Indecomposability of graded modules over a graded ring

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Abstract

Let $R = \bigoplus_{i \geq 0} R_i$ be a Noetherian commutative non-negatively graded ring such that $(R_0, \mathfrak{m}_0)$ is a Henselian local ring. Let $\mathfrak{m}$ be its unique graded maximal ideal $\mathfrak{m}_0 + \bigoplus_{i>0} R_i$. Let T be a module-finite (non-commutative) graded R-algebra. Let $T\,\mathrm{grmod}$ denote the category of finite graded left T-modules, and $M \in T\,\mathrm{grmod}$. Then the following are equivalent: (1) $\hat{M}$ is an indecomposable $\hat{T}$-module, where $\widehat{(-)}$ denotes the $\mathfrak{m}$-adic completion; (2) $M_{\mathfrak{m}}$ is an indecomposable $T_{\mathfrak{m}}$-module; (3) M is an indecomposable T-module; (4) M is indecomposable as a graded T-module. As a corollary we prove that for two finite graded left T-modules M and N, the following are equivalent: (1) If $M = M_1 \oplus \cdots \oplus M_s$ and $N = N_1 \oplus \cdots \oplus N_t$ are decompositions into indecomposable objects in $T\,\mathrm{grmod}$, then $s = t$, and there exist some permutation $\sigma \in \mathfrak{S}_s$ and integers $d_1, \ldots, d_s$ such that $N_i \cong M_{\sigma i}(d_i)$, where $-(d_i)$ denotes the shift of degree; (2) $M \cong N$ as T-modules; (3) $M_{\mathfrak{m}} \cong N_{\mathfrak{m}}$ as $T_{\mathfrak{m}}$-modules; (4) $\hat{M} \cong \hat{N}$ as $\hat{T}$-modules. As an application, we compare the FFRT property of rings of characteristic p in the graded sense and in the local sense.

1. Introduction

In many cases, a theory in Noetherian local rings has its graded version or *version. Moreover, such a graded version often determines the ring-theoretic property of the graded ring A as a non-graded ring.

We give an easy illustrative example. Let $A = \bigoplus_{i \geq 0}$ be a finitely generated positively graded (that is, $A_0 = k$) commutative algebra over a field k. Then we can define a homogeneous system of parameters of A to be a sequence $x_1, \ldots, x_d$ of homogeneous elements of positive degrees such that $\dim_k A/(x_1, \ldots, x_d) < \infty$, and the length of the sequence d is small as possible among such sequences. It is well-known that a sequence $x_1, \ldots, x_r$ of homogeneous elements of positive degree is a homogeneous system of parameters if and only if their images in the local ring $A_{\mathfrak{m}}$ is a system of parameters in

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the local sense. In particular, if $x_1, \dots, x_d$ is a homogeneous system of parameters, then we have $d = \dim A_\mathfrak{m}$. What is interesting is $\dim A = \dim A_\mathfrak{m}$ in fact [BH, Section 1.5]. Moreover, there is a homogeneous system of parameters which is a regular sequence if and only if $A_\mathfrak{m}$ is Cohen–Macaulay if and only if A is Cohen–Macaulay [MR].

Some important results in this direction can be found in the textbook [BH].

In this paper, we compare the indecomposability of finitely generated module M over a module-finite algebra over a graded ring along this line, and we get the following. Namely, we have

Theorem 3.8. *Let $R = \bigoplus_{i \geq 0} R_i$ be a Noetherian commutative graded ring such that $(R_0, \mathfrak{m}_0)$ is Henselian local, $R_+ = \bigoplus_{i>0} R_i$, $\mathfrak{m} = R_+ + \mathfrak{m}_0$, and T a $\mathbb{Z}$-graded module-finite (non-commutative) R-algebra. Let M be a finitely generated graded left T-module. Then the following are equivalent.*

