the form

$$\mathcal{L}u = \lambda |u|^2 u, \qquad t > 0,\ x \in \mathbb{R}, \tag{2.1}$$

where λ is a constant. What is interesting in (2.1) is that the large-time behavior of the solution is actually affected by the coefficient λ even if the initial data is sufficiently small, smooth and decaying fast as $|x| \to \infty$. If $\lambda \in \mathbb{R}$, it is shown by Ozawa [32] and Hayashi–Naumkin [5] that the solution to (2.1) with small data behaves like

$$u(t,x) = \frac{1}{\sqrt{it}} \alpha(x/t) e^{i\{\frac{x^2}{2t} - \lambda |\alpha(x/t)|^2 \log t\}} + o(t^{-1/2}) \quad \text{as} \ \ t \to +\infty$$

with a suitable $\mathbb{C}$-valued function $\alpha(y)$. An important consequence of this asymptotic expression is that the solution decays like $O(t^{-1/2})$ in $L^\infty(\mathbb{R}_x)$, while it does not behave like free solutions unless $\lambda = 0$. In other words, the additional logarithmic factor in the phase reflects the long-range character of the cubic nonlinear Schrödinger equations in one space dimension. If $\lambda \in \mathbb{C}\backslash\mathbb{R}$ in (2.1), another kind of long-range effect can be observed. For instance, according to [37] (see also [19], [10], [3], etc.), the small data solution $u(t,x)$ to (2.1) decays like $O(t^{-1/2}(\log t)^{-1/2})$ in $L^\infty(\mathbb{R}_x)$ as $t \to +\infty$ if $\operatorname{Im}\lambda < 0$. This gain of additional logarithmic time decay should be interpreted as another kind of long-range effect (see also [1], [2], [3], [4], [6], [7], [8], [10], [11], [13], [15], [17], [19], [21], [22], [25], [31], [35], [36], and so on). Time decay in L^2-norm is also investigated by several authors. Among others, it is pointed out by Kita-Sato [18] that the optimal L^2-decay rate is $O((\log t)^{-1/2})$ in the case of (2.1) with $\operatorname{Im}\lambda < 0$. We are interested in extending L^2-decay results of this kind to derivative nonlinearity case or system case.

3 Nonlinear Schrödinger equations of derivative type: Weak dissipativity

In this section, we focus on the initial value problem in the form

$$\mathcal{L}u = N(u, \partial_x u), \qquad t > 0,\ x \in \mathbb{R} \tag{3.1}$$

with

$$u(0,x) = \varphi(x), \qquad x \in \mathbb{R}, \tag{3.2}$$

where φ is a prescribed $\mathbb{C}$-valued function on $\mathbb{R}$. The nonlinear term $N(u, \partial_x u)$ is a cubic homogeneous polynomial in $(u, \overline{u}, \partial_x u, \overline{\partial_x u})$ with complex coefficients. If φ is $O(\varepsilon)$ in $H^3 \cap H^{2,1}$ with $0 < \varepsilon \ll 1$, what we can expect for general cubic nonlinear Schrödinger equations in $\mathbb{R}$ is the lower estimate for the lifespan T_ε in the form $T_\varepsilon \geq \exp(c/\varepsilon^2)$ with some $c > 0$ not depending on ε, and this is best possible in general (see [14] for an example of small data blow-up). More precise information on the lifespan is available under the restriction

$$N(e^{i\theta}, 0) = e^{i\theta} N(1, 0), \qquad \theta \in \mathbb{R} \tag{3.3}$$

and the initial condition

$$u(0, x) = \varepsilon \psi(x), \qquad x \in \mathbb{R}, \tag{3.4}$$

instead of (3.2), where $\psi \in H^3 \cap H^{2,1}$ is independent of ε. In fact we have the following.

