Hochster–Eagon type theorem for Serre's (S_n) condition

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Abstract

Let $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a pure homomorphism between Noetherian commutative rings. If $B/\mathfrak{m}B$ is an Artinian ring, then we have $\dim A = \dim B$ and $\operatorname{depth} A \geq \operatorname{depth} B$. Using this version of Hochster–Eagon theorem, we prove the following. Let $A \to B$ be a pure homomorphism between Noetherian commutative rings. Assume that the fiber ring $\kappa(\mathfrak{p}) \otimes_A B$ is Artinian for each $\mathfrak{p} \in \operatorname{Spec} A$, and B satisfies Serre's (S_n) condition. Then A also satisfies Serre's (S_n) condition. In particular, if a finite group G acts on B and the order $|G|$ of G is invertible in B, and if B is Noetherian with the (S_n) condition, then the ring of invariants $A = B^G$ also satisfies the (S_n) condition.

1. Introduction

We say that a homomorphism of commutative rings $f : A \to B$ is *pure* if for any A-module W, the map $j_W : W \to B \otimes_A W$ given by $j_W(w) = 1 \otimes w$ is injective. If $f : A \to B$ is pure, many good ring-theoretic properties of B are inherited by A. For example, if B is Noetherian (resp. a normal domain), then so is A [HR, (6.15)]. The Hochster–Eagon theorem [HE, Proposition 12], a striking result in this direction appeared in 1971, states that if $f : A \to B$ is an integral homomorphism between commutative rings, B is Noetherian and Cohen–Macaulay, and A is a direct summand subring of B through f (that is, there is a left inverse $g : B \to A$ as an A-linear map (that is, $gf = 1_A$)), then A is also Noetherian and Cohen–Macaulay. As an application, Hochster and Eagon proved that the non-modular invariant subring of a finite group action on a Cohen–Macaulay ring is again Cohen–Macaulay [HE]. The assumption of integral extension is necessary, as any finitely generated domain over

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a field, Cohen–Macaulay or not, is a direct summand subring of a finitely generated Cohen–Macaulay domain by Kawasaki's arithmetic Macaulayfication theorem [Kaw, Theorem 1.3]. Although Cohen–Macaulay property is not inherited by a pure subring in genral, some important results in this direction are known, see [Bou, HH, HR, Sch].

In this paper, we prove an analog of Hochster–Eagon theorem for Serre's (S_n) condition. We say that a Noetherian commutative ring R satisfies the condition (S_n) if $\operatorname{depth} R_{\mathfrak{p}} \geq \min(n, \dim R_{\mathfrak{p}})$ for any $\mathfrak{p} \in \operatorname{Spec} R$. This is equivalent to say that for any $\mathfrak{p} \in \operatorname{Spec} R$ such that $\operatorname{depth} R_{\mathfrak{p}} < n$, we have $R_{\mathfrak{p}}$ is Cohen–Macaulay. Our main theorem is the following.

Theorem 2.8. *Let $f : A \to B$ be a pure ring homomorphism. Let B be a Noetherian ring which satisfies Serre's (S_n) condition. If the fiber ring $B \otimes_A \kappa(\mathfrak{p})$ is zero-dimensional for any prime ideal $\mathfrak{p}$ of A, then A satisfies (S_n), too.*

If $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ is a pure local homomorphism such that $B/\mathfrak{m}B$ is Artinian, then we have that $\dim A = \dim B$ and $\operatorname{depth} A \geq \operatorname{depth} B$ (Proposition 2.6), and the proof of Theorem 2.8 is reduced to this version of Hochster–Eagon theorem. As a corollary, we have a result in non-modular invariant theory of finite groups; if a finite group G acts on a Noetherian ring B and the order $|G|$ of G is invertible in B, the (S_n) property of B is inherited by A for any $n \geq 1$ (Corollary 2.9).

2. The results

(2.1) Let R be a commutative ring and $f : M \to N$ be an R-linear map between R-modules. We say that f is *pure* or R-pure if for any R-module W, the map $f \otimes 1_W : M \otimes_R W \to N \otimes_R W$ is injective. Clearly, a pure linear map is injective. A ring homomorphism between commutative rings $\varphi : A \to B$ is said to be pure if it is so as an A-linear map. For each A-module W, we denote by $j_W : W = A \otimes_A W \to B \otimes_A W$ the map $\varphi \otimes 1_W$. So φ is pure if and only if j_W is injective for any A-module W. If A is a subring of B and the inclusion map $A \hookrightarrow B$ is pure, then we say that A is a pure subring of B.