(1) *$\hat{M}$ is indecomposable as a $\hat{T}$-module, where $\widehat{(-)}$ denotes the $\mathfrak{m}$-adic completion.*

(2) *$M_\mathfrak{m}$ is indecomposable as a $T_\mathfrak{m}$-module.*

(3) *M is indecomposable as a T-module.*

(4) *M is indecomposable as a graded T-module.*

As a corollary, we prove

Corollary 3.9. *Let M and N be objects in $T\,\mathrm{grmod}$, the category of finitely generated graded left T-modules. Then the following are equivalent.*

(1) *If $M = M_1 \oplus \cdots \oplus M_s$ and $N = N_1 \oplus \cdots \oplus N_t$ are decompositions into indecomposable objects in $T\,\mathrm{grmod}$, then $s = t$, and there exist some permutation $\sigma \in \mathfrak{S}_s$ and integers $d_1, \dots, d_s$ such that $N_i \cong M_{\sigma i}(d_i)$, where $-(d_i)$ denotes the shift of degree.*

(2) *$M \cong N$ as T-modules.*

(3) *$M_\mathfrak{m} \cong N_\mathfrak{m}$ as $T_\mathfrak{m}$-modules.*

(4) *$\hat{M} \cong \hat{N}$ as $\hat{T}$-modules.*

As an application, we prove a comparison theorem for the finite F-representation type (FFRT for short) property of an F-finite non-negatively graded ring $R = \bigoplus_{i \geq 0} R_i$ such that $(R_0, \mathfrak{m}_0)$ is an F-finite Henselian local ring of a prime characteristic p in the graded sense and the FFRT property of the $\mathfrak{m}$-adic completion of R, where $\mathfrak{m} = \mathfrak{m}_0 + \bigoplus_{i>0} R_i$ (Corollary 3.10). See Corollary 3.10 for the definition of FFRT. This property for rings of characteristic p was defined by K. E. Smith and M. Van den Bergh [SVdB], and has been studied extensively [AK, DQ, HB, HO, Shi1, Shi2].

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2. Preliminaries

(2.1) For a ring A, we denote the set of units of A by $A^\times$. We say that A is *local* if $A \setminus A^\times$ is an additive subgroup of A. This is equivalent to say that $A \neq 0$, and $A \setminus A^\times$ is closed under addition. If so, $A \setminus A^\times$ is the unique maximal left ideal of A. It is also the unique maximal right ideal of A, and agrees with the radical $\operatorname{rad} A$. Note that A is local if and only if $A/\operatorname{rad} A$ is a division ring, see [L, (19.1)].

(2.2) Let $B = \bigoplus_{i\in\mathbb{Z}} B_i$ be a graded ring, and $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a graded left module. A left graded submodule N is said to be *maximal if N is a maximal element of the set $\{N \subset M \mid N \text{ is a graded submodule of } M \text{ and } N \neq M\}$. The intersection of all the *maximal graded submodules is denoted by $^*\operatorname{rad} M$. If $f : M \to M'$ is a homomorphism of graded left B-modules, then $f(^*\operatorname{rad} M) \subset {}^*\operatorname{rad} M'$. In particular, if $b \in B_i$, then $(^*\operatorname{rad} B)b \subset {}^*\operatorname{rad}(B(i)) = (^*\operatorname{rad} B)(i)$, and $^*\operatorname{rad} B$ is a two-sided ideal.