Theorem 3.1 ([33], [38], [39]). *Assume that* $\psi \in H^3 \cap H^{2,1}$. *Suppose that the nonlinear term* N *satisfies* (3.3). *Let* T_ε *be the supremum of* $T > 0$ *such that the initial value problem* (3.1)–(3.4) *admits a unique solution in* $C([0, T); H^3 \cap H^{2,1})$. *Then it holds that*

$$\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon \geq \frac{1}{2 \sup_{\xi \in \mathbb{R}} (|\hat{\psi}(\xi)|^2 \mathrm{Im}\, \nu(\xi))} \tag{3.5}$$

with the convention $1/0 = +\infty$, *where the function* $\nu : \mathbb{R} \to \mathbb{C}$ *is defined by*

$$\nu(\xi) = \frac{1}{2\pi i} \oint_{|z|=1} N(z, i\xi z) \frac{dz}{z^2}, \quad \xi \in \mathbb{R}, \tag{3.6}$$

and $\hat{\psi}$ *denotes the Fourier transform of* ψ, *i.e.,*

$$\hat{\psi}(\xi) = \mathcal{F}\psi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iy\xi} \psi(y)\, dy, \quad \xi \in \mathbb{R}.$$

Note that (3.3) excludes just the worst terms u^3, $|u|^2\overline{u}$, $\overline{u}^3$. It is known that these three terms are quite difficult to handle in the present setting, and we do not pursue this case here (cf. [28]).

In view of the right-hand side in (3.5), it may be natural to expect that the sign of $\mathrm{Im}\, \nu(\xi)$ has something to do with global behavior of small data solutions to (3.1). In fact, it has been pointed out in [33] that typical results on small data global existence and large-time asymptotic behavior for (3.1) under (3.3) can be summarized in terms of $\mathrm{Im}\, \nu(\xi)$ as follows:

- Small data global existence holds in $C([0, \infty); H^3 \cap H^{2,1})$ under the condition

  $$\mathrm{Im}\, \nu(\xi) \leq 0, \quad \xi \in \mathbb{R}. \tag{$\mathbf{A}$}$$

  (See also Theorem A.1 in Appendix.)

- The global solution has (at most) logarithmic phase correction if

  $$\mathrm{Im}\, \nu(\xi) = 0, \quad \xi \in \mathbb{R}. \tag{$\mathbf{A}_0$}$$

  Also it is not difficult to see that there is no L^2-decay under ($\mathbf{A}_0$) for generic initial data of small amplitude.

- L^2-decay of the global solution occurs under the condition

$$\sup_{\xi \in \mathbb{R}} \operatorname{Im} \nu(\xi) < 0. \qquad (\mathbf{A}_+)$$

(See also Theorem A.2 in Appendix.)

Note that $\nu(\xi) = \lambda$ if $N = \lambda |u|^2 u$. So these results cover the results in the power-type nonlinearity case mentioned in Section 2. However, as pointed out in [23], an interesting case is not covered by these classifications, that is the case where $(\mathbf{A})$ is satisfied but $(\mathbf{A}_0)$ and $(\mathbf{A}_+)$ are violated. For example, if $N = -i|\partial_x u|^2 u$, we can easily check that $\operatorname{Im} \nu(\xi) = -\xi^2 \leq 0$, while the inequality is not strict because of vanishing at $\xi = 0$. This is what we are interested in.

To going further, let us remember the fact that, if $(\mathbf{A})$ is satisfied but $(\mathbf{A}_0)$ and $(\mathbf{A}_+)$ are violated, then there exist $c_0 > 0$ and $\xi_0 \in \mathbb{R}$ such that $\operatorname{Im} \nu(\xi) = -c_0(\xi - \xi_0)^2$. The converse is also true. This fact naturally leads us to the following definition of *the weak dissipativity.*

Definition 3.2. We say that a cubic nonlinear term N is *weakly dissipative* if the following two conditions (i) and (ii) are satisfied:

(i) $N(e^{i\theta}, 0) = e^{i\theta} N(1, 0)$ for $\theta \in \mathbb{R}$.

(ii) There exist $c_0 > 0$ and $\xi_0 \in \mathbb{R}$ such that $\operatorname{Im} \nu(\xi) = -c_0(\xi - \xi_0)^2$.

The following two results reveal the L^2-decay property in the weakly dissipative case.

Theorem 3.3 ([24]). *Suppose that N is weakly dissipative and that $\varepsilon = \|\varphi\|_{H^3 \cap H^{2,1}}$ is sufficiently small. Then there exists a positive constant C, not depending on ε, such that the global solution u to* (3.1)–(3.2) *satisfies*

$$\|u(t)\|_{L^2_x} \leq \frac{C\varepsilon}{(1 + \varepsilon^2 \log(t + 1))^{1/4}}$$

for $t \geq 0$.

