Generalized F-signatures of the rings of invariants of finite group schemes

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Abstract

Let k be a perfect field of prime characteristic p, G a finite group scheme over k, and V a finite-dimensional G-module. Let $S = \operatorname{Sym} V$ be the symmetric algebra with the standard grading. Let M be a $\mathbb{Q}$-graded S-finite S-free (G,S)-module, and L be its S-reflexive graded (G,S)-submodule. Assume that the action of G on V is small in the sense that there exists some G-stable Zariski closed subset F of V of codimension two or more such that the action of G on $V \setminus F$ is free. Generalizing the result of P. Symonds and the first author, we describe the Frobenius limit $\mathrm{FL}(L^G)$ of the S^G-module L^G. In particular, we determine the generalized F-signature $s(M, S^G)$ for each indecomposable gradable reflexive S^G-module M. In particular, we prove the fact that the F-signature $s(S^G) = s(S^G, S^G)$ equals $1/\dim k[G]$ if G is linearly reductive (already proved by Watanabe–Yoshida, Carvajal-Rojas–Schwede–Tucker, and Carvajal-Rojas) and 0 otherwise (some important cases has already been proved by Broer, Yasuda, Liedtke–Martin–Matsumoto).

1. Introduction

Study of Frobenius maps and their iterations has long been important in studying Noetherian commutative rings of prime characteristic p [Hun].

For simplicity, let p be a prime number, k a perfect field of characteristic p, and A be a complete Noetherian local ring of characteristic p whose residue field is k (in the graded case, let $A = \bigoplus_{n \geq 0} A_n$ be a finitely generated positively graded k-algebra with $A_0 = k$) in this introduction.

Huneke and Leuschke [HL] defined the F-signature $s(A)$ of A using the iteration $F_A^e : A \to {}^eA$ of the Frobenius map, where ${}^eA = A$. We can decompose ${}^eA = A^{a_e} \oplus M_e$ as an A-module, where M_e does not have a free summand, and

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$s(A)$ is defined to be $\lim_{e\to\infty} a_e/p^{de}$, where $d = \dim A$. Tucker [Tuc] proved that the limit exists and $s(A)$ is well-defined. As Yao [Yao] pointed out, the F-signature agrees with the minimal relative Hilbert–Kunz multiplicity defined by Watanabe and Yoshida [WY]. A is regular if and only if $s(A) = 1$ [HL], [WY]. Moreover, $s(A) > 0$ if and only if A is strongly F-regular [AL].

Let G be a finite group and V a d-dimensional representation of G over k. Let $S = \operatorname{Sym} V$ be the symmetric algebra, and $A = S^G$. We say that the action of G on V is small if $G \to GL(V)$ is injective, and its image does not have a pseudo-reflection. Watanabe and Yoshida proved that if the action is small and G is linearly reductive (or equivalently, p does not divide the order $|G|$ of G), then $s(A) = 1/|G|$. From the result of Broer [Bro] and Yasuda [Yas], $s(A) = 0$ if G is small but not linearly reductive.

It is natural to ask the asymptotic behavior of other indecomposable summands than the free one of ${}^eA$. The first author and Yusuke Nakajima [HN] named it the generalized F-signature, and calculated them for the invariant subring A for the case that G is small and linearly reductive [HN]. After that, the first author and P. Symonds [HS] defined the Frobenius limit $\mathrm{FL}([A]) = \lim_{e\to\infty}[{}^eA]/p^{de}$ of $[A]$, where the limit is taken in certain normed space whose $\mathbb{R}$-basis is the set of isomorphism classes of indecomposable $\mathbb{Q}$-graded A-modules (up to shiftings), see (3.2). They proved that

$$\mathrm{FL}([A]) = \frac{1}{|G|}[S] = \frac{1}{|G|}\sum_{i=1}^{r}\frac{\dim_k V_i}{\dim_k \operatorname{End}_G V_i}[M_i],$$

where $V_1, \dots, V_r$ is the complete set of representatives of the isomorphism classes of simple G-modules, P_i the projective cover of V_i, and $M_i = (P_i \otimes_k S)^G$ [HS, Theorem 5.1]. This information is enough to deduce that the generalized F-signature $s(M_i, A) = \lim_{e\to\infty} c_{i,e}/p^{de}$ agrees with $\frac{\dim_k V_i}{|G| \dim_k \operatorname{End}_G V_i}$, where ${}^eA = M_i^{c_{i,e}} \oplus N_{i,e}$ such that $N_{i,e}$ does not have M_i as a direct summand. If k is algebraically closed, then we simply have that $s(M_i, A) = (\dim_k V_i)/|G|$.

The purpose of this paper is to extend these results on the (generalized) F-signature and the Frobenius limit to the case that G is a finite group scheme, rather than a constant finite group. We say that the action of G on V is small if there exists some closed subset F of $V/\!/G = \operatorname{Spec} k[V]^G$ of codimension two or more such that $\pi : V \setminus \pi^{-1}(F) \to V/\!/G \setminus F$ is a principal G-bundle (G-torsor), where $\pi : V \to V/\!/G$ is the canonical map. Note that the definition of the smallness is the natural generalization of the definition for the constant group G (that is, a faithful action without pseudo-reflection). Our main theorem is the following.

Theorem 3.16. *Let k be a perfect field of characteristic $p > 0$, G be a finite k-group scheme over k, and V a finite-dimensional G-module. Let $S = \operatorname{Sym} V$ be the symmetric algebra of V, and we assume that S is graded so that each* *element of V is homogeneous of degree one. Assume that the action of G on S is small. Let M be a $\mathbb{Q}$-graded S-finite S-free (G,S)-module, and L be its graded (G,S)-submodule which is reflexive as an S-module. Let $k = V_1, \dots, V_r$ be the simple G-modules, and let P_i be the projective cover of V_i. Then we have*

$$\mathrm{FL}(L^G) = \frac{\operatorname{rank}_S L}{\dim_k k[G]}[S''] = \frac{\operatorname{rank}_S L}{\dim_k k[G]} \sum_{i=1}^{r} \frac{\dim V_i}{\dim \operatorname{End}_G V_i}[(P_i \otimes_k S)^G],$$

where $S'' = (S \otimes_k k[G])^G$ is S viewed as an A-module, where $A = S^G$.

In particular, we have that $s(A) = 1/\dim_k k[G]$ if G is linearly reductive, and $s(A) = 0$ otherwise. This fact for the case that G is linearly reductive is deduced easily from [Car, Theorem 4.8]. The case that G is not linearly reductive ($s(A) = 0$, or A is not strongly F-regular) is proved in [LMM, Proposition 7.2] for very small actions.

In section 2, we review some basic facts for small actions of group schemes. Some of them are found in [Has3] in very general forms which are much more than we need here, and we gave shorter proofs for some of them here for convenience of readers. In section 3, we prove our main theorem and some corollaries.

After the first version of this paper put on arXiv, we got aware that [LY, Theorem 1.6] by Liedtke and Yasuda has some overlap with Corollary 3.24.

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2. Preliminaries

(2.1) Throughout this article, k denotes a field. For a k-scheme X, we denote by $k[X]$ the k-algebra $H^0(X, \mathcal{O}_X)$.

(2.2) Let G be an affine k-group scheme. For an affine G-scheme X, G acts on $k[X]$, the coordinate ring of X, by $(gf)(x) = f(g^{-1}x)$. In other words, letting $\psi : k[X] \to k[G] \otimes k[X]$ be the map corresponding to the action $G \times X \to X$, the comodule structure of $k[X]$ is the composite

$$k[X] \xrightarrow{\psi} k[G] \otimes k[X] \xrightarrow{T} k[X] \otimes k[G] \xrightarrow{1_{k[X]} \otimes \mathcal{S}} k[X] \otimes k[G],$$

where $T(\alpha \otimes f) = f \otimes \alpha$, and $\mathcal{S}$ is the antipode. Note that $k[X]$ is a $k[G]$-comodule algebra (k-algebra G-module such that the product $k[X] \otimes k[X] \to k[X]$ and the unit map $k \to k[X]$ are G-linear).