Lemma 2.3. *Let B be as above. Then any nonzero finitely generated left graded module M has a *maximal submodule. In particular, $M \neq {}^*\operatorname{rad} M$.*

Proof. Let Γ be the set of graded proper submodules of M. As M is nonzero, Γ is nonempty. If Ω is a non-empty chain of Γ, then $\sum_{N\in\Omega} N \neq M$, and $\sum_{N\in\Omega} N \in \Gamma$. Indeed, if $\sum_{N\in\Omega} N = M$, then there exists some N that contains all the generators of M. This implies $N = M \in \Omega \subset \Gamma$ and this is a contradiction. So by Zorn's lemma, Γ has a maximal element, and this is what we wanted to prove. □

Corollary 2.4. *Let B be as above, and assume that $B \neq 0$. Then B has a left *maximal ideal. In particular, $B \neq {}^*\operatorname{rad} B$.* □

Lemma 2.5. *Let B be as above, $i \in \mathbb{Z}$, and $b \in B_i$. Then the following are equivalent.*

(1) *For any $c \in B_{-i}$, $1 + cb \in B_0^\times$.*

(2) *For any $c \in B_{-i}$, $1 + cb \in B^\times$.*

(3) *$b \in {}^*\operatorname{rad} B$.*

(4) *$b \in {}^*\operatorname{rad} B^{\mathrm{op}}$, where B^{op} is the opposite ring of B.*

(5) *For any $c \in B_{-i}$, $1 + bc \in B^\times$.*

(6) *For any $c \in B_{-i}$, $1 + bc \in B_0^\times$.*

Proof. It is obvious that $B_0^\times \subset B_0 \cap B^\times$. Conversely, if $x \in B_0 \cap B^\times$ and $xy = 1$ with $y = \sum y_j$ ($y_j \in B_j$), then $xy_0 = 1$, and hence $x \in B_0^\times$. So (1)⇔(2) and (5)⇔(6) are obvious.

If $b \notin {}^*\operatorname{rad} B$, then $b \notin \mathfrak{m}$ for some *maximal left ideal $\mathfrak{m}$ of B. So $B = Bb + \mathfrak{m}$, and $1 + cb \in \mathfrak{m}$ for some $c \in B_{-i}$. Thus we have (2)⇒(3). Similarly, (5)⇒(4) is proved.

(3)⇒(2). If $b \in {}^*\operatorname{rad} B$ and $c \in B_{-i}$, then $cb \in (\operatorname{rad} B)_0$. So $1 + cb \in B_0$ is not contained in any *maximal ideal of B, and hence $B(1 + cb) = B$. So there exists some

$d \in B_0$ such that $d(1+cb) = 1$. If $d \in \mathfrak{m}$ for some *maximal ideal $\mathfrak{m}$ of B, then $d + dcb \in \mathfrak{m}$, since $dcb \in {}^*\operatorname{rad} B$. This shows that $1 = d(1+cb) \in \mathfrak{m}$, which is absurd. So $d \in B^\times$, and hence $1 + cb \in B^\times$.

(3)$\Rightarrow$(5). As $b \in {}^*\operatorname{rad} B$ and ${}^*\operatorname{rad} B$ is a two-sided ideal, $bc \in {}^*\operatorname{rad} B$. So by the assertion (3)$\Rightarrow$(2), which we have already proved, we have that $1 + bc \in B^\times$.

(4)$\Rightarrow$(2) follows from (3)$\Rightarrow$(5) above, applied to the graded ring B^{op}. □

Lemma 2.6. *Let B be as above. Then the following are equivalent.*

1. *B_0 is local.*
2. *B has a unique *maximal left ideal.*
3. *B has a unique *maximal right ideal.*
4. *For each $i \in \mathbb{Z}$, $B_i \setminus B^\times$ is an additive subgroup of B_i.*

Proof. (1)$\Rightarrow$(2). If $B = 0$, then $B_0 = 0$, and B_0 is not local. So $B \neq 0$, and B has a *maximal left ideal by Lemma 2.3. If B has two *maximal left ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ with $\mathfrak{m}_1 \neq \mathfrak{m}_2$, then there exists some $a_1 \in \mathfrak{m}_1 \cap B_0$ and $a_2 \in \mathfrak{m}_2 \cap B_0$ such that $a_1 + a_2 = 1$, and B_0 is not local.

