INVARIANT INTEGRAL STRUCTURES IN PSEUDO H-TYPE LIE ALGEBRAS: CONSTRUCTION AND CLASSIFICATION

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Abstract. Pseudo H-type Lie algebras are a special class of 2-step nilpotent metric Lie algebras, intimately related to Clifford algebras $\mathrm{Cl}_{r,s}$. In this work we propose the classification method for integral orthonormal structures of pseudo H-type Lie algebras. We apply this method for the full classification of these structures for $r \in \{1, \dots, 16\}$, $s \in \{0, 1\}$ and irreducible Clifford modules. The latter cases form the basis for the further extensions by making use of the Atiyah-Bott periodicity. The existence of integral structures gives rise to the integral discrete uniform subgroups of the pseudo H-type Lie groups.

Contents

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1. Introduction

Two-step nilpotent Lie algebras attracted the attention of G. Métivier [M80́] in an attempt to describe hypoelliptic operators in a non-Euclidean setting. The condition of hypo-ellipticity required the adjoint map with the value on the center to be surjective. This type of Lie algebras was studied under different names and for different purposes, for instance, in [Ebe94, LT99, MS04, OW10, GMKMV18]. A. Kaplan [Kap80] showed that if the adjoint operator is an isometry, then the sub-Laplacian on two-step nilpotent Lie groups, admits a fundamental solution, reminiscent of that in Euclidean space. His result extended a theorem obtained by G. Folland on the Heisenberg group [Fol73]. Therefore, the class of these Lie algebras received the name H(eisenberg)-type Lie algebras. The H-type Lie algebras are in a bijective relation to Clifford algebras $\mathrm{Cl}_{r,0}$, generated by the Euclidean space $\mathbb{R}^r$ [Rei01a]. The definition of H-type Lie algebras related to Clifford algebras $\mathrm{Cl}_{r,s}$, $s > 0$, generated by pseudo Euclidean spaces $\mathbb{R}^{r,s}$ was extended by P. Ciatti [Cia00] and received the name pseudo H-type Lie algebras, see also [GMKM13]. The pseudo H-type Lie algebras, which will be denoted by $\mathfrak{n}_{r,s}$ is a fruitful source for studies of Damek-Ricci spaces [BTV95], Iwasawa decomposition of symmetric spaces [CDKR98], Riemannian nilmanifolds [Kap81], rigidity problems [Rei01b], properties of PDE on Lie groups [CS12, MR92, BFM20] and many others topics in geometry, analysis, and geometric measure theory. The classification of the pseudo H-type Lie algebras was completed in [FM17, FM19].

Our work is motivated by the study of uniform discrete subgroups on nilpotent Lie groups, which are crucial for the study of homogeneous spaces, compact nilmanifolds, and spectral problems. The existence of a uniform subgroup is guaranteed by a presence of a rational structure on the associated Lie algebra by seminal work of A. I. Malčev [Mc49]. The existence of rational structures on pseudo H-type Lie algebras was proved in [CD02, Ebe03, FM14]. A complete classification of rational structures in the class of pseudo H-type Lie algebras exists only on the Heisenberg algebra (related to the Clifford algebra $\mathrm{Cl}_{1,0}$) [GW86]. Some progress in the study of lattices can be found in [CP08].

In the present work, we describe the set of invariant integral structures, which are at the core of rational structures of the Lie algebras. An invariant integral structure is a span over $\mathbb{Z}$ of an orthonormal basis, constructed as an action of a subgroup $G(B_{r,s})$ of the invertible elements $\mathrm{Pin}(r,s)$ in the Clifford algebra $\mathrm{Cl}_{r,s}$ on a suitably chosen normal vector $v \in V$ in the Clifford module V, see Section 3 and Section 3.2. As a result, the basis of the Clifford module V is invariant under the action of $G(B_{r,s})$ and the non-vanishing structure constants of the H-type Lie algebra are equal to ± 1. We emphasize that invariant integral structures are particular cases of integral structures (having structure constants ± 1) that are included in a general class of rational structures on a Lie algebra (having rational structure constants). Two invariant integral structures are orthogonally isomorphic, if and only if the isotropy subgroups $\mathcal{S}_v^{(1)} \subset \mathrm{Cl}_{r,s}$ and $\mathcal{S}_v^{(2)} \subset \mathrm{Cl}_{r,s}$ of $v \in V$ belongs to the same equivalence class, see Definition 4.3 in Section 4. Section 6 is dedicated to showing the isomorphism properties of invariant integral structures on the H-type Lie algebras concerning the equivalence of the isotropy subgroups. The isomorphism of invariant integral structures of the Lie algebras leads to the isomorphism of uniform discrete subgroups on the corresponding Lie groups, which is always extended to an automorphism of ambient pseudo H-type Lie groups, see [Rag72].