(2.3) Let G be a k-group scheme of finite type, and N its normal closed subgroup scheme. Let $f : X \to Y$ be a k-morphism between k-schemes of finite type on which G acts. We say that f is a G-enriched principal N-bundle if f is faithfully flat, G-equivariant, and locally a trivial N-bundle with respect to the fppf topology. As f itself is fppf (i.e., faithfully flat and of finite presentation), this is equivalent to say that the G-equivariant X-morphism $\Psi : N \times X \to X \times_Y X$ given by $\Psi(n, x) = (nx, x)$ is an isomorphism, where G acts on N by the conjugate action.

Lemma 2.4. *Assume that G, N, X, and Y are all affine, and $f : X \to Y$ is a G-enriched principal N-bundle. Then $\mathcal{G} := (-)^N : \mathrm{Mod}(G, k[X]) \to \mathrm{Mod}(G/N, k[Y])$ is an equivalence. The quasi-inverse is given by $\mathcal{F} := k[X] \otimes_{k[Y]} -$, and this is an equivalence of monoidal categories.*

Proof. It is easy to see that $\mathcal{G}$ is right adjoint to $\mathcal{F}$. Indeed,

$$\begin{aligned} \mathrm{Hom}_{G,k[X]}(k[X] \otimes_{k[Y]} L, M) &\cong \mathrm{Hom}_{k[Y]}(L, M)^G \\ &= (\mathrm{Hom}_{k[Y]}(L, M)^N)^{G/N} = \mathrm{Hom}_{G/N,k[Y]}(L, M^N) \end{aligned}$$

in a natural way for $L \in \mathrm{Mod}(G/N, k[Y])$ and $M \in \mathrm{Mod}(G, k[X])$.

To verify that the unit of adjunction $u : L \to (k[X] \otimes_{k[Y]} L)^N$ ($u(\alpha) = 1 \otimes \alpha$) and the counit of adjunction $\varepsilon : k[X] \otimes_{k[Y]} M^N \to M$ ($\varepsilon(f \otimes m) = fm$) are isomorphisms, we may and shall assume that $G = N$ and G/N is trivial.

As $k[Y] \to k[X]'$ is flat, where $k[X]'$ is the k algebra $k[X]$ with the trivial N-action, we have that $k[X]' \otimes_{k[Y]} (-)^N \to (k[X]' \otimes_{k[Y]} -)^N$ is an isomorphism between the functors $\mathrm{Mod}(N, k[Y]) \to \mathrm{Mod}\, k[X]'$. So taking the base change by $X' \to Y$, where X' is the k-scheme X with the trivial N-action, we may assume that $X = N \times Y$ is the trivial N-bundle.

As the counit of adjunction $\varepsilon : \mathcal{F}\mathcal{G} \to \mathrm{Id}$ is given by $\mathcal{F}\mathcal{G}M = k[X] \otimes_{k[Y]} M^N \cong k[N] \otimes_k M^N \to M = \mathrm{Id}M$, where the last map is given by $f \otimes m \mapsto fm$ (note that M is an $(N, k[N])$-module), which is an isomorphism, see [Swe, Theorem 4.1.1]. Conversely, the unit of adjunction $L \to \mathcal{G}\mathcal{F}L = (k[X] \otimes_{k[Y]} L)^N \cong (k[N] \otimes_k L)^N$ given by $n \mapsto 1 \otimes n$ is an isomorphism for $L \in \mathrm{Mod}\, k[Y]$. Thus $\mathcal{F}$ and $\mathcal{G}$ are quasi-inverse each other.

As $\mathcal{F}$ preserves the monoidal structure, the equivalence is that of monoidal categories. □

(2.5) Let G be a k-group scheme of finite type, N its normal closed subgroup scheme, and $f : X \to Y$ a G-enriched almost principal N-bundle if f is G-equivariant, the action of N on Y is trivial, there exist G-stable open subset V of Y and U of $f^{-1}(V)$ such that $\mathrm{codim}(Y \setminus V, Y) \geq 2$, $\mathrm{codim}(X \setminus U, X) \geq 2$, and $f_U : U \to V$ is a G-enriched principal N-bundle. Considering the case that $G = N$, a G-enriched almost principal G-bundle is simply called an almost principal G-bundle. For basics on almost principal G-bundles, see [Has3].

Lemma 2.6. *Let N be a normal closed subgroup scheme of G. Assume that $f : X \to Y$ is a G-enriched almost principal N-bundle with G, X and Y are all affine. Assume that both X and Y are normal. Then*

(1) *The canonical map* $k[Y] \to k[X]^N$ *is an isomorphism of* G/N*-algebras.*

(2) *The functors* $\mathcal{G} = (-)^N : \mathrm{Ref}(G, k[X]) \to \mathrm{Ref}(G/N, k[Y])$ *and* $\mathcal{F} = (k[X] \otimes_{k[Y]} -)^{**} : \mathrm{Ref}(G/N, k[Y]) \to \mathrm{Ref}(G, k[X])$ *are quasi-inverse each other, and give an equivalence of monoidal categories, where* $\mathrm{Ref}(G/N, k[Y])$ *denotes the category of* $(G/N, k[Y])$*-modules which are finitely generated reflexive as* $k[Y]$*-modules, and* $\mathrm{Ref}(G, k[X])$ *denotes the category of* $(G, k[X])$*-modules which are finitely generated reflexive as* $k[X]$*-modules. The counit* $\varepsilon : (k[X] \otimes_{k[Y]} M^G)^{**} \to M$ *is the double dual of* $\alpha \otimes m \mapsto \alpha m$*, and the unit* $u : L \to ((k[X] \otimes_{k[Y]} -)^{**})^N$ *is the composite*

$$L \cong k[Y] \otimes_{k[Y]} L \to (k[X] \otimes_{k[Y]} L)^N \to ((k[X] \otimes_{k[Y]} L)^{**})^N.$$

(3) *The equivalence preserves the rank of modules.*

Proof. We may assume that $G = N$. We may discuss componentwise, and we may assume that Y is connected, and hence $k[Y]$ is a normal domain. There is an open subset V of Y and an N-stable open subset U of $f^{-1}(V)$ such that $f : U \to V$ is a principal N-bundle, $\mathrm{codim}(Y \setminus V, Y) \geq 2$, and $\mathrm{codim}(X \setminus U, X) \geq 2$.

First consider the case that $V = Y$. As $G = N$ is affine, $f : U \to Y$ is affine, and hence U is affine. As X is normal and $\mathrm{codim}(X \setminus U, X) \geq 2$, we have that $U = X$. Thus f itself is a principal N-bundle. Then $M \in \mathrm{Mod}(N, k[X])$ is finitely generated as a $k[X]$-module and is reflexive if and only if $M^N \in \mathrm{Mod}(N, k[Y])$ is finitely generated as a $k[Y]$-module and is reflexive. Indeed, As $k[Y] \to k[X]$ is faithfully flat and $M \cong k[X] \otimes_{k[Y]} M^N$, we have that M is finitely generated if and only if M^N is finitely generated. If this is the case,

$$(M^*)^N = \mathrm{Mod}(N, k[X])(M, k[X]) \cong \mathrm{Mod}(k[Y])(M^N, k[X]^N) = (M^N)^*.$$

In particular, M is reflexive if and only if M^N is so. So the assertion of the lemma follows from Lemma 2.4 this case. In particular, the lemma is true if $\dim Y \leq 1$. So the lemma is true for $f_P : X_P \to Y_P$ for every $P \in \mathrm{Spec}\, k[Y]$ with $\mathrm{ht}\, P \leq 1$, where $f_P : X_P = \mathrm{Spec}\, k[X]_P \to Y_P = \mathrm{Spec}\, k[Y]_P$ is the base change of f.