(2)$\Rightarrow$(4). Let $\mathfrak{m}$ be the unique *maximal left ideal. If $b \in B_i \setminus B^\times$, then $b \in Bb \subset \mathfrak{m}$, and $B_i \setminus B^\times = B_i \cap \mathfrak{m}$.

(4)$\Rightarrow$(1). This is trivial.

(1)$\Leftrightarrow$(3) follows from (1)$\Leftrightarrow$(2), which already has been proved, applied to B^{op}. □

(2.7) We say that B is *local if B satisfies the equivalent conditions in Lemma 2.6.

Lemma 2.8 (graded Nakayama's lemma)**.** *Let B be as above, and $J = {}^*\operatorname{rad} B$. If M is a finitely generated graded left B-module and $JM = M$, then $M = 0$.*

Proof. For any homogeneous element m of M, $x \mapsto xm$ is a graded homomorphism of left B-modules $B \to M(-\deg m)$, and ${}^*\operatorname{rad} B$ is mapped to ${}^*\operatorname{rad} M$ by this map. This shows that $JM \subset {}^*\operatorname{rad} M \subset M$. By assumption, we have that ${}^*\operatorname{rad} M = M$. By Lemma 2.3, we have that $M = 0$. □

3. Main results

Lemma 3.1. *Let $(A, \mathfrak{m})$ be a Henselian local ring, and Λ a module-finite (non-commutative) A-algebra. Then Λ is semi-perfect, and the category of finite left Λ-modules is Krull–Schmidt.*

For Henselian local rings, see [LW, Section A.3] and [M].

Proof. Let V be a simple left Λ-module. Let $\bar{\Lambda} = \Lambda / \operatorname{rad} \Lambda$. As $\mathfrak{m}\Lambda \subset \operatorname{rad} \Lambda$ (by Nakayama's lemma applied to the finite A-module Λ), we have that $\bar{\Lambda}$ is a finite-dimensional $A/\mathfrak{m}$-algebra with the trivial radical. So $\bar{\Lambda}$ is semi-simple. As V is a simple $\bar{\Lambda}$-module, there exists some idempotent $\bar{e}$ of $\bar{\Lambda}$ such that $V = \bar{\Lambda}\bar{e}$. By [LW, Theorem A.30], $\bar{e}$ lifts to an idempotent e of Λ. Let $P = \Lambda e$. Then $P/JP = (\Lambda/J)e = \bar{\Lambda}\bar{e} = V$. By Nakayama's lemma, JP is the unique maximal submodule of P, and hence the canonical surjective map $P \to P/JP \cong V$ is a projective cover by [Kra, Lemma 3.6]. So Λ is semi-perfect, and the category of left Λ-modules is Krull–Schmidt, see [Kra, Proposition 4.1]. $\square$

(3.2) Let $R = \bigoplus_{i\geq 0} R_i$ be a Noetherian $\mathbb{Z}_{\geq 0}$-graded commutative ring such that $(R_0, \mathfrak{m}_0)$ is Henselian local. We set $\mathfrak{m} = R_+ + \mathfrak{m}_0$, where $R_+ = \bigoplus_{i>0} R_i$.

Lemma 3.3. *Let M be a finite graded R-module. Then $\bigcap_{r\geq 1} \mathfrak{m}^r M = 0$.*

Proof. Let $N = \bigcap_{r\geq 1} \mathfrak{m}^r M$. Note that N is a finite graded R-submodule of M. By Artin–Rees lemma, there exists some $c \geq 1$ such that

$$N = N \cap \mathfrak{m}^{c+1} M = \mathfrak{m}(N \cap \mathfrak{m}^c M) \subset \mathfrak{m}N \subset N.$$

As $\mathfrak{m}N = N$, $N = 0$ by graded Nakayama's lemma. $\square$

Let $T = \bigoplus_{i\in\mathbb{Z}} T_i$ be a graded R-algebra which is a finite R-module. Let J be the *radical $^*\operatorname{rad} T$ of T.