We apply the classification algorithm to isotropy groups $\mathcal{S}_v$ for parameters $r \in \{3, \dots, 16\}$ and $s \in \{0, 1\}$ in Section 5. We note that the restricted range of r and s in the construction of the list of non-equivalent isotropy groups corresponds to the first and the second period in r of pseudo H-type Lie groups concerning the Atiyah-Bott periodicity of the Clifford algebras. The reader can notice that the second period $r \in \{9, \dots, 16\}$ contains more non-equivalent subgroups with phenomena, such as disconnectedness, that can not appear in the first period $r \in \{3, \dots, 8\}$ due to the lack of dimension of the center of the Lie algebra. The forthcoming paper will be dedicated to the study of new features in the increasing of the parameter s and the study of the periodicity in both r and s of the construction of non-equivalent isotropy groups. Despite this, most of the theorems and the characterizations proved in Sections 3, 4, and 6 are valid for arbitrary parameters (r, s).

2. Clifford algebras and pseudo H-type Lie algebras

In this section we remind some classical objects and introduce the main ones of our interest.

2.1. **Clifford algebras.** We denote by $\mathbb{R}^{r,s}$ the pseudo Euclidean space, that is the vector space $\mathbb{R}^{r+s}$ endowed with the non-degenerate symmetric bilinear form

$$\langle x, y\rangle_{r,s} = \sum_{k=1}^{r} x_k y_k - \sum_{k=r+1}^{r+s} x_k y_k.$$

Let $\mathrm{Cl}_{r,s}$ be a Clifford algebra over $\mathbb{R}$ generated by $\mathbb{R}^{r,s}$. Remind that $\mathrm{Cl}_{r,s}$ is a quotient of the tensor algebra

$$\mathcal{T}(U) := \mathbb{R} \oplus \mathbb{R}^{r,s} \oplus \Big(\overset{2}{\otimes}\, \mathbb{R}^{r,s}\Big) \oplus \Big(\overset{3}{\otimes}\, \mathbb{R}^{r,s}\Big) \oplus \Big(\overset{4}{\otimes}\, \mathbb{R}^{r,s}\Big) \oplus \dots$$

by a two sided ideal $I_{r,s}$ generated by elements of the form

$$x \otimes x + \langle x, x\rangle_{r,s}\mathbf{1}, \quad x \in \mathbb{R}^{r+s},$$

and $\mathbf{1}$ is the identity element of the Clifford algebra $\mathrm{Cl}_{r,s}$. Consider a representation of $\mathrm{Cl}_{r,s}$ on a real vector space V

$$J\colon \mathrm{Cl}_{r,s} \to \mathrm{End}(V).$$

We call V the $\mathrm{Cl}_{r,s}$-module, or simply module if we do not want to specify the signature (r, s), and will denote by $J_z v$ the action of $z \in \mathbb{R}^{r,s}$ on $v \in V$. Assume also that the module V is equipped with a non-degenerate symmetric bilinear form $\langle . , .\rangle_V$ satisfying the condition

$$\langle J_z u, v\rangle_V + \langle u, J_z v\rangle_V = 0 \quad \text{for any} \quad z \in \mathbb{R}^{r,s} \quad \text{and} \quad u, v \in V. \tag{2.1}$$

We call such a module $V = (V, \langle . , .\rangle_V)$ an *admissible module* of the Clifford algebra $\mathrm{Cl}_{r,s}$. We write $V_{min} = (V_{min}, \langle . , .\rangle_V)$ or simply V_{min} for an admissible $\mathrm{Cl}_{r,s}$-module of the minimal dimension and call it a *minimal admissible module*. The reader can find more about analogous constructions of 2 step nilpotent Lie algebras, not related to representation of Clifford algebras in [Ebe04].