Theorem 3.4 ([24]). *Suppose that N is weakly dissipative and that the Fourier transform of ψ does not vanish at the point ξ_0 coming from* (ii) *in* Definition 3.2. *Then we can choose $\varepsilon_0 > 0$ such that the global solution u to* (3.1)–(3.4) *satisfies*

$$\liminf_{t \to +\infty} ((\log t)^{1/4} \|u(t)\|_{L^2_x}) > 0$$

for $\varepsilon \in (0, \varepsilon_0]$.

Remark 3.5. According to [18], the optimal L^2-decay rate is $O((\log t)^{-1/2})$ in the case where $N = \lambda |u|^2 u$ with $\operatorname{Im} \lambda < 0$. This should be contrasted with Theorems 3.3 and 3.4, because these tell us that the optimal L^2-decay rate in the weakly dissipative case is $O((\log t)^{-1/4})$.

Now, let us explain heuristically why L^2-decay rate should be $O((\log t)^{-1/4})$ if $\hat{\psi}(\xi_0) \neq 0$. For this purpose, let us first remember the fact that the solution u^0 to the free Schrödinger equation (i.e., the case of $N = 0$) behaves like

$$\partial_x^k u^0(t,x) \sim \left(\frac{ix}{t}\right)^k \frac{e^{-i\pi/4}}{\sqrt{t}} \hat{\varphi}\left(\frac{x}{t}\right) e^{i\frac{x^2}{2t}} + \cdots$$

as $t \to +\infty$ for $k = 0, 1, 2, \ldots$. Viewing it as a rough approximation of the solution u for (3.1), we may expect that $\partial_x^k u(t,x)$ could be better approximated by

$$\left(\frac{ix}{t}\right)^k \frac{1}{\sqrt{t}} A\left(\log t, \frac{x}{t}\right) e^{i\frac{x^2}{2t}}$$

with a suitable function $A(\tau, \xi)$, where $\tau = \log t$, $\xi = x/t$ and $t \gg 1$. Note that

$$A(0, \xi) = e^{-i\pi/4}\, \hat{\varphi}(\xi)$$

and that the extra variable $\tau = \log t$ is responsible for possible long-range nonlinear effect. Substituting the above expression into (3.1) and keeping only the leading terms, we can see (at least formally) that $A(\tau, \xi)$ should satisfy the ordinary differential equation

$$i\partial_\tau A = \nu(\xi)|A|^2 A + \cdots$$

under (3.3). If N is weakly dissipative, we see that

$$\partial_\tau |A|^2 = -2c_0(\xi - \xi_0)^2 |A|^4 + \cdots.$$

Then it follows that

$$|A(\tau, \xi)|^2 = \frac{|\hat{\varphi}(\xi)|^2}{1 + 2c_0(\xi - \xi_0)^2 |\hat{\varphi}(\xi)|^2 \tau} + \cdots,$$

whence

$$\|u(t)\|_{L^2_x} \sim \|A(\log t)\|_{L^2_\xi} \sim \left(\int_{\mathbb{R}} \frac{|\hat{\varphi}(\xi)|^2}{1 + 2c_0(\xi - \xi_0)^2 |\hat{\varphi}(\xi)|^2 \log t}\, d\xi\right)^{1/2} \quad (t \to +\infty).$$

By considering the behavior as $t \to +\infty$ of this integral carefully, we see that L^2-decay rate in the weakly dissipative case should be just $O((\log t)^{-1/4})$ if $\hat{\varphi}(\xi_0) \neq 0$. Indeed, we have the following lemma.

Lemma 3.6 ([24])**.** *Let $\theta \in L^\infty$ and $\xi_0 \in \mathbb{R}$. We set*

$$S(\tau) = \int_{\mathbb{R}} \frac{|\theta(\xi)|^2}{1 + (\xi - \xi_0)^2 |\theta(\xi)|^2 \tau}\, d\xi$$

for $\tau \geq 1$.