As $k[X]$ is normal, we can write $k[X] = B_1 \times \cdots \times B_r$, where B_i is a normal domain. Let $M \in \mathrm{Ref}(G, k[X])$, and Q be a height-one prime ideal of $k[X]$. Then Q as a point of X lies in U. As $f : U \to Y$ is flat, we have that $\mathrm{ht}(Q \cap k[Y]) \leq 1$. Let $M_i = B_i \otimes_{k[X]} M$. Then

$$M_i \subset \bigcap_{P \in \mathrm{Spec}\, k[Y],\ \mathrm{ht}\, P \leq 1} (M_i)_P$$
$$= \bigcap_{P \in \mathrm{Spec}\, k[Y],\ \mathrm{ht}\, P \leq 1} \; \bigcap_{Q \in \mathrm{Spec}\, B_i,\ \mathrm{ht}\, Q \leq 1,\ Q \cap k[Y] \subset P} (M_i)_Q = \bigcap_{Q \in \mathrm{Spec}\, B_i,\ \mathrm{ht}\, Q \leq 1} (M_i)_Q = M_i,$$

where the intersection is taken in $Q(B_i) \otimes_{B_i} M_i$. This shows that

$$M = \prod_{i=1}^{r} M_i = \prod_{i=1}^{r} \Big(\bigcap_{P \in \operatorname{Spec} k[Y],\ \operatorname{ht} P \leq 1} (M_i)_P \Big)$$
$$= \bigcap_{P \in \operatorname{Spec} k[Y],\ \operatorname{ht} P \leq 1} \prod_{i=1}^{r} (M_i)_P = \bigcap_{P \in \operatorname{Spec} k[Y],\ \operatorname{ht} P \leq 1} M_P.$$

So
$$M^N = (\bigcap_P M_P)^N = \bigcap_P (M_P)^N = \bigcap_P (M^N)_P.$$

As Y is a Noetherian space and hence the subset $\{P \in \operatorname{Spec} k[Y] \mid \operatorname{ht} P = 1\}$ of $\operatorname{Spec} k[Y]$ is quasi-compact, we have that there is a finitely generated $k[Y]$-submodule L of M^N such that $L_P = (M^N)_P$ for each P. So $M^N = \bigcap_P (M^N)_P = \bigcap_P L_P = L^{**}$ is also finitely generated. As $(M^N)_P \cong (M_P)^N$ is reflexive and hence is free, $M^N \cong \bigcap_P (M^N)_P$ is also reflexive. Thus $\mathcal{G}$ is well-defined. Note also that $\mathcal{F}$ is also well-defined, since a double dual of a finitely generated module over a Noetherian ring is a second syzygy.

We want to prove that $\mathcal{F}L \xrightarrow{u} \mathcal{F}\mathcal{G}\mathcal{F}L \xrightarrow{\varepsilon} \mathcal{F}L$ is the identity, and $\mathcal{G}M \xrightarrow{u} \mathcal{G}\mathcal{F}\mathcal{G}M \xrightarrow{\varepsilon} \mathcal{G}M$ is the identity. As the canonical map

$$\operatorname{Hom}_{k[Y]}(\mathcal{G}M, \mathcal{G}M) \to \prod_P \operatorname{Hom}_{k[Y]_P}(\mathcal{G}_P M_P, \mathcal{G}_P M_P)$$

is injective, where the product runs through all the minimal prime ideals P of $k[Y]$, and $\mathcal{G}_P : \operatorname{Ref}(G, k[X]_P) \to \operatorname{Ref}(k[Y]_P)$ is the functor $(-)^G$, to see that the element $\mathrm{id}_{\mathcal{G}M} - \varepsilon u$ is zero, we may assume that $k[Y]$ is a field, and this case is trivial, where εu is the composite

$$\mathcal{G}M \xrightarrow{u} \mathcal{G}\mathcal{F}\mathcal{G}M \xrightarrow{\varepsilon} \mathcal{G}M.$$

Using a similar argument, we can prove that $\mathrm{id}_{\mathcal{F}N} - \varepsilon u$ is zero, and thus $\varepsilon u = \mathrm{id}$. So $\mathcal{G}$ is right adjoint to $\mathcal{F}$. Moreover, since $\mathcal{F}\mathcal{G}M$ is reflexive if M is reflexive, the counit $\varepsilon : \mathcal{F}\mathcal{G}M \to M$ is an isomorphism if and only if $\varepsilon_P : \mathcal{F}_P\mathcal{G}_P M_P \to M_P$ is an isomorphism for $P \in \operatorname{Spec} k[Y]$ whose height is at most one. As $f_P : X_P \to Y_P$ is a principal G-bundle and a reflexive $k[X]_P$-module and a reflexive $k[Y]_P$-module are free over $k[X]_P$ and $k[Y]_P$, respectively for $P \in \operatorname{Spec} k[Y]$ whose height is at most one, we have that the counit $\varepsilon : \mathcal{F}\mathcal{G} \to \mathrm{Id}_{\operatorname{Ref}(G,k[X])}$ is an isomorphism. Similarly, the unit of adjunction $u : \mathrm{Id}_{\operatorname{Ref}(k[Y])} \to \mathcal{G}\mathcal{F}$ is also an isomorphism. Thus $\mathcal{F}$ and $\mathcal{G}$ are quasi-inverse each other.

As the unit of adjunction $u : k[Y] \to \mathcal{G}\mathcal{F}k[Y] = k[X]^N$ is an isomorphism, the first assertion is now clear.

Let $L, L' \in \operatorname{Ref}(G/N, k[Y])$, and consider the canonical map

$$(k[X] \otimes_{k[Y]} (L \otimes_{k[Y]} L'))^{**} \to (k[X] \otimes_{k[Y]} (L \otimes_{k[Y]} L')^{**})^{**}.$$

This is a morphism in $\mathrm{Ref}(G, k[X])$. To see that this is an isomorphism, we may localize at $P \in \operatorname{Spec} k[Y]$ with $\operatorname{ht} P = 1$, and this case is obvious. Similarly, to see that

$$((k[X] \otimes_{k[Y]} L) \otimes_{k[X]} (k[X] \otimes_{k[Y]} L'))^{**} \to ((k[X] \otimes_{k[Y]} L)^{**} \otimes_{k[X]} (k[X] \otimes_{k[Y]} L')^{**})^{**}$$

is an isomorphism, it suffices to show that this is bijective, and we may localize at a height one prime P of $k[Y]$ again, and this case is obvious.

Thus $\mathcal{F}$ preserves the monoidal structure and the rank of modules. $\square$

(2.7) From now on, we assume that k is a perfect field of characteristic $p > 0$, and G be a finite k-group scheme. We set $N = G^\circ$, the identity component of G, and $H = G_{\mathrm{red}}$. As k is perfect, H is a closed subgroup scheme of G, and is étale over k. Note that the composite $H \hookrightarrow G \to G/N \cong \pi_0 G$ is an isomorphism, where $\pi_0 G$ is the unique maximal étale quotient of G. In other words, $k[\pi_0 G]$ is the k-subalgebra of $k[G]$ generated by all the étale k-subalgebras of $k[G]$. So G is the semidirect product $G = N \rtimes H$.

Lemma 2.8. *There exists some $e_0 \geq 1$ such that $B^{p^{e_0}} \subset B^N \subset B$, where $B^{p^{e_0}}$ is the image of the eth Frobenius map $F^e : B \to B$. In particular, $B^N \to B$ is finite, and B^N is finitely generated over k.*

Proof. Let $J = \operatorname{Ker} \varepsilon$ be the kernel of the counit map $k[N] \to k$. Then $k[N] = k \oplus J$ as a k-vector space. As N is infinitesimal (that is, $N_{\mathrm{red}} = \operatorname{Spec} k$), J is a nilpotent ideal, and hence there exists some $e_0 \geq 1$ such that $J^{[p^{e_0}]} = 0$, where $J^{[p^{e_0}]}$ is the ideal of $k[N]$ generated by $\{a^{p^{e_0}} \mid a \in J\}$. Then for $c + a \in k \oplus J = k[N]$, we have that $F^{e_0}(c+a) = c^{p^{e_0}} = F^{e_0}(u\varepsilon(c+a))$, where $u : k \to k[N]$ is the unit map. Hence $F^{e_0} = F^{e_0} u \varepsilon$. So for $b \in B$,

$$\omega_B(b^{p^{e_0}}) = (\omega_B b)^{p^{e_0}} = \sum_{(b)} b_{(0)}^{p^{e_0}} \otimes b_{(1)}^{p^{e_0}} = \sum_{(b)} (b_{(0)} \varepsilon(b_{(1)}))^{p^{e_0}} \otimes 1 = b^{p^{e_0}} \otimes 1.$$

That is, $b^{p^{e_0}} \in B^N$, and the first assertion has been proved.