Lemma 3.4. *$J^r \subset \mathfrak{m}T \subset J$ for some $r \geq 1$.*

Proof. If $\mathfrak{m}T \not\subset J$, then there exists some *maximal left ideal $\mathfrak{n}$ of T such that $\mathfrak{n} + \mathfrak{m}T = T$. By graded Nakayama's lemma, $\mathfrak{n} = T$, and this is absurd. So $\mathfrak{m}T \subset J$.

To prove that $J^r \subset \mathfrak{m}T$ for some $r \geq 1$, we may assume that $\mathfrak{m} = 0$. Then T is a finite-dimensional $R_0/\mathfrak{m}_0$-algebra. As $J^r = J^{r+1}$ for some $r \geq 1$, $J^r = 0$ by graded Nakayama's lemma again. $\square$

Lemma 3.5. *Let e be an idempotent of $\hat{T}$, where $\hat{T}$ is the $\mathfrak{m}$-adic completion of T. If $e \in J\hat{T} = \hat{J}$, then $e = 0$. In particular, if e is an idempotent of T such that $e \in J$, then $e = 0$.*

Proof. By Lemma 3.4, we can take $r \geq 1$ such that $J^r \subset \mathfrak{m}T$. So $e = e^r \in \mathfrak{m}\hat{T}$. As $e = e^n \in \mathfrak{m}^n\hat{T}$ for any $n \geq 1$, so $e \in \bigcap_{n\geq 1} \mathfrak{m}^n \hat{T} = 0$. Since $T \to \hat{T}$ is injective, the last assertion follows immediately. $\square$

Lemma 3.6. *If T is *local, then $J = \operatorname{rad} T_0 + \bigoplus_{i\neq 0} T_i$, and $T/J \cong T_0/\operatorname{rad} T_0$ is a division ring.*

Proof. Replacing T by T/J, we may assume that $J = 0$. Then $\mathfrak{m}T \subset J = 0$. Replacing R by $R/\mathfrak{m}$, we may assume that $R = R_0$ is a field concentrated in degree zero. Then T is a finite-dimensional R-algebra. If $i \neq 0$, $a \in R_i$, and a is a unit, then $R_{ir} \neq 0$ for $r \in \mathbb{Z}$, and R cannot be finite-dimensional. So $T_i \subset {}^*\operatorname{rad} T$ for $i \neq 0$. On the other hand, we have

$$({}^*\operatorname{rad} T)_0 = T_0 \setminus T^\times = T_0 \setminus T_0^\times = \operatorname{rad} T_0.$$

So ${}^*\operatorname{rad} T = \operatorname{rad} T_0 + \bigoplus_{i\neq 0} T_i$. Hence $T/{}^*\operatorname{rad} T \cong T_0/\operatorname{rad} T_0$. As T_0 is local, this is a division ring. $\square$

(3.7) Let $T\,\mathrm{grmod}$ denote the category of finitely generated graded left T-modules. We say that $M \in T\,\mathrm{grmod}$ is *indecomposable if it is an indecomposable object of $T\,\mathrm{grmod}$. For $M \in \mathrm{grmod}\,T$, the endomorphism ring of M as an object of $T\,\mathrm{grmod}$ is $E_0 = (\mathrm{End}_T\, M)_0$, the degree zero component of $E = \mathrm{End}_T\, M$, the endomorphism ring of M as a (non-graded) T-module. Note that E_0 is finite as an R_0-module. As we assume that R_0 is Henselian local, E_0 is semi-perfect, and hence the additive category $T\,\mathrm{grmod}$ is Krull–Schmidt by Lemma 3.1. In particular, M is *indecomposable if and only if E is *local, that is, E_0 is local.