(1) *We have*

$$S(\tau) \le 4\|\theta\|_{L^\infty}\tau^{-1/2}, \qquad \tau \ge 1.$$

(2) *Assume that there exists an open interval I with $I \ni \xi_0$ such that $\inf_{\xi\in I}|\theta(\xi)| > 0$. Then we can choose a positive constant C_*, which is independent of $\tau \ge 1$ (but may depend on θ and ξ_0), such that*

$$S(\tau) \ge C_*\tau^{-1/2}, \qquad \tau \ge 1.$$

Our strategy of the proof of Theorems 3.3 and 3.4 is to justify the above heuristic argument, which has been carried out in [23] and [24]. The key is to concentrate on the function

$$\alpha(t,\xi) = \mathcal{F}[\mathcal{U}(-t)u(t,\cdot)](\xi),$$

which is expected to play the role of $A(\log t, \xi)$ in the above argument. For the details, see [23] and [24].

4 A two-component system of nonlinear Schrödinger equations

In this section, we turn our attentions to the system case. Our goal is to reveal that the system case is much more delicate than the single case by considering the specific two-component system

$$\begin{cases} \mathcal{L}u_1 = -i|u_2|^2u_1, \\ \mathcal{L}u_2 = -i|u_1|^2u_2, \end{cases} \quad (t,x) \in (0,\infty)\times\mathbb{R} \tag{4.1}$$

under the initial condition

$$u_j(0,x) = \varphi_j^0(x), \qquad x\in\mathbb{R},\ j=1,2. \tag{4.2}$$

When $\varphi_1^0 = \varphi_2^0$, the system (4.1)–(4.2) is reduced to the single equation (2.1) with $\lambda = -i$, so we can adapt the previous approach to see that $\|u(t)\|_{L^2_x} \to 0$ as $t\to+\infty$. However, as we will see below, this is an exceptional case. It turns out that highly non-trivial behavior can be observed in (4.1)–(4.2) for generic small initial data.

4.1 The initial value problem for (4.1) with generic small data

We start the discussion with the following basic result.

Theorem 4.1 ([21])**.** *Suppose that $\varphi^0 = (\varphi_1^0,\varphi_2^0) \in H^2\cap H^{1,1}$ and that $\|\varphi^0\|_{H^2\cap H^{1,1}}$ is suitably small. Let $u = (u_1,u_2) \in C([0,\infty); H^2\cap H^{1,1})$ be the solution to (4.1)–(4.2). Then there exists $\varphi^+ = (\varphi_1^+,\varphi_2^+) \in L^2$ with $\hat{\varphi}^+ = (\hat{\varphi}_1^+,\hat{\varphi}_2^+) \in L^\infty$ such that*

$$\lim_{t\to+\infty}\|u_j(t) - \mathcal{U}(t)\varphi_j^+\|_{L^2_x} = 0, \quad j=1,2. \tag{4.3}$$

Moreover we have

$$\hat{\varphi}_1^+(\xi)\cdot\hat{\varphi}_2^+(\xi) = 0, \quad \xi\in\mathbb{R}. \tag{4.4}$$

The global existence part of this asssertion is just a special case of more general result (see Theorem A.1 below). On the other hand, we emphasize that (4.4) should be regarded as a consequence of non-trivial long-range nonlinear interactions because such a phenomenon does not occur in the usual short-range situation. Note also that the system (4.1) possesses two conservation laws

$$\frac{d}{dt}\Big(\|u_1(t)\|_{L_x^2}^2 + \|u_2(t)\|_{L_x^2}^2\Big) = -4\int_{\mathbb{R}} |u_1(t,x)|^2 |u_2(t,x)|^2\, dx$$

and

$$\frac{d}{dt}\Big(\|u_1(t)\|_{L_x^2}^2 - \|u_2(t)\|_{L_x^2}^2\Big) = 0. \tag{4.5}$$

However, these are not enough to assert that the solution $u = (u_1, u_2)$ is asymptotically free in the sense of (4.3). It is worthwhile to mention that (4.5) tells us that at least one component $u_1(t)$ or $u_2(t)$ does not decay as $t \to +\infty$ in $L^2(\mathbb{R}_x)$ if $\|\varphi_1^0\|_{L^2} \neq \|\varphi_2^0\|_{L^2}$. In particular, it is far from obvious whether or not both $u_1(t)$ and $u_2(t)$ can behave like non-trivial free solutions as $t \to +\infty$. That is why we are interested in (non-)triviality of each component of the scattering state.

4.2 Criteria for (non-)triviality of the scattering state

To investigate the relation (4.4) in more detail, let us point out that we also have the following proposition.