As B is finitely generated over a perfect field, B is F-finite. So B is finite over $B^{p^{e_0}}$. Hence, both $B^{p^{e_0}} \to B^N$ and $B^N \to B$ are also finite, and we are done. $\square$

Lemma 2.9. *B^G is a finitely generated k-algebra, and $B^G \to B$ is finite.*

Proof. As $B^G = (B^N)^H$, we may assume that either $G = N$ or $G = H$. The case that $G = N$ is done in Lemma 2.8. The case that $G = H$ is reduced to the case that k is algebraically closed. In that case, G is a constant finite group, and this is well-known. $\square$

(2.10) For an H-module W, we denote its restriction by the canonical homomorphism $G \to G/N \cong H$ by W'. Thus the restriction of W' on N is trivial, while the restriction of W' on H is W. We adopt this notation for G-modules M. We regard M as its restriction to H, and then consider the G-module M'. Thus M' is the restriction of the G-module M to G with respect to the homomorphism $\rho : G \to G/N \cong H \hookrightarrow G$.

(2.11) The group scheme G viewed as the G-scheme with the left (resp. right) regular G-action is denoted by G_l (resp. G_r). That is, the action $G \times G_l \to G_l$ is given by $(g, g_1) \mapsto gg_1$ (resp. $G \times G_r \to G_r$ is given by $(g, g_2) \mapsto g_2 g^{-1}$). Note that the inverse $\iota : G_l \to G_r$ is the isomorphism of G-schemes. Note also that the coaction of $k[G_l]$ is given by $f \mapsto \sum_{(f)} f_{(2)} \otimes \mathcal{S} f_{(1)}$, where $\mathcal{S} : k[G] \to k[G]$ is the antipode. The coaction of $k[G_r]$ is the coproduct $f \mapsto \sum_{(f)} f_{(1)} \otimes f_{(2)}$.

(2.12) We consider that $G = N \rtimes H$ acts on N_l by $(nh)(n_1) = nhn_1h^{-1}$, and on H_l by $(nh)(h_1) = hh_1$. Note that the product $N_l \times H_l$ is isomorphic to G_l by $(n_1, h_1) \mapsto n_1 h_1$. Similarly, $H_r \times N_r \to G_r$ given by $(h, n) \mapsto hn$ is an isomorphism, where H acts on N_r by $(h, n) \mapsto hnh^{-1}$, and N acts on H_r trivially. In particular, we get an isomorphism of G-modules $k[G_r] \cong k[H_r]' \otimes k[N_r]$.

(2.13) A G-module W is both an N-module and an H-module, and the composite of the actions $h \circ n \circ h^{-1}$ agrees with the action of $hnh^{-1} \in N$. The converse is also true, and a G-linear map is nothing but an N-linear H-linear mapping.

Lemma 2.14. *For a G-module W, the map $\square : W \otimes k[N_r] \to W' \otimes k[N_r]$ given by $w \otimes \alpha \mapsto \sum_{(w)} w_{(0)} \otimes w_{(1)} \alpha$ is an isomorphism of $(G, k[N_r])$-modules.*

Proof. As

$$\begin{aligned}\beta\square(w \otimes \alpha) = \beta(\sum_{(w)} w_{(0)} \otimes w_{(1)}\alpha) &= \sum_{(\beta)} \sum_{(w)} \beta_{(1)} w_{(0)} \otimes \beta_{(2)} w_{(1)} \alpha \\ &= \sum_{(\beta w)} (\beta w)_{(0)} \otimes (\beta w)_{(1)} \alpha = \square(\beta(w \otimes \alpha)),\end{aligned}$$

we have that $\square$ is $k[N_r]$-linear. It is easy to see that $w' \otimes \beta \mapsto \sum_{(w')} w'_{(0)} \otimes (\mathcal{S} w'_{(1)})\beta$ is the inverse of $\square$, and $\square$ is bijective.

It remains to show that the map $\square$ is H-linear and N-linear. As

$$\begin{aligned}(\square \otimes 1_{k[H]})\omega(w \otimes \alpha) &= (\square \otimes 1_{k[H]}) \left(\sum_{(w),(\alpha)} w_{(0)} \otimes \alpha_{(2)} \otimes w_{(1)} (\mathcal{S}\alpha_{(1)}) \alpha_{(3)} \right) \\ &= \sum_{(w),(\alpha)} w_{(0)} \otimes w_{(1)} \alpha_{(2)} \otimes w_{(2)} (\mathcal{S}\alpha_{(1)}) \alpha_{(3)}\end{aligned}$$

and

$$\begin{aligned}\omega\square(w\otimes\alpha) &= \omega(\sum_{(w)} w_{(0)}\otimes w_{(1)}\alpha)\\ &= \sum_{(w),(\alpha)} w_{(0)}\otimes w_{(3)}\alpha_{(2)}\otimes w_{(1)}(\mathcal{S}w_{(2)})w_{(3)}(\mathcal{S}\alpha_{(1)})\alpha_{(3)}\\ &= \sum_{(w),(\alpha)} w_{(0)}\otimes w_{(1)}\alpha_{(2)}\otimes w_{(2)}(\mathcal{S}\alpha_{(1)})\alpha_{(3)},\end{aligned}$$

we have that $\square$ is H-linear. The fact that $\square$ is N-linear is checked more easily, and thus $\square$ is G-linear. $\square$

(2.15) Let W be a k-vector space. For $e \in \mathbb{Z}$, we define ${}^eW$ to be the additive group W with a new k action given by $\alpha \cdot w = \alpha^{p^e} w$ for $\alpha \in k$ and $w \in W$. For $w \in W$, we denote the element $w \in W = {}^eW$ by ${}^ew$. It is easy to see that ${}^e(-) : \operatorname{Mod} k \to \operatorname{Mod} k$ is an auto-equivalence of monoidal categories. For a k-algebra B, ${}^eB$ is a k-algebra, and the Frobenius map $F^e : B \to {}^eB$ is a k-algebra map. If $X = \operatorname{Spec} B$, then we denote ${}^eX = \operatorname{Spec} {}^eB$. For a B-module M, ${}^eM$ is a ${}^eB$-module in a natural way, and hence is a B-module.

(2.16) Let $f : R \to R'$ be a k-algebra map between k-algebras of finite type. Then f is étale if and only if ${}^eR \otimes_R R' \to {}^eR'$ given by ${}^er \otimes x \mapsto {}^e(rx^{p^e})$ is an isomorphism [Radu]. In particular, since H is étale over k, the Frobenius map $F^e : {}^eH \to H$ is an isomorphism of groups. If W is an H-module, then we have a homomorphism of k-groups $\psi : H \to GL(W)$. Note that ${}^eGL(W)$ acts on ${}^eW$ via the action ${}^eGL(W) \times {}^eW \cong {}^e(GL(W) \times W) \to {}^eW$. The coaction of ${}^eW$ is

$${}^eW \xrightarrow{{}^e\omega} {}^e(W \otimes k[H]) \cong {}^eW \otimes {}^ek[H] \xrightarrow{1\otimes F^{-e}} {}^eW \otimes k[H].$$

In particular, if H is a constant finite group, $h \in H$, $w_1, \ldots, w_n$ a k-basis of W, and $hw_j = \sum_i c_{ij} w_i$ ($c_{ij} \in k$), then ${}^ew_1, \ldots, {}^ew_n$ is a k-basis of ${}^eW$, and $h\,{}^ew_j = \sum_i c_{ij}^{1/p^e}\,{}^ew_i$.