(2.2) Let $\varphi : A \to B$ be a pure ring homomorphism, and $\mathfrak{p} \in \operatorname{Spec} A$. Then $\kappa(\mathfrak{p}) = A \otimes_A \kappa(\mathfrak{p}) \to B \otimes_A \kappa(\mathfrak{p})$ is injective, and hence the fiber ring $B \otimes_A \kappa(\mathfrak{p})$ is not zero, where $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is the residue field of $A_{\mathfrak{p}}$. It follows that ${}^a\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective, since $({}^a\varphi)^{-1}(\mathfrak{p}) \cong \operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p})) \neq \emptyset$ for each $\mathfrak{p}$. In particular, a pure ring homomorphism between local rings is a local homomorphism.

(2.3) Let $\varphi : A \to B$ be a pure homomorphism. If B is Noetherian, then so is A [HR, (6.15)].

Lemma 2.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring, and $f : A \to B$ be a pure ring homomorphism. Then there exists some maximal ideal $\mathfrak{n}$ of B such that $\mathfrak{n} \cap A = \mathfrak{m}$, and $A \to B_{\mathfrak{n}}$ is pure.*

Proof. Let $E = E_A(A/\mathfrak{m})$ be the injective hull of the residue field of A. Then $j_E(1) \neq 0$ in $B \otimes_A E$ [HR, (6.11)], where 1 is the image of $1 \in A/\mathfrak{m}$ in E. So there exists some maximal ideal $\mathfrak{n}$ of B such that $j_E(1)$ is still nonzero in $B_\mathfrak{n} \otimes_A E$. Then $A \to B_\mathfrak{n}$ is pure by [HR, (6.11)] again. As $\operatorname{Spec} B_\mathfrak{n} \to \operatorname{Spec} A$ is surjective, we must have $\mathfrak{n} \cap A = \mathfrak{m}$. □

Lemma 2.5. *If A is a Noetherian ring and $f : A \to B$ is a pure ring homomorphism, then* $\dim A \leq \dim B$.

Proof. Let $\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_2 \supsetneq \cdots \supsetneq \mathfrak{p}_d$ be a chain of prime ideals of A. Note that $A_{\mathfrak{p}_0} \to B_{\mathfrak{p}_0}$ is pure by [HR, (6.2)]. By Lemma 2.4, there exists some prime ideal P_0 of B lying over $\mathfrak{p}_0$ such that $A_{\mathfrak{p}_0} \to (B_{\mathfrak{p}_0})_{P_0 B_{\mathfrak{p}_0}} = B_{P_0}$ is pure. Using this argument to the pure ring homomorphism $A_{\mathfrak{p}_0} \to B_{P_0}$, we know that there exists some prime ideal $P_1 \subset P_0$ such that $P_1 \cap A = \mathfrak{p}_1$ and $A_{\mathfrak{p}_1} \to B_{P_1}$ is pure. Continuing this, we can take a chain of prime ideals

$$P_0 \supsetneq P_1 \supsetneq P_2 \supsetneq \cdots \supsetneq P_d$$

such that $P_i \cap A = \mathfrak{p}_i$ and $A_{\mathfrak{p}_i} \to B_{P_i}$ is pure. This proves the lemma. □

Proposition 2.6. *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a pure local ring homomorphism such that B is Noetherian and the fiber ring $B/\mathfrak{m}B$ is Artinian. Then* $\dim A = \dim B$ *and* $\operatorname{depth} A \geq \operatorname{depth} B$.

Proof. We have $\dim A \leq \dim B \leq \dim A + \dim B/\mathfrak{m}B = \dim A$ by Lemma 2.5 and the assumption that $\dim B/\mathfrak{m}B$ is Artinian.

Let $r = \operatorname{depth} A$ and $s = \operatorname{depth} B$. We prove that $r \geq s$ by induction on s. If $s = 0$, then the assertion is trivial. Assume that $s > 0$. If $r = 0$, then there exists some $x \in \mathfrak{m}$ such that $0 :_A x = \mathfrak{m}$. As f is an injective map, $\mathfrak{m}B \subset 0 :_B x \subset \mathfrak{n}$. As $B/\mathfrak{m}B$ is Artinian, $B/(0 :_B x)$ is Artinian and is nonzero. Hence

$$\operatorname{Ass}_B B \supset \operatorname{Ass}_B Bx = \operatorname{Ass}_B B/(0 :_B x) = \{\mathfrak{n}\},$$

and $s = 0$. This is a contradiction. So $r > 0$. For each $P \in \operatorname{Ass} B$, we have $P \cap A \neq \mathfrak{m}$, since $\mathfrak{n} \cap A = \mathfrak{m}$, $P \subsetneq \mathfrak{n}$, and $B/\mathfrak{m}B$ is zero-dimensional. By prime avoidance, there exists some

$$y \in \mathfrak{m} \setminus \left(\Big(\bigcup_{P \in \operatorname{Ass} B} (P \cap A) \Big) \cup \Big(\bigcup_{\mathfrak{p} \in \operatorname{Ass} A} \mathfrak{p} \Big) \right).$$