We emphasise the difference between an irreducible Clifford module and a minimal admissible module. Not all irreducible modules can be equipped with a non-degenerate bilinear symmetric form, satisfying (2.1). For instance, lack of dimension of an irreducible module can make any bilinear symmetric form degenerate. An admissible module V of $\mathrm{Cl}_{r,s}$ has an even dimension $\dim(V) = 2n = N$. It is isometric to $\mathbb{R}^{n,n}$ if $s > 0$ and it is isometric to $\mathbb{R}^{\pm N,0}$ if $s = 0$, see [Cia00, Theorem 3.1] and [FM17, Proposition 1]. Any admissible $\mathrm{Cl}_{r,s}$-module can be decomposed into an orthogonal direct sum of minimal admissible modules [FM19, Proposition 2.3 (2)].

2.2. Pseudo H-type Lie algebras and Lie groups.

Definition 2.1. *Let $(V, \langle . , .\rangle_V)$ be an admissible module of a Clifford algebra $Cl_{r,s}$ with the representation map J. Define the Lie bracket on $V \times \mathbb{R}^{r,s}$ by*

$$\langle J_z u, v\rangle_V = \langle z, [u, v]\rangle_{r,s}, \quad z \in \mathbb{R}^{r,s}, \quad u, v \in V. \tag{2.2}$$

The pseudo H-type Lie algebra $\mathfrak{n}_{r,s}(V) = (V \oplus \mathbb{R}^{r,s}, [. , .])$ is a Lie algebra whose non-vanishing Lie bracket is defined in (2.2).

Note that the Lie algebra $\mathfrak{n}_{r,s}(V)$ is 2-step nilpotent where $\mathbb{R}^{r,s}$ is the centre. Property (2.1) and the representation property $J_z^2 v = -\langle z, z\rangle_{r,s} v$ for $v \in V$ imply

$$\langle J_z u, J_z v\rangle_{r,s} = \langle z, z\rangle_{r,s}\langle u, v\rangle_V, \quad \langle J_z u, J_w u\rangle_{r,s} = \langle z, w\rangle_{r,s}\langle u, u\rangle_V. \tag{2.3}$$

The connected simply connected Lie group $\mathbb{N}_{r,s}(V)$ of the Lie algebra $\mathfrak{n}_{r,s}(V)$ is called the pseudo H-type Lie group. The exponential map $\exp\colon \mathfrak{n}_{r,s}(V) \to \mathbb{N}_{r,s}(V)$ is a global analytic diffeomorphism [CG90, Theorem 1.2.1]. It allows to induce the coordinates on the Lie group from the Lie algebra by means of Backer-Campbell-Hausdroff formula. Points $g \in \mathbb{N}_{r,s}(V)$ are considered as vectors $g = (u, z) \in V \oplus \mathbb{R}^{r,s} = \mathfrak{n}_{r,s}(V)$. The group product $*$ on $\mathbb{N}_{r,s}(V)$ is given by

$$
\begin{aligned}
&*\colon \mathbb{N}_{r,s}(V) \times \mathbb{N}_{r,s}(V) \to \mathbb{N}_{r,s}(V),\\
&\quad (u_1, z_1) * (u_2, z_2) = \Big(u_1 + u_2,\ z_1 + z_2 + \frac{1}{2}[u_1, u_2]\Big).
\end{aligned}
$$

2.3. **Automorphisms of pseudo H-type Lie algebras.** Since automorphisms of a Lie algebra define the automorphisms of its connected simply connected Lie group, we consider only the automorphisms of Lie algebras. The complete description of the group of automorphisms of pseudo H-type Lie algebras can be found in [Rie82, Saa96, FM21], see also [AS14].

The automorphisms of pseudo H-type Lie algebras are generated by the following ones:

[1] The transformations $\delta_\lambda(u, z) = (\lambda u, \lambda^2 z)$, calling the dilations.