Although ${}^eW$ is an H-module again for any H-module W, it seems that there is no canonical way to make ${}^eM$ a G-module for a G-module M.

3. F-signatures of the rings of invariants

(3.1) Let k be a perfect field of characteristic $p > 0$, $A = \bigoplus_{i\geq 0} A_i$ be a finitely generated commutative graded k-algebra such that A_0 is finite over k. Let $T = \bigoplus_{i\geq 0} T_i$ be a finite graded A-algebra which might not be commutative. We define $\mathcal{C}(T)$ the category of $\mathbb{Q}$-graded finitely generated left T-modules. Let $\Theta^*(T)$ denote $(\bigoplus_{M\in \operatorname{Ob}(\mathcal{C}(T))} \mathbb{R}\cdot[M])/([L]-[M]-[N])$, the $\mathbb{R}$-vector space with the set of objects $[M]$ of $\mathcal{C}(T)$ its basis modulo the relations $[L]-[M]-[N]$ for objects $L, M, N \in \mathcal{C}(T)$ such that $L \cong M \oplus N$. As the endomorphism ring of any object of $\mathcal{C}(T)$ is a finite dimensional algebra, $\Theta^*(T)$ has $\mathrm{Ind}(\mathcal{C}(T))$, the set of isomorphism classes of indecomposable objects of $\mathcal{C}(T)$ as its basis by the Krull–Schmidt theorem. We set $\Theta^\circ(T) = \Theta^*(T)/([M] - [M(c)] \mid M \in \mathcal{C}(T),\ c \in \mathbb{Q})$. Note that $\Theta^\circ(T)$ has $\mathrm{Ind}^\circ(\mathcal{C}(T)) = \mathrm{Ind}(\mathcal{C}(T))/\sim$ as its basis, where $\sim$ is the equivalence relation of $\mathrm{Ind}(\mathcal{C}(T))$ such that $[M] \sim [N]$ if and only if there exists some $c \in \mathbb{Q}$ such that $N \cong M(c)$.

(3.2) For $\alpha = \sum_{M \in \mathrm{Ind}^\circ(\mathcal{C}(T))} c_M[M] \in \Theta^\circ(T)$ ($c_M \in \mathbb{R}$), we define the norm $\|\alpha\|$ of α by $\|\alpha\| := \sum_M |c_M| u_T(M)$, where $u_T(M) = \ell_T(M/JM)$, where J is the graded radical of T (note that T/J is a finite-dimensional algebra). It is easy to see that $(\Theta^*(T), \|{-}\|)$ is a normed space.

(3.3) Let $B = \bigoplus_{i \geq 0} B_i$, $B_0 = k$, be a finitely generated positively graded k-algebra. Let G be a finite k-group scheme. We denote by $|G|$ the dimension $\dim k[G]$ of $k[G]$. Let G act on B. Assume that the action of G on B is degree-preserving. That is, each B_i is a G-submodule of B for any i. As B is a module algebra over the dual Hopf algebra $k[G]^*$ of $k[G]$, we can define the crossed product (smash product) $T = k[G]^* \# B$, see [Mon, Chapter 4]. By Lemma 2.9, T is a finite algebra over $A := B^G$. We denote $\mathcal{C}(T)$ and $\Theta^\circ(T)$ by $\mathcal{C}(G, B)$ and $\Theta^\circ(G, B)$, respectively. Thus an element α of $\mathcal{C}(G, B)$ can be written as $\alpha = \sum_M c_M[M]$ uniquely, where M runs through $\mathrm{Ind}^\circ((G, B))$. We say that $\alpha \geq 0$ if $c_M \geq 0$ for each M. We define $\mathcal{C}^+(G, B) = \{\alpha \in \mathcal{C}(G, B) \mid \alpha \geq 0\}$.

(3.4) For $M, L \in \mathcal{C}(G, B)$, we define $\mathrm{sum}(M, L)$ to be the supremum of $r \in \mathbb{Z}_{\geq 0}$ such that M^r is a direct summand of L. More generally, for $M \in \mathcal{C}(G, B)$ and $\alpha \in \mathcal{C}^+(G, B)$, we define $\mathrm{sum}(M, \alpha)$ to be the supremum of $r \in \mathbb{Z}_{\geq 0}$ such that $\alpha - r[M] \geq 0$. If M is indecomposable and $\alpha = \sum_N c_N[N] \in \mathcal{C}^+(G, B)$, then $\mathrm{sum}(M, \alpha) := c_M$.

(3.5) Let $M, L \in \mathcal{C}(G, B)$, and $f : M \to L$ be a G-linear B-linear mapping. We say that f is gradable if there exists some direct decomposition $M = \bigoplus_{i=1}^r M_i$ in $\mathcal{C}(G, B)$ and $c_1, \ldots, c_r \in \mathbb{Q}$ such that $f : \bigoplus_{i=1}^r M_i(c_i) \to L$ is a morphism in $\mathcal{C}(G, B)$ (that is, degree-preserving).

(3.6) Let H be an étale k-group scheme. Let $\mathcal{C} = \mathcal{C}(H, B)$, where $B = \bigoplus_{i \geq 0} B_i$ with $B_0 = k$ is a finitely generated positively graded k-algebra with a degree-preserving H-action. It is easy to see that if $M \in \mathcal{C}$, then ${}^eM \in \mathcal{C}$, and ${}^e(-) : \mathcal{C} \to \mathcal{C}$ is an exact functor. Note also that ${}^e(M(c)) \cong {}^eM(c/p^e)$. So we have a well-defined $\mathbb{R}$-linear map ${}^e(-) : \Theta^\circ(H, B) \to \Theta^\circ(H, B)$ given by ${}^e[M] = [{}^eM]$.

(3.7) Let $d = \dim B$. We define the Frobenius limit [HS] of $\alpha \in \Theta^\circ(H, B)$ by $\mathrm{FL}\, \alpha = \lim_{e \to \infty} {}^e\alpha / p^{de}$, if the limit in the right-hand side exists.

We define the H-equivariant generalized F-signature, denoted by $s_H(M, \alpha)$, to be $\lim_{e \to \infty} \mathrm{sum}(M, {}^e\alpha)/p^{de}$ if the limit exists. If the Frobenius limit $\mathrm{FL}(\alpha)$

exists, then $s_H(M,\alpha) = \mathrm{sum}(M, \mathrm{FL}(\alpha))$. If $H = \{e\}$ is trivial, then we simply denote $s_{\{e\}}(M,B)$ by $s(M,B)$.

(3.8) Let k be a perfect field of characteristic $p > 0$, and G be a finite k-group scheme.

Lemma 3.9. *Assume that the action of G on B is small, and B is a normal domain.*

(1) $M \mapsto M^G$ from the category of $\mathbb{Q}$-graded finitely generated reflexive (G,B)-modules $\mathcal{R}(G,B)$ to the category of $\mathbb{Q}$-graded finitely generated reflexive B^G-modules $\mathcal{R}(B^G)$ is an equivalence, whose quasi-inverse is given by $L \mapsto (L \otimes_{B^G} B)^{**}$.

(2) Assume that G is étale. Then for an indecomposable object $M \in \mathcal{R}(G,B)$, we have that $s_G(M,B) = s(M^G, B^G)$, provided $s_G(M,B)$ exists.

Proof. (1). This is an obvious extension of Lemma 2.6.