Theorem 3.8. *Let $R = \bigoplus_{i\geq 0} R_i$ be a Noetherian commutative graded ring such that $(R_0, \mathfrak{m}_0)$ is Henselian local, $R_+ = \bigoplus_{i>0} R_i$, $\mathfrak{m} = R_+ + \mathfrak{m}_0$, and T a $\mathbb{Z}$-graded module-finite (non-commutative) R-algebra. Let M be a finitely generated graded left T-module. Then the following are equivalent.*

(1) *$\hat{M}$ is indecomposable as a $\hat{T}$-module, where $\widehat{(-)}$ denotes the $\mathfrak{m}$-adic completion.*

(2) *$M_\mathfrak{m}$ is indecomposable as a $T_\mathfrak{m}$-module.*

(3) *M is indecomposable as a T-module.*

(4) *M is indecomposable as a graded T-module.*

Proof. (1)⇒(2). If $M_\mathfrak{m} \cong N_1 \oplus N_2$ as a $T_\mathfrak{m}$ module with $N_1 \neq 0$ and $N_2 \neq 0$, then taking the completion, $\hat{M} \cong \hat{N}_1 \oplus \hat{N}_2$, and $\hat{N}_1 \neq 0$ and $\hat{N}_2 \neq 0$, and $\hat{M}$ is not indecomposable. If $M_\mathfrak{m} = 0$, then $\hat{M} = 0$, and $\hat{M}$ is not indecomposable.

(2)⇒(3). If $M = 0$, then $M_\mathfrak{m} = 0$. Assume that there is a decomposition $M = M_1 \oplus M_2$ with $M_1 \neq 0$ and $M_2 \neq 0$ in the category of T-modules. Let m_1 and m_2 be non-zero elements of M_1 and M_2, respectively. Then it is easy to see that there exists some $r \geq 1$ such that both m_1 and m_2 are nonzero in $M/\mathfrak{m}^r M$ by Lemma 3.3. Then $M_1/\mathfrak{m}^r M_1 \neq 0$ and $M_2/\mathfrak{m}^r M_2 \neq 0$. This shows $(M_1)_\mathfrak{m} \neq 0$ and $(M_2)_\mathfrak{m} \neq 0$. This contradicts the indecomposability of $M_\mathfrak{m}$, since $M_\mathfrak{m} = (M_1)_\mathfrak{m} \oplus (M_2)_\mathfrak{m}$.

(3)⇒(4). Set $E = \mathrm{End}_T\, M$. Then $E \neq 0$, and E does not have a non-trivial idempotent. So $E_0 \neq 0$, and E_0 does not have a non-trivial idempotent. As the endomorphism ring of M as an object of $T\,\mathrm{grmod}$ is E_0, we have that M is indecomposable as an object of $T\,\mathrm{grmod}$.

(4)⇒(1). Let $\hat{e}$ be an idempotent of $\hat{E} = \mathrm{End}_{\hat{T}}\, \hat{M}$. If $\hat{e} \in \hat{J}$, then $\hat{e} = 0$ by Lemma 3.5. If $1 - \hat{e} \in \hat{J}$, then $1 - \hat{e} = 0$. So if $\hat{e}$ is nontrivial, then the image of $\hat{e}$ in $\hat{E}/\hat{J} \cong E_0/J_0$ must be still nontrivial, but this is absurd, since E_0/J_0 is a division ring where there is no nontrivial idempotent. □

Corollary 3.9. *Let M and N be objects in $T\,\mathrm{grmod}$, the category of finitely generated graded left T-modules. Then the following are equivalent.*

(1) *If $M = M_1 \oplus \cdots \oplus M_s$ and $N = N_1 \oplus \cdots \oplus N_t$ are decompositions into indecomposable objects in $T\,\mathrm{grmod}$, then $s = t$, and there exists some permutation $\sigma \in \mathfrak{S}_s$ and integers $d_1, \ldots, d_s$ such that $N_i \cong M_{\sigma i}(d_i)$, where $-(d_i)$ denotes the shift of degree.*