So y is A-regular B-regular. By [HR, (6.2)], $A/yA \to B/yB$ is pure, and $(B/yB)/\mathfrak{m}(B/yB)$ is Artinian. By induction assumption, we have

$$r - 1 = \operatorname{depth} A/yA \geq \operatorname{depth} B/yB = s - 1$$

and hence $r \geq s$, as required. □

Theorem 2.7 (Hochster–Eagon [HE]). *Let $f : A \to B$ be a pure ring homomorphism. Let B be Noetherian and Cohen–Macaulay. If the fiber ring $B \otimes_A \kappa(\mathfrak{m})$ is zero-dimensional for any maximal ideal $\mathfrak{m}$ of A, then A is Cohen–Macaulay.*

Proof. It suffices to show that $A_{\mathfrak{m}}$ is a Cohen–Macaulay local ring for any maximal ideal $\mathfrak{m}$ of A. Replacing f by $f_{\mathfrak{m}} : A_{\mathfrak{m}} \to B_{\mathfrak{m}}$, we may assume that $(A, \mathfrak{m})$ is local. Then we can find a maximal ideal $\mathfrak{n}$ of B lying over $\mathfrak{m}$ such that $A \to B_{\mathfrak{n}}$ is pure. As a localization of a zero-dimensional ring is still zero-dimensional, we may assume that f is a local homomorphism between local rings. By the Cohen–Macaulay property of B and Proposition 2.6, we have

$$\operatorname{depth} A \geq \operatorname{depth} B = \dim B = \dim A \geq \operatorname{depth} A,$$

and the assertion follows. □

Theorem 2.8. *Let $f : A \to B$ be a pure ring homomorphism. Let B be a Noetherian ring which satisfies Serre's (S_n) condition. If the fiber ring $B \otimes_A \kappa(\mathfrak{p})$ is zero-dimensional for any prime ideal $\mathfrak{p}$ of A, then A satisfies (S_n), too.*

Proof. Let $\mathfrak{p}$ be a prime ideal of A such that $\operatorname{depth} A_{\mathfrak{p}} < n$. Then we can find a prime ideal P of B lying over $\mathfrak{p}$ such that the local homomorphism $A_{\mathfrak{p}} \to B_P$ is pure. Note that the dimension of the fiber ring $\kappa(\mathfrak{p}) \otimes_{A_{\mathfrak{p}}} B_P$ is zero-dimensional. Since $\operatorname{depth} B_P \leq \operatorname{depth} A_{\mathfrak{p}} < n$ by Proposition 2.6 and B satisfies the (S_n) condition, we have that B_P is Cohen–Macaulay. By Theorem 2.7, we have that $A_{\mathfrak{p}}$ is Cohen–Macaulay. This shows that A satisfies the (S_n) condition. □

Corollary 2.9. *Let G be a finite group acting on a Noetherian ring B. Assume that the order $|G|$ of G belongs to $B^\times$, the unit group of B. If B satisfies Serre's (S_n) condition, then so does the ring of invariants B^G.*

Proof. For $b \in B$, the polynomial $f(t) = \prod_{g \in G}(t - gb)$ is monic and lies in $B^G[t]$. As $f(b) = 0$, the element b is integral over B^G. This shows that $B^G \hookrightarrow B$ is an integral extension, and all the fiber rings are zero-dimensional.

Let $R = \mathbb{Z}[|G|^{-1}]$, and define the Reynolds operator to be $\rho = |G|^{-1} \sum_{g \in G} g \in RG$. It is a central idempotent of the group algebra RG of G over R. As RG acts on B, the G-linear map $\rho : B \to B$ is defined. It is easy to see that $\rho(B) \subset B^G$ and ρ is the identity map on B^G. So $\rho(B) = B^G$, and $\rho : B \to B^G$ is the left inverse of the inclusion $j : B^G \to B$. Moreover,

$$\rho(ab) = |G|^{-1} \sum_{g \in G} a(gb) = a(\rho b)$$

for $a \in B^G$ and $b \in B$. Namely, ρ is the left inverse of j as a B^G-linear map, and thus B^G is a direct summand subring of B. It follows that B^G is a pure subring of B. The corollary follows from Theorem 2.8. □

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