(2). Note that the G-invariance $(-)^G$ induces an $\mathbb{R}$-linear map $[(-)^G] : \Theta^\circ(G,B) \to \Theta^\circ(B^G)$. By (1), it is an isomorphism from the subspace $\Theta^\circ_{\mathrm{Ref}}(G,B)$ of $\Theta^\circ(G,B)$ generated by the reflexive objects to the subspace $\Theta^\circ_{\mathrm{Ref}}(B^G)$ of $\Theta^\circ(B^G)$ generated by the reflexive objects. Note that $({}^eB)^G \cong {}^e(B^G)$. As $[(-)^G]$ is continuous,

$$\mathrm{FL}(B^G) = \lim_{e\to\infty} {}^e[B^G]/p^{de} = (\lim_{e\to\infty} {}^e[B]/p^{de})^G = \mathrm{FL}(B)^G,$$

where $d = \dim B$. We can write $\mathrm{FL}(B) = \sum_{L \in \mathrm{ind}^\circ(B^G)} c_L[L]$ ($c_L \in \mathbb{R}$). Then $\mathrm{FL}(B^G) = \mathrm{FL}(B)^G = \sum_L c_L[L^G]$. By (1), we have that $s(M^G, B^G) = c_M = s_G(M,B)$. $\square$

(3.10) Let G be a finite group scheme over k, and V a finite-dimensional G-module. Let $S = \mathrm{Sym}\, V^*$ be the symmetric algebra of V, and we identify V with $\mathrm{Spec}\, S$. We have that $G = G^\circ \rtimes G_{\mathrm{red}}$. By [Has2, (10.8)], the action of G on V is small if and only if the action of G° on V is small and the action of G_{red} on $V/\!/G^\circ = \mathrm{Spec}\, S^{G^\circ}$ is small.

In what follows, we assume that the action of G is small. In particular, $G \to GL(V)$ is a closed immersion.

(3.11) Let k be a perfect field of characteristic $p > 0$, G be an étale k-group scheme over k, and V a finite-dimensional G-module. Let $S = \mathrm{Sym}\, V$ be the symmetric algebra of V, and we assume that S is graded so that each element of V is homogeneous of degree one. Assume that the action of G on S is small. Let $\tilde{G} = G \times \mathbb{G}_m$. Let $X = \mathrm{Spec}\, S$. Then $\tilde{G}$ acts on X. Let U be the étale locus of the quotient map $\pi : X \to Y = \mathrm{Spec}\, S^G$. Then U is a dense open subset of X.

Lemma 3.12. *There is a split monomorphism $W \hookrightarrow S$ of $\tilde{G}$-modules, where W is a $\tilde{G}$-module whose underlying G-module is isomorphic to $k[G]$.*

Proof. Let $g = \dim k[G]$. If there is no k-rational point of U, then since U is a dense open subset of an affine space, $k = \mathbb{F}_q$ is a finite field. In this case, we can take a prime number $\ell > g$ such that U has a $k_1 := \mathbb{F}_{q^\ell}$-valued point x. Then the image y of x by $\pi : X \to Y$ is again a k_1-valued point. As $G \times U \to U \times_Y U$ is an isomorphism, we have that $G \times \{x\} \to Gx$ is an isomorphism. Or equivalently, $S \xrightarrow{\omega} S \otimes k[G] \to k_1[G]$ is surjective. As $k_1[G]$ as a G-module is nothing but $k[G]^\ell$, there is a surjective G-linear map $\rho : S \to k[G]$. As $k[G]^*$ is a finite dimensional Hopf algebra, it is a Frobenius algebra [SY, Theorem 3.6]. So the injective G-module ($kG = k[G]^*$-module) $k[G]$ is also projective. So ρ splits, and $k[G]$ is a summand of S as a G-module. So there exists some $r \geq 0$ such that $k[G]$ is a summand of $\bigoplus_{i=0}^r S_i$, where S_r denotes the homogeneous component of S of degree r. By the Krull–Schmidt theorem, $\bigoplus_{i=0}^r S_i$ has a graded summand which is isomorphic to $k[G]$ as a G-module, and this is what we wanted to prove. $\square$

(3.13) Just mimicking the proof in [HS, section 4], we can extend [HS, Theorem 4.13] to the actions of étale group schemes. Lemma 3.12 above can be used to extend [HS, Lemma 4.11] to the case that G is a general étale group scheme.

Proposition 3.14. *Let k be a perfect field of characteristic $p > 0$, G be an étale k-group scheme, and V be a finite-dimensional G-module. Let $S = \operatorname{Sym} V$ be the symmetric algebra of V, and we assume that S is graded so that each element of V is homogeneous of degree one. Let M be a $\mathbb{Q}$-graded S-finite S-free (G, S)-module. Let $k = V_1, \ldots, V_r$ be the simple G-modules, and let P_i be the projective cover of V_i. Then we have*

$$\mathrm{FL}(M) = \frac{\operatorname{rank}_S M}{\dim_k k[G]} [k[G] \otimes_k S].$$

In particular, we have

$$\mathrm{FL}(M^G) = \frac{\operatorname{rank}_S M}{\dim_k k[G]} [S] = \frac{\operatorname{rank}_S M}{\dim_k k[G]} \sum_{i=1}^r \frac{\dim V_i}{\dim \operatorname{End}_G V_i} [(P_i \otimes_k S)^G].$$

Using Proposition 3.14 above, we can prove the following.

Proposition 3.15. *Let the notation be as in Proposition 3.14. Let L be a graded (G, S)-submodule of M, and assume that L is reflexive as an S-module. Then*

$$\mathrm{FL}(L) = \frac{\operatorname{rank}_S L}{\dim_k k[G]} [k[G] \otimes_k S].$$

In particular, we have

$$\mathrm{FL}(L^G) = \frac{\operatorname{rank}_S L}{\dim_k k[G]} [S] = \frac{\operatorname{rank}_S L}{\dim_k k[G]} \sum_{i=1}^r \frac{\dim V_i}{\dim \operatorname{End}_G V_i} [(P_i \otimes_k S)^G].$$

Proof. We set $d = \dim_k V$, $g = \dim_k k[G]$, $m = \operatorname{rank}_S M$ and $\ell = \operatorname{rank}_S L$. For $e \geq 1$, we set $\gamma_e := \operatorname{sum}(k[G] \otimes_k S, {}^e M)$. By Proposition 3.14, we have that $\lim_{e\to\infty} \gamma_e/p^{de} = m/g$.

Let $A = S^G$. Then there is a gradable injective A-linear mapping $A^\ell \to L^G$, which induces a gradable injective (G, S)-linear mapping $h : S^\ell \to L$. For each $e \geq 1$, consider the composite

$$\delta_e : {}^e S^\ell \xrightarrow{{}^e h} {}^e L \hookrightarrow {}^e M \to (k[G] \otimes_k S)^{\gamma_e} \to (k[G] \otimes_k S)^{\gamma_e}/\mathfrak{m}(k[G] \otimes_k S)^{\gamma_e} \cong k[G]^{\gamma_e},$$

where $\mathfrak{m} = S_+$ is the irrelevant ideal. Let $W_e := \operatorname{Im} \delta_e$, and we write $W_e = k[G]^{\mu_e} \oplus U_e$ such that U_e does not have $k[G]$ as a direct summand. Then since $k[G]$ is a projective G-module, the surjective map ${}^e S^\ell \xrightarrow{\delta_e} W_e \to k[G]^{\mu_e}$ has a gradable G-linear splitting $\psi_e : k[G]^{\mu_e} \to {}^e S^\ell$. This induces a unique gradable (G, S)-linear map $\tilde{\psi}_e : (k[G] \otimes S)^{\mu_e} \to {}^e S^\ell$, and

$$\lambda_e : (k[G] \otimes_k S)^{\mu_e} \xrightarrow{\tilde{\psi}_e} {}^e S^\ell \to {}^e L \to {}^e M \to (k[G] \otimes_k S)^{\gamma_e}$$

induces a split monomorphism $\bar{\lambda}_e = \lambda \otimes S/\mathfrak{m} : k[G]^{\mu_e} \to k[G]^{\gamma_e}$. Then it is easy to see that λ_e is injective, and the cokernel of λ_e is S-free. As $k[G] \otimes_k S$ is an injective object in the exact category of S-free modules in $\mathcal{C}(G, S)$, it follows that λ_e is a split monomorphism in $\mathcal{C}(G, S)$. This shows that $\operatorname{sum}(k[G] \otimes_k S, {}^e L) \geq \mu_e$.