Proposition 4.2 ([21]). *We put* $\varphi_j^+ = \lim_{t\to+\infty} \mathcal{U}(-t)u_j(t)$ *in* L^2, $j = 1, 2$, *for the global solution* $u = (u_1, u_2)$ *to* (4.1)–(4.2), *whose existence is guaranteed by* Theorem 4.1. *There exists a function* $m : \mathbb{R} \to \mathbb{R}$ *such that the following holds for each* $\xi \in \mathbb{R}$

- $m(\xi) > 0$ *implies* $\hat{\varphi}_1^+(\xi) \neq 0$ *and* $\hat{\varphi}_2^+(\xi) = 0$;
- $m(\xi) < 0$ implies $\hat{\varphi}_1^+(\xi) = 0$ and $\hat{\varphi}_2^+(\xi) \neq 0$;
- $m(\xi) = 0$ implies $\hat{\varphi}_1^+(\xi) = \hat{\varphi}_2^+(\xi) = 0$.

In fact, $m(\xi)$ *has the following expression:*

$$m(\xi) = |\alpha_1(2,\xi)|^2 - |\alpha_2(2,\xi)|^2 + \int_2^\infty \rho(\tau,\xi)\,d\tau,$$

where

$$\alpha_j(t,\xi) = \mathcal{F}\big[\mathcal{U}(-t)u_j(t,\cdot)\big](\xi), \tag{4.6}$$

$$\rho(t,\xi) = 2\mathrm{Re}\left[\overline{\alpha_1(t,\xi)}R_1(t,\xi) - \overline{\alpha_2(t,\xi)}R_2(t,\xi)\right],$$

$$R_1 = \frac{1}{t}|\alpha_2|^2\alpha_1 - \mathcal{F}\mathcal{U}(-t)\big[|u_2|^2 u_1\big], \quad R_2 = \frac{1}{t}|\alpha_1|^2\alpha_2 - \mathcal{F}\mathcal{U}(-t)\big[|u_1|^2 u_2\big]. \tag{4.7}$$

Note that (4.4) follows from Propisition 4.2 immediately. In other words, Propisition 4.2 is more precise than the relation (4.4), and the function $m(\xi)$ plays an important role in it. This indicates that better understanding of $m(\xi)$ will bring us more precise information on the scattering state φ^+. To address this point, we put a small parameter ε in front of the initial data to distinguish information on the amplitude from the others, that is, we replace the initial condition (4.2) by

$$u_j(0,x) = \varepsilon\psi_j(x), \quad j = 1,2, \tag{4.8}$$

where $\psi_j \in H^2 \cap H^{1,1}$ is independent of ε. Then we have the following.

Theorem 4.3 ([22])**.** *Let m be the function given in* Proposition 4.2 *with the initial condition* (4.2) *replaced by* (4.8)*. We have*

$$m(\xi) = \varepsilon^2\big(|\hat{\psi}_1(\xi)|^2 - |\hat{\psi}_2(\xi)|^2\big) + O(\varepsilon^4)$$

as $\varepsilon \to +0$ uniformly in $\xi \in \mathbb{R}$.

As a consequence of Theorem 4.3, we have the following criteria for (non-)triviality of the scattering state $\varphi^+ = (\varphi_1^+, \varphi_2^+)$ for the initial value problem (4.1)–(4.8).

Corollary 4.4 ([22])**.** *Assume that there exist points $\xi^* \in \mathbb{R}$ and $\xi_* \in \mathbb{R}$ such that*

$$|\hat{\psi}_1(\xi^*)| > |\hat{\psi}_2(\xi^*)| \tag{4.9}$$

and

$$|\hat{\psi}_1(\xi_*)| < |\hat{\psi}_2(\xi_*)|, \tag{4.10}$$

respectively. Then, for sufficiently small ε, we have $\|\varphi_1^+\|_{L^2} > 0$ and $\|\varphi_2^+\|_{L^2} > 0$.

Corollary 4.5 ([22])**.** *Assume that*

$$|\hat{\psi}_1(\xi)| > |\hat{\psi}_2(\xi)| \tag{4.11}$$

for all $\xi \in \mathbb{R}$. Then, for sufficiently small ε, φ_2^+ vanishes almost everywhere on $\mathbb{R}$, while $\|\varphi_1^+\|_{L^2} > 0$.