[2] Let $A\colon V \to V$ be a nonsingular linear map and $C \in \mathrm{O}(r, s)$ an orthogonal transformation of $\mathbb{R}^{r,s}$. Then the map $A \oplus C$ is a pseudo H-type Lie algebra automorphism, if and only if

$$
A^\tau \circ J_z \circ A = J_{C^\tau(z)}, \quad z \in \mathbb{R}^{r,s}, \tag{2.4}
$$

where A^τ, C^τ are transpose maps with respect to the respective bilinear forms

$$
\langle A^\tau u, v\rangle_V = \langle u, Av\rangle_V, \quad \langle C^\tau z, w\rangle_{r,s} = \langle z, Cw\rangle_{r,s}.
$$

[3] Let $B\colon V \to \mathbb{R}^{r,s}$ be a linear map, then $(v, z) \mapsto \big(v, z + Bv\big)$ is an automorphism.

2.4. **Rational structures, uniform discrete subgroups, lattices.** We refer to [Rag72, CG90] for the details discussed in this section.

Definition 2.2. *A Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ over rational numbers $\mathbb{Q}$ is called the rational structure of a real Lie algebra $\mathfrak{g}$ if $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$.*

A real Lie algebra $\mathfrak{g}$ has a rational structure if and only if there is a basis for $\mathfrak{g}$ such that the structure constants of the Lie algebra are rational numbers.

Definition 2.3. *Let G be a Lie group. A subgroup Γ is called uniform subgroup if Γ is discrete and G/Γ is a compact space.*

Definition 2.4. *Let G be a Lie group with a measure μ. A subgroup Λ is called lattice if $\mu(G/\Lambda) < \infty$.*

Let G be a nilpotent Lie group and μ the Haar measure on it. Then a discrete subgroup Γ is lattice if and only if it is a uniform subgroup, i.e $\mu(G/\Gamma) < \infty$ implies that G/Γ is compact. From now on we will not distinguish the lattices and uniform subgroups. A result from [Mc49] can be formulated as follows.

- If Γ is a uniform subgroup of G, then $\mathfrak{g}$ has a rational structure $\mathfrak{g}_{\mathbb{Q}}$ such that $\mathfrak{g}_{\mathbb{Q}} = \operatorname{span}_{\mathbb{Q}}\{\log(\Gamma)\}$.
- If $\mathfrak{g}$ has a rational structure $\mathfrak{g}_{\mathbb{Q}}$, then G has a uniform subgroup Γ such that $\log(\Gamma) \subseteq \mathfrak{g}_{\mathbb{Q}}$.

Theorem 2.5. [Rag72] *Let $\Gamma_i \subset G_i$, $i = 1, 2$ be uniform subgroups of simply connected nilpotent Lie groups G_i. An isomorphism $\varphi\colon \Gamma_1 \to \Gamma_2$ of discrete subgroups, can be extended to the smooth isomorphism $\tilde{\varphi}\colon G_1 \to G_2$ of the Lie groups.*

3. Invariant basis of a Clifford module

3.1. Definition of invariant integral structure and uniform subgroups.

From now on we will consider only **minimal admissible modules** of Clifford algebras $\mathrm{Cl}_{r,s}$, denoting them either by $V^{r,s}$ or simply by V. Let $\mathfrak{n}_{r,s}(V) = (V \oplus \mathbb{R}^{r,s}, [.\,,.])$ be a pseudo H-type Lie algebra with $B_{r,s}$ a basis for $\mathbb{R}^{r,s}$ and $\mathfrak{B}(V)$ a basis for V. Note that $\mathbb{R}^{r,s}$ is the centre of $\mathfrak{n}_{r,s}(V)$. We write the structure constants c_{ij}^l for $\mathfrak{n}_{r,s}(V)$ with respect to bases $\mathfrak{B}(V)$ and $B_{r,s}$ by

$$[v_i, v_j] = \sum_{l=1}^{r+s} c_{ij}^l z_l. \tag{3.1}$$

Definition 3.1. *A basis $\{\mathfrak{B}(V), B_{r,s}\}$ for $\mathfrak{n}_{r,s}(V)$ is called integral if the structure constants c_{ij}^l in (3.1) take the values in $\{-1, 0, 1\}$.*