So it suffices to prove that $\lim_{e\to\infty} \mu_e/p^{de} = \ell/g$.

Let κ denote the composite map $S^\ell \xrightarrow{h} L \to M$, and $Q := \operatorname{Coker} \kappa$. Then

$$\begin{aligned}\lim_{e\to\infty} \mu_S(\operatorname{Im} {}^e\kappa)/p^{de} &= \lim_{e\to\infty} \mu_S({}^e M)/p^{de} - \lim_{e\to\infty} \mu_S({}^e Q)/p^{de} \\ &= e^S_{\mathrm{HK}}(M) - e^S_{\mathrm{HK}}(Q) = \operatorname{rank}_S M - \operatorname{rank}_S Q = \operatorname{rank}_S S^\ell = \ell,\end{aligned}$$

where μ_S deontes the number of generators, and e^S_{HK} denotes the Hilbert–Kunz multiplicity (as S is the polynomial ring, $e^S_{\mathrm{HK}} = \operatorname{rank}_S$). Note that

$$0 \leq \mu_S(\operatorname{Im} {}^e\kappa) - \dim_k W_e \leq \operatorname{rank}_S {}^e M - \operatorname{rank}_S (k[G] \otimes_k S)^{\gamma_e} = p^{de}(m - g\gamma_e).$$

So $\lim_{e\to\infty}(\mu_S(\operatorname{Im} {}^e\kappa) - \dim_k W_e)/p^{de} = 0$. This shows $\lim_{e\to\infty} \dim_k W_e/p^{de} = \ell$.

We can write ${}^e S = (k[G] \otimes S)^{\rho_e} \oplus Z_e$ such that Z_e does not have a summand isomorphic to $k[G]$. Note that $\lim_{e\to\infty} \rho_e/p^{de} = 1/g$. The restriction of δ_e : ${}^e S^\ell \to W_e$ to $(k[G] \otimes S)^{\rho_e \ell}$ induces a G-linear map

$$\eta_e : k[G]^{\rho_e \ell} \cong (k[G] \otimes S)^{\rho_e \ell}/\mathfrak{m}(k[G] \otimes S)^{\rho_e \ell} \to W_e.$$

Let Y_e be its image. Then $0 \leq \dim_k W_e - \dim_k Y_e \leq \ell \operatorname{rank}_S Z_e = \ell(p^{de} - g\rho_e)$. So $\lim_{e\to\infty} \frac{1}{p^{de}}(\dim_k W_e - \dim_k Y_e) = 0$. It follows that $\lim_{e\to\infty} \frac{1}{p^{de}} \dim_k Y_e = \ell$. Let I_e be the kernel of η_e, and J_e be its injective hull, which is a G-submodule of $k[G]^{\rho_e \ell}$. As $\dim J_e \leq g \dim I_e$, we have that $\dim J_e/p^{de} \to 0$. So when we set

$$\tau_e := \max\{r \mid k[G]^r \text{ is a submodule of } k[G]^{\rho_e \ell}/J_e\},$$

we have that $\lim_{e\to\infty} \tau_e/p^{de} = \ell/g$. As $\tau_e \leq \mu_e \leq \dim W_e/g$, we have that $\lim_{e\to\infty} \mu_e/p^{de} = \ell/g$. This is what we wanted to prove. $\square$

Theorem 3.16. *Let k be a perfect field of characteristic $p > 0$, G be a finite k-group scheme over k, and V a finite-dimensional G-module. Let $S = \operatorname{Sym} V$ be the symmetric algebra of V, and we assume that S is graded so that each element of V is homogeneous of degree one. Assume that the action of G on S is small. Let M be a $\mathbb{Q}$-graded S-finite S-free (G,S)-module, and L be its graded (G,S)-submodule which is reflexive as an S-module. Let $k = V_1, \dots, V_r$ be the simple G-modules, and let P_i be the projective cover of V_i. Then we have*

$$\mathrm{FL}(L^G) = \frac{\operatorname{rank}_S L}{\dim_k k[G]}[S''] = \frac{\operatorname{rank}_S L}{\dim_k k[G]} \sum_{i=1}^{r} \frac{\dim V_i}{\dim \operatorname{End}_G V_i}[(P_i \otimes_k S)^G],$$

where $S'' = (S \otimes_k k[G])^G$ is S viewed as an A-module, where $A = S^G$.

(3.17) We set $N := G^\circ$ and $H := G/N \cong G_{\mathrm{red}}$. We define $\psi : G \to G$ by the composite $G \to G/N = H = G_{\mathrm{red}} \hookrightarrow G$. For a G-module W, we define the G-module $\operatorname{res}_\psi W$ by W', see (2.10).

Note that the coordinate algebra $k[N]$ is a G-module, or a right $k[G]$-comodule by the coaction $k[N] \to k[N] \otimes k[G]$ induced by the map $k[G] \to k[N] \otimes k[G]$ given by $f \mapsto \sum_{(f)} n_{(2)} \otimes (\mathcal{S} n_{(1)}) \cdot n_{(3)} \in k[N] \otimes k[G]$ (this map is well-defined, since N is a normal subgroup scheme of G). It is easy to see that for a G-module W, the coproduct $\omega_W : W \to W' \otimes k[N]$ is G-linear.

(3.18) We set $B = S^N$ and $A = S^G = B^H$. By Lemma 2.8, we have that there exists some $e_0 \geq 1$ such that $S^{p^{e_0}} \subset B$.

(3.19) As G is the semidirect product $G = N \rtimes H$, the composite

$$k[G] \xrightarrow{\Delta_G} k[G] \otimes k[G] \xrightarrow{\pi_N \otimes \pi_H} k[N] \otimes k[H]'$$

is an isomorphism, where $\pi_N : k[G] \to k[N]$ and $\pi_H : k[G] \to k[H]'$ are the canonical maps corresponding to the canonical closed immersion $N \hookrightarrow G$ and $H \hookrightarrow G$, respectively. Note that $B \subset S' \subset {}^{e_0}B$, and that

$$\begin{aligned}({}^{e_0}(L^G) \otimes_A S)^{**} &\cong (({}^{e_0}(L^G) \otimes_A B)^{\star\star} \otimes_B S)^{**} \cong ({}^{e_0}(L^N) \otimes_B S)^{**} \\ &\cong ({}^{e_0}(L^N) \otimes_{S'} (S' \otimes_B S)^{**})^{**} \cong ({}^{e_0}(L^N) \otimes_{S'} S' \otimes_k k[N])^{**} \\ &\cong ({}^{e_0}(L^N) \otimes_k k[N])^{**} \cong F' \otimes_k k[N],\end{aligned}$$

where $(-)^* = \operatorname{Hom}_S(-,S)$, $(-)^\star = \operatorname{Hom}_B(-,B)$, and $F = {}^{e_0}(L^N)$. Note that $F \in \operatorname{Ref}(H, S')$ and it is a graded (H,S')-submodule M'. As M' is finite free as an S'-module, we have that

$$\mathrm{FL}(F) = \frac{1}{|H|}\frac{p^{de_0}\operatorname{rank}_S L}{|N|}[S' \otimes_k k[H]] = \frac{p^{de_0}\operatorname{rank}_S L}{|G|}[S' \otimes_k k[H]]$$

in $\Theta^\circ(H, S')$ by Proposition 3.15, where $|H| = \dim_k k[H]$, $|N| = \dim_k k[N]$, and $|G| = \dim_k k[G]$.