(2) *$M \cong N$ as T-modules.*

(3) *$M_{\mathfrak{m}} \cong N_{\mathfrak{m}}$ as $T_{\mathfrak{m}}$-modules.*

(4) *$\hat{M} \cong \hat{N}$ as $\hat{T}$-modules.*

Proof. (1)⇒(2)⇒(3)⇒(4) is trivial. We prove (4)⇒(1). As $\hat{R}$ is a Noetherian complete local ring and $\hat{T}$ is its module-finite algebra, the category of finite $\hat{T}$-modules is Krull–Schmidt. As $\hat{M} = \hat{M}_1 \oplus \cdots \oplus \hat{M}_s$ and $\hat{N} = \hat{N}_1 \oplus \cdots \oplus \hat{N}_t$ are decompositions into indecomposable $\hat{T}$-modules by Theorem 3.8, we have that $s = t$, and after change of indices (if necessary), there are isomorphisms $\hat{M}_i \cong \hat{N}_i$ for $i = 1, \ldots, s$. So we may assume that $s = t = 1$.

Let $\hat{\phi} : \hat{M} \to \hat{N}$ be a $\hat{T}$-isomorphism, and let $\hat{\psi} : \hat{N} \to \hat{M}$ be its inverse. We can write $\hat{\phi} = \sum_i \hat{a}_i \phi_i$ and $\hat{\psi} = \sum_j \hat{b}_j \psi_j$ with $\phi_i \in \operatorname{Hom}_T(M, N)_{u_i}$, $u_i \in \mathbb{Z}$, $\hat{a}_i \in \hat{R}$, $\psi_j \in \operatorname{Hom}_T(N, M)_{v_j}$, $v_j \in \mathbb{Z}$, and $\hat{b}_j \in \hat{R}$. So $1_{\hat{M}} = \hat{\psi}\hat{\phi} = \sum_{ij} \hat{a}_i \hat{b}_j \psi_j \phi_i$. As $1_{\hat{M}} \notin \hat{J}$, there exists some (i, j) such that $\psi_j \phi_i \notin J$, where $J = {}^*\operatorname{rad} \operatorname{End}_T M$. As $\psi_j \phi_i$ is a homogeneous element of $E = \operatorname{End}_T M$, which is *local, we have that $\psi_j \phi_i$ is a unit of E. In particular, ψ_j is a split epimorphism. As N is also indecomposable, we have that ψ_j is a T-isomorphism. So we have that $\psi_j : N \to M(v_j)$ is an isomorphism in $T \operatorname{grmod}$. □

Corollary 3.10. *Let $R = \bigoplus_{i \geq 0} R_i$ be a Noetherian $\mathbb{Z}_{\geq 0}$-graded commutative ring such that $(R_0, \mathfrak{m}_0)$ is an F-finite Henselian local ring of prime characteristic p. Let $\mathfrak{m} = \mathfrak{m}_0 + R_+$, where $R_+ = \bigoplus_{i > 0} R_i$. Let $\hat{R}$ be the $\mathfrak{m}$-adic completion of R. Let $M_1, \ldots, M_r$ be finitely generated $\mathbb{Q}$-graded R-modules. Then the following are equivalent.*

(1) *R has finite F-representation type (FFRT for short) in the graded sense with $M_1, \ldots, M_r$. That is,*

(1-a) *For each i, M_i is indecomposable;*

(1-b) *For each i, there exists some $e \geq 0$ and $c \in \mathbb{Q}$ such that $M_i(c)$ is a direct summand of ${}^eR$;*

(1-c) *For each $e \geq 0$, any indecomposable direct summand of ${}^eR$ is isomorphic to $M_i(c)$ for some $1 \leq i \leq r$ and $c \in \mathbb{Q}$.*

(2) *The local ring $\hat{R}$ has FFRT with $\hat{M}_1, \ldots, \hat{M}_r$. That is,*

(2-a) *For each i, $\hat{M}_i$ is indecomposable;*

(2-b) *For each i, there exists some $e \geq 0$ such that $\hat{M}_i$ is a direct summand of ${}^e\hat{R}$;*

(2-c) *For each $e \geq 0$, any indecomposable direct summand of ${}^e\hat{R}$ is isomorphic to $\hat{M}_i$ for some $1 \leq i \leq r$.*

In particular, R has FFRT in the graded sense if and only if $\hat{R}$ has FFRT.