It follows from (4.3) and Corollary 4.4 that both $u_1(t)$ and $u_2(t)$ behave like non-trivial free solutions as $t \to +\infty$. In particular, we see that L^2 decay does not occur for $u_1(t)$ and $u_2(t)$ under (4.9) and (4.10). To the contrary, Corollary 4.5 tells us that only the second component $u_2(t)$ is dissipated as $t \to +\infty$ in the sense of L^2 under (4.11). We emphasize again that such phenomena do not occur in the usual short-range settings. In this sense, the dynamics for the system (4.1) is much more delicate than that for the single Schrödinger equation (2.1) with a dissipative cubic nonlinear term.

At the end of this subsection, let us mention the sketch of the proof of Theorem 4.3 briefly. The key is to focus on the function α_j given by (4.6). By the reduction similar to that in the previous section, we see that the leading part of u_j as $t \to +\infty$ can be given by

$$\frac{1}{\sqrt{it}}\alpha_j\left(t, \frac{x}{t}\right) e^{i\frac{x^2}{2t}}$$

and that the evolution of $\alpha = (\alpha_1, \alpha_2)$ is governed by the system

$$\partial_t \alpha_1 = -\frac{|\alpha_2|^2}{t}\alpha_1 + R_1, \qquad \partial_t \alpha_2 = -\frac{|\alpha_1|^2}{t}\alpha_2 + R_2,$$

where R_1 and R_2 are given by (4.7). If R_1 and R_2 are shown to be harmless, we have

$$\sup_{\xi\in\mathbb{R}}\left| m(\xi) - \big(|\alpha_1(2,\xi)|^2 - |\alpha_2(2,\xi)|^2\big)\right| \le C\varepsilon^4. \tag{4.12}$$

Moreover we can show that

$$\begin{aligned}\alpha_j(2,\xi) &= \hat{u}_j(0,\xi) - i\int_0^2 \mathcal{F}\big[\mathcal{U}(-t)\mathcal{L}u_j(t,\cdot)\big](\xi)\,dt \\ &= \varepsilon\hat{\psi}_j(\xi) + O(\varepsilon^3)\end{aligned} \tag{4.13}$$

as $\varepsilon \to +0$, uniformly in $\xi \in \mathbb{R}$, $j = 1, 2$, provided that we have a good control of u. By (4.12) and (4.13), we reach the conclusion. For the technical details, see [21] and [22] (see also [30] and [29] for closely related works on the wave equation case.)

4.3 The final state problem for (4.1)

To see the role of the relation (4.4) from a different angle, let us consider the final state problem for (4.1), that is, finding a solution $u = (u_1, u_2)$ to (4.1) which satisfies

$$\lim_{t\to+\infty} \|u_j(t) - \mathcal{U}(t)\psi_j^+\|_{L_x^2} = 0, \qquad j = 1, 2 \tag{4.14}$$

for a prescribed final state $\psi^+ = (\psi_1^+, \psi_2^+)$. Roughly speaking, the propositions below imply that (4.14) holds if and only if

$$\hat{\psi}_1^+(\xi)\cdot\hat{\psi}_2^+(\xi) = 0, \quad \xi\in\mathbb{R}. \tag{4.15}$$

Remember that (4.14) should hold in the short-range case regardless of whether (4.15) is true or not. In other words, our problem must be distinguished from the usual short-range situation.

The precise statements are as follows.

Proposition 4.6 ([21]). *Let $T_0 \ge 1$ be given, and let u be a solution to* (4.1) *for $t \ge T_0$ satisfying*

$$\sup_{t\ge T_0}\Big(t^{-\gamma}\|\mathcal{U}(-t)u(t)\|_{H_x^{1,1}} + \|\mathcal{F}\mathcal{U}(-t)u(t)\|_{L_\xi^\infty}\Big) < \infty$$

with some $\gamma \in (0, 1/12)$. If there exists $\psi^+ \in L^2$ with $\hat{\psi}^+ \in L^\infty$ such that (4.14) *holds, then we must have* (4.15).

Proposition 4.7 ([21])**.** *Suppose that* ψ^+ *satisfies* $\hat{\psi}^+ \in H^{0,s} \cap L^\infty$ *with some* $s > 1$*, and that* $\delta = \|\hat{\psi}^+\|_{L^\infty}$ *is suitably small. If* (4.15) *holds, then there exist* $T \geq 1$ *and a unique solution* u *to* (4.1) *for* $t \geq T$ *satisfying* $\mathcal{U}(-t)u \in C([T,\infty); H^{0,1})$ *and* (4.14)*.*

The proof of Proposition 4.6 is based on a contradiction argument. Proposition 4.7 can be shown by rewriting the system (4.1) in the form of integral equations and applying a suitable fixed point argument. See [21] and [20] for the details of the proof.