Note that for a $\mathbb{Q}$-graded (H, S)-module Q, $Q' \otimes_k k[N]$ is a (G, S)-module with the S-action

$$s \cdot (q \otimes h) = \sum_{(s)} s_{(1)} q \otimes s_{(0)} h.$$

Lemma 3.20. *$S' \otimes k[H]' \otimes k[N]$ is isomorphic to $S \otimes k[G]$ as a (G, S)-module.*

Proof. The maps in the sequence

$$S \otimes k[G] \xrightarrow{\square} S'' \otimes k[G] \xrightarrow{\delta} S'' \otimes k[H]' \otimes k[N] \xrightarrow{\square_H^{-1}} S' \otimes k[H]' \otimes k[N]$$

are all isomorphisms, where $\square(s \otimes f) = \sum_{(s)} s_{(0)} \otimes s_{(1)} f$, $\delta(s \otimes f) = \sum_{(f)} s \otimes \pi_H(f_{(1)}) \otimes \pi_N(f_{(2)})$, and $\square_H^{-1}(s \otimes h) = \sum_{(s)} s_{(0)} \otimes \mathcal{S}_H(s_{(1)}) h$, where $\pi_H : k[G] \to k[H]$ and $\pi_N : k[G] \to k[N]$ are the canonical surjective homomorphisms of k-Hopf algebras associated with the inclusions $H \hookrightarrow G$ and $N \hookrightarrow G$, respectively. These maps are isomorphisms of G-modules and isomorphisms of k-algebras. Thus by $s(s' \otimes f) = ss' \otimes f$, $s(s'' \otimes f) = \sum_{(s)} s_{(0)} s'' \otimes s_{(1)} f$, $s(s'' \otimes h \otimes r) = \sum_{(s)} s_{(0)} s'' \otimes s_{(1)} h \otimes s_{(2)} r$, and $s(s'' \otimes h \otimes r) = s_{(0)} s'' \otimes h \otimes s_{(1)} r$, the k-algebras $S \otimes k[G]$, $S'' \otimes k[G]$, $S'' \otimes k[H]' \otimes k[N]$, and $S' \otimes k[H]' \otimes k[N]$ are G-algebras, and the maps $\square$, δ, and $\square_H^{-1}$ above are all isomorphisms of (G, S)-modules. $\square$

(3.21) Note that the (H, S')-module $F = {}^{e_0}(M^N)$ is reflexive and is an (H, S')-submodule of ${}^{e_0}M'$, which is finite free as an S'-module. By Proposition 3.15, we have that

$$\mathrm{FL}(F) = \frac{\mathrm{rank}_{S'} F}{|H|}[S' \otimes k[H]] = \frac{p^{de_0} \mathrm{rank}_S L}{|G|}[S' \otimes k[H]]$$

in $\Theta^\circ(H, S')$. Hence

$$\begin{aligned} \mathrm{FL}(L^G) = \frac{1}{p^{de_0}} \mathrm{FL}({}^{e_0}(L^G)) &= \frac{1}{p^{de_0}} \mathrm{FL}((({}^{e_0}(L^G) \otimes_A S)^{**})^G) \\ &= \frac{1}{p^{de_0}} \mathrm{FL}((F' \otimes_k k[N])^G) \\ &= \frac{\mathrm{rank}_S L}{|G|}[(S' \otimes_k k[H]' \otimes_k k[N])^G] = \frac{\mathrm{rank}_S L}{|G|}[(S \otimes_k k[G])^G], \end{aligned}$$

and Theorem 3.16 has been proved. $\square$

Corollary 3.22. *Let the notation be as in Theorem 3.16. Then for every indecomposable $\mathbb{Q}$-graded finite A-module N,*

$$s(N, A) = \begin{cases} \dfrac{\dim V_i}{\dim k[G] \cdot \dim \mathrm{End}_G V_i} & (N \cong (P_i \otimes_k S)^G) \\ 0 & (\textit{there is no } i \textit{ such that } N \cong (P_i \otimes_k S)^G) \end{cases}.$$

The following corollaries for the case that G is linearly reductive was proved by Watanabe–Yoshida [WY], Carvajal-Rojas–Schwede–Tucker [CST], and Carvajal-Rojas [Car]. For some important cases that G is not linearly reductive was proved by Broer [Bro], Yasuda [Yas], and Liedtke–Martin–Matsumoto [LMM].

Corollary 3.23. *Let k be a perfect field, V a finite-dimensional k-vector space, and $G \subset GL(V)$ be a small finite subgroup scheme. Let $S := \operatorname{Sym} V$, and $A := S^G$. Then we have*

$$s(A) := s(A, A) = \begin{cases} \dfrac{1}{\dim k[G]} & (G \text{ is linearly reductive}) \\ 0 & (\text{otherwise}) \end{cases}.$$

Proof. Note that $A = S^G \cong (P_i \otimes_k S)^G$ up to degree shifting if and only if $S \cong P_i \otimes_k S$ up to degree shifting if and only if $k \cong P_i$ as G-modules. Considering the case that $N = A$ in Corollary 3.22, the corollary follows, since G is linearly reductive if and only if $k \cong P_i$ for some i. □

Corollary 3.24. *Let the notation be as in Corollary 3.23. The following are equivalent.*

(1) *G is linearly reductive.*

(2) *Both G° and G_{red} are linearly reductive.*

(3) *$\bar{G}^\circ \cong \mu_{p^{e_1}} \times \cdots \times \mu_{p^{e_r}}$ for some $e_1, \dots, e_r \geq 1$, and $|\bar{G}_{\mathrm{red}}|$ is not divisible by p, where $\bar{G} = \bar{k} \otimes_k G$ is the base change of G to the algebraic closure $\bar{k}$ of k.*

(4) *S^G is a direct summand subring of S.*

(5) *S^G is strongly F-regular.*

(6) *S^G is F-regular.*

(7) *S^G is weakly F-regular.*

(8) *The F-signature $s(S^G)$ is positive.*

Proof. For the proof of (1)⇔(2)⇔(3), we may assume that k is an algebraically closed field.

For the equivalence (1)⇔(2), see [Has2, Lemma 2.2].

We prove (2)⇒(3). As G° is finite, connected and linearly reductive, we have $G^\circ \cong \mu_{p^{e_1}} \times \cdots \times \mu_{p^{e_r}}$ for some $e_1, \dots, e_r \geq 1$ by [Swe2, Theorem 4.1]. Moreover, as G_{red} is linearly reductive, $|G_{\mathrm{red}}|$ is not divisible by p by Maschke's theorem.

(3)⇒(2). As $G^\circ = \mu_{p^{e_1}} \times \cdots \times \mu_{p^{e_r}} = \operatorname{Spec} k\Lambda$, where Λ is the abelian group $\mathbb{Z}/p^{e_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{e_r}\mathbb{Z}$, the category of G°-modules is equivalent to the category of Λ-graded k-vector spaces [Jan, (I.2.11)], and hence G° is linearly reductive. On the other hand, G_{red} is linearly reductive by Maschke's theorem.

We prove (1)⇒(4). For a G-module V, let $U(V) = \sum_{S \subset V} S$, where the sum is taken over all the non-trivial simple G-submodules of V. Then we have a functorial decomposition $V = V^G \oplus U(V)$. The projection $V \to V^G$ is a G-linear splitting of the inclusion $V^G \to V$, and it is a Reynolds operator. So S^G is a direct summand subring of S.

(4)⇒(5). As S is a polynomial ring over k, it is strongly F-regular. Hence its direct summand S^G is also strongly F-regular, see [Has, Lemma 3.17].

For (5)⇒(6), see [Has, Corollary 3.7].

(6)⇒(7) is obvious by the definitions of the F-regularity and the weak F-regularity, see [HH, (4.5)].

(7)⇒(4). Let $A = S^G$, and A^+ be the integral closure of A in the algebraic closure K of the field of fractions $Q(A)$ of A. Then as in the proof of [Smi, Proposition 2.14], we have that $IA^+ \cap A \subset I^*$, where I^* denotes the tight closure. By the definition of weak F-regularity, we have that $IA^+ \cap A = I$. As I is arbitrary, we have that $A \hookrightarrow A^+$ is cyclically pure. In particular, $A \hookrightarrow S$ is also cyclically pure. As A is normal, we have that $A \to S$ is pure by [Hoc]. By [HR, Corollary 5.3], A is a direct summand subring of S.

The equivalence (5)⇔(8) is well-known, see [AL, Theorem 0.2].

(8)⇒(1) follows immediately from Corollary 3.23. □

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