Proof. By Theorem 3.8, (1-a) and (2-a) are equivalent.

Note that for each e,

$$ {}^e\hat{R} \cong \varprojlim {}^e(R/\mathfrak{m}^n) = \varprojlim {}^eR/{}^e(\mathfrak{m}^n) = \varprojlim {}^eR/{}^e((\mathfrak{m}^n)^{[p^e]}) = \varprojlim {}^eR/(\mathfrak{m}^n)^eR = \widehat{{}^eR}, $$

where $\mathfrak{m}^{[p^r]}$ is the ideal of R generated by $\{a^{p^r} \mid a \in \mathfrak{m}\}$.

(1-a)$\Rightarrow$(2-a). If $M_i(c)$ is a direct summand of ${}^eR$, then $\hat{M}_i$ is a direct summand of $\widehat{{}^eR} = {}^e\hat{R}$.

(2-a)$\Rightarrow$(1-a). Assume that $\hat{M}_i$ is a direct summand of ${}^e\hat{R}$. If we have a decomposition

$$ {}^eR \cong N_1 \oplus \cdots \oplus N_r, \tag{1} $$

where each N_j is an indecomposable $\mathbb{Q}$-graded R-modules, then

$$ {}^e\hat{R} \cong \widehat{{}^eR} \cong \hat{N}_1 \oplus \cdots \oplus \hat{N}_r. \tag{2} $$

This is the decomposition into indecomposable modules by Theorem 3.8. So by the Krull–Schmidt property of the category of $\hat{R}$-modules, $\hat{M}_i \cong \hat{N}_j$ for some j. By Corollary 3.9, $N_j \cong M_i(c)$ for some $c \in \mathbb{Q}$.

(1-c)$\Rightarrow$(2-c). Let $\hat{N}$ be a direct summand of ${}^e\hat{R}$. By assumption, there is a decomposition (1) such that each N_j is isomorphic to $M_{i(j)}(c_j)$ for some $1 \le i(j) \le r$ and $c_j \in \mathbb{Q}$. Then the isomorphism (2) holds, and by the Krull–Schmidt property, $\hat{N} \cong \hat{N}_j \cong \hat{M}_{i(j)}$.

(2-c)$\Rightarrow$(1-c). Let L be a direct summand of ${}^eR$. Then $\hat{L}$ is a direct summand of ${}^e\hat{R}$. So $\hat{L} \cong \hat{M}_i$ for some i. Hence by Corollary 3.9, $L \cong M_i(c)$ for some $c \in \mathbb{Q}$.

We prove the last assertion. The 'only if' part is clear from what we have proved above. We prove the 'if' part. Let $\hat{R}$ have FFRT with the finite indecomposable $\hat{R}$-modules $\hat{L}_1, \dots, \hat{L}_r$. So for each i, $\hat{L}_i$ is a direct summand of ${}^e\hat{R}$ for some e. Now let (1) be the decomposition of ${}^eR$ into indecomposable $\mathbb{Q}$-graded R-modules. Then we have an isomorphism (2), which is a decomposition into indecomposables by Theorem 3.8. By the Krull–Schmidt, $\hat{L}_i \cong \hat{N}_j$ for some j. In particular, there is a finite indecomposable $\mathbb{Q}$-graded module M_i such that $\hat{M}_i \cong \hat{L}_i$. By what we have proved above, R has FFRT with $M_1, \dots, M_r$ in the graded sense, as required. □

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