A Appendix: General nonlinear Schrödinger systems of derivative type

For the convenience of the readers, we collect the results obtained in [25] without proof. We consider general n-component nonlinear Schrödinger systems in the form

$$\begin{cases} \mathcal{L}_{m_j} u_j = N_j(u, \partial_x u), & t > 0,\ x \in \mathbb{R},\ j = 1, \ldots, n, \\ u_j(0,x) = \varphi_j(x), & x \in \mathbb{R},\ j = 1, \ldots, n, \end{cases} \tag{A.1}$$

where $\mathcal{L}_{m_j} = i\partial_t + \frac{1}{2m_j}\partial_x^2$, $m_j \in \mathbb{R}\backslash\{0\}$, and $u = (u_j(t,x))_{1\leq j\leq n}$ is a $\mathbb{C}^n$-valued unknown function. The nonlinear term $N = (N_j)_{1\leq j \leq n}$ is assumed to be a cubic homogeneous polynomial in $(u, \partial_x u, \overline{u}, \overline{\partial_x u})$. We set $I_n = \{1, \ldots, n\}$ and $I_n^\sharp = \{1, \ldots, n, n+1, \ldots, 2n\}$. For $z = (z_j)_{j\in I_n} \in \mathbb{C}^n$, we write

$$z^\sharp = (z_k^\sharp)_{k\in I_n^\sharp} := (z_1, \ldots, z_n, \overline{z_1}, \ldots, \overline{z_n}) \in \mathbb{C}^{2n}.$$

Then general cubic nonlinear term $N = (N_j)_{j\in I_n}$ can be written as

$$N_j(u, \partial_x u) = \sum_{l_1,l_2,l_3=0}^{1} \sum_{k_1,k_2,k_3\in I_n^\sharp} C_{j,k_1,k_2,k_3}^{l_1,l_2,l_3} (\partial_x^{l_1} u_{k_1}^\sharp)(\partial_x^{l_2} u_{k_2}^\sharp)(\partial_x^{l_3} u_{k_3}^\sharp)$$

with suitable $C_{j,k_1,k_2,k_3}^{l_1,l_2,l_3} \in \mathbb{C}$. With this expression of N, we define $p = (p_j(\xi;Y))_{j\in I_n} : \mathbb{R}\times\mathbb{C}^n \to \mathbb{C}^n$ by

$$p_j(\xi;Y) := \sum_{l_1,l_2,l_3=0}^{1} \sum_{k_1,k_2,k_3\in I_n^\sharp} C_{j,k_1,k_2,k_3}^{l_1,l_2,l_3} (i\tilde{m}_{k_1}\xi)^{l_1}(i\tilde{m}_{k_2}\xi)^{l_2}(i\tilde{m}_{k_3}\xi)^{l_3} Y_{k_1}^\sharp Y_{k_2}^\sharp Y_{k_3}^\sharp$$

for $\xi \in \mathbb{R}$ and $Y = (Y_j)_{j\in I_n} \in \mathbb{C}^n$, where

$$\tilde{m}_k = \begin{cases} m_k & (k = 1, \ldots, n), \\ -m_{(k-n)} & (k = n+1, \ldots, 2n). \end{cases}$$

We denote by $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ the standard scalar product in $\mathbb{C}^n$, i.e.,

$$\langle z, w \rangle_{\mathbb{C}^n} = \sum_{j=1}^{n} z_j \overline{w_j}$$

for $z = (z_j)_{j\in I_n}$ and $w = (w_j)_{j\in I_n} \in \mathbb{C}^n$.

With these notations, let us introduce the following conditions:

(a) For all $j \in I_n$ and $k_1, k_2, k_3 \in I_n^\sharp$,

$$m_j \neq \tilde{m}_{k_1} + \tilde{m}_{k_2} + \tilde{m}_{k_3} \ \text{ implies } \ C^{l_1,l_2,l_3}_{j,k_1,k_2,k_3} = 0, \quad l_1, l_2, l_3 \in \{0, 1\}.$$

(b_0) There exists an $n \times n$ positive Hermitian matrix $\mathcal{H}$ such that

$$\operatorname{Im} \langle p(\xi; Y), \mathcal{H}Y\rangle_{\mathbb{C}^n} \leq 0$$

for all $(\xi, Y) \in \mathbb{R} \times \mathbb{C}^n$.

(b_1) There exist an $n\times n$ positive Hermitian matrix $\mathcal{H}$ and a positive constant C_* such that

$$\operatorname{Im} \langle p(\xi; Y), \mathcal{H}Y\rangle_{\mathbb{C}^n} \leq -C_*|Y|^4$$

for all $(\xi, Y) \in \mathbb{R} \times \mathbb{C}^n$.

(b_2) There exist an $n \times n$ positive Hermitian matrix $\mathcal{H}$ and a positive constant C_{**} such that

$$\operatorname{Im} \langle p(\xi; Y), \mathcal{H}Y\rangle_{\mathbb{C}^n} \leq -C_{**}\langle\xi\rangle^2|Y|^4$$

for all $(\xi, Y) \in \mathbb{R} \times \mathbb{C}^n$.

(b_3) $p(\xi; Y) = 0$ for all $(\xi, Y) \in \mathbb{R} \times \mathbb{C}^n$.

We have the following.

Theorem A.1 ([25])**.** *Assume the conditions* (a) *and* (b_0) *are satisfied. Let* $\varphi = (\varphi_j)_{j\in I_n} \in H^3 \cap H^{2,1}$*, and assume* $\varepsilon := \|\varphi\|_{H^3} + \|\varphi\|_{H^{2,1}}$ *is sufficiently small. Then* (A.1) *admits a unique global solution* $u = (u_j)_{j\in I_n} \in C([0,\infty); H^3 \cap H^{2,1})$*. Moreover we have*

$$\|u(t)\|_{L^\infty_x} \leq \frac{C\varepsilon}{\sqrt{1+t}}, \qquad \|u(t)\|_{L^2_x} \leq C\varepsilon$$

for $t \geq 0$*, where* C *is a positive constant not depending on* ε*.*

Theorem A.2 ([25], [23])**.** *Assume the conditions* (a) *and* (b_1) *are satisfied. Let* u *be the global solution to* (A.1)*, whose existence is guaranteed by* Theorem A.1*. Then we have*

$$\|u(t)\|_{L^\infty_x} \leq \frac{C\varepsilon}{\sqrt{(1+t)\{1+\varepsilon^2\log(2+t)\}}}$$

for $t \geq 0$*, where* C *is a positive constant not depending on* ε*. We also have*

$$\|u(t)\|_{L^2_x} \leq \frac{C\varepsilon}{(1+\varepsilon^2\log(2+t))^{3/8-\delta}},$$

where $\delta > 0$ *can be taken arbitrarily small.*

Theorem A.3 ([25]). *Assume the conditions* (a) *and* (b_2) *are satisfied. Let u be as above. Then we have*

$$\|u(t)\|_{L^2_x} \leq \frac{C\varepsilon}{\sqrt{1+\varepsilon^2 \log(2+t)}}$$

for $t \geq 0$, where C is a positive constant not depending on ε.

Theorem A.4 ([25]). *Assume the conditions* (a) *and* (b_3) *are satisfied. Let u be as above. For each $j \in I_n$, there exists $\varphi_j^+ \in L^2(\mathbb{R}_x)$ with $\hat{\varphi}_j^+ \in L^\infty(\mathbb{R}_\xi)$ such that*

$$u_j(t) = e^{i\frac{t}{2m_j}\partial_x^2}\varphi_j^+ + O(t^{-1/4+\delta}) \quad \text{in } L^2(\mathbb{R}_x)$$

and

$$u_j(t,x) = \sqrt{\frac{m_j}{it}}\,\hat{\varphi}_j^+\left(\frac{m_j x}{t}\right)e^{i\frac{m_j x^2}{2t}} + O(t^{-3/4+\delta}) \quad \text{in } L^\infty(\mathbb{R}_x)$$

as $t \to +\infty$, where $\delta > 0$ can be taken arbitrarily small.

In the system case, many interesting problems are left unsolved. For more recent progress or related issues, we refer the readers to [9], [12], [11], [13], [16], [26], [27], [34], and so on.

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