We want to study a special class of integral bases of $\mathfrak{n}_{r,s}(V)$. To describe it, we fix an orthonormal basis $B_{r,s} = \{z_1, \ldots, z_r, z_{r+1}, \ldots, z_{r+s}\}$ of $\mathbb{R}^{r,s}$, where

$$\begin{cases} z_1, \ldots, z_r \quad \text{are positive, i.e., } \langle z_i,\, z_i\rangle_{r,s} = 1, \\ z_{r+1}, \ldots, z_{r+s} \text{ are negative, i.e., } \langle z_i,\, z_i\rangle_{r,s} = -1. \end{cases} \tag{3.2}$$

Denote by $G(B_{r,s})$ a finite subgroup of the Pin group in $\mathrm{Cl}_{r,s}$ defined by

$$\begin{aligned} G(B_{r,s}) = \{ \quad & \pm\mathbf{1},\ \pm z_1,\ \ldots,\ \pm z_{r+s},\ \pm z_{i_1}\cdots z_{i_k} \mid \\ & 1 \le i_1 < \cdots < i_k \le r+s, \quad k = 2, \ldots, r+s\}. \end{aligned}$$

Thus the generators of the group $G(B_{r,s})$ are $\{-1, B_{r,s}\}$. Elements $\sigma \in G(B_{r,s})$ satisfy the properties: either $\sigma^2 = \mathbf{1}$ or $\sigma^2 = -\mathbf{1}$.

We proceed to the construction of bases $\mathfrak{B}(V^{r,s})$ for the minimal admissible module $V^{r,s}$. In Table (1) the reader finds dimensions of $V^{r,s}$. We indicated by red colour the Clifford algebras, where the minimal admissible modules differ from the irreducible modules. With the subscript $_{\times 2}$ we indicated the presence of two non-equivalent minimal admissible modules.

TABLE 1. Dimensions of minimal admissible modules

8	16	32	64	$64_{\times 2}$	128	128	128	$128_{\times 2}$	256
7	16	32	64	64	128	128	128	128	256
6	16	$16_{\times 2}$	32	32	64	$64_{\times 2}$	128	128	256
5	16	16	16	16	32	64	128	128	256
4	8	8	8	$8_{\times 2}$	16	32	64	$64_{\times 2}$	128
3	8	8	8	8	16	32	64	64	128
2	4	$4_{\times 2}$	8	8	16	$16_{\times 2}$	32	32	64
1	2	4	8	8	16	16	16	16	32
0	1	2	4	$4_{\times 2}$	8	8	8	$8_{\times 2}$	16
s/r	0	1	2	3	4	5	6	7	8

(1) If a minimal admissible module $V^{r,s}$ is irreducible, then the set

$$O_v = G(B_{r,s}).v := \{J_\sigma v \mid \ \sigma \in G(B_{r,s})\} \tag{3.3}$$

contains a basis $\mathfrak{B}(V^{r,s})$ for any non-zero vector $v \in V^{r,s}$.

(2) If a minimal admissible module $V^{r,s}$ is reducible, then set (3.3) contains $\mathfrak{B}(V^{r,s})$ for any non-zero and non-null vector $v \in V^{r,s}$.

Thus we obtain that $V^{r,s} = \operatorname{span}_{\mathbb{R}}\{O_v\} = \operatorname{span}_{\mathbb{R}}\{\mathfrak{B}(V^{r,s})\}$. If $v \in V^{r,s}$ is a null vector, then the orbit O_v depends on the choice of v, but even in this case, one can make a special choice of a null vector $v \in V^{r,s}$, that generates an entire orbit O_v including $\mathfrak{B}(V^{r,s})$. From the other side if $V^{r,s} = V_1^{r,s} \oplus V_2^{r,s}$ is a decomposition of a minimal admissible module on irreducible modules, then the bilinear form $\langle . , . \rangle_{V^{r,s}}$ vanishes identically on $V_i^{r,s}$, $i = 1, 2$. In this case only the union $\bigcup_{i=1}^{2} \{J_\sigma v_i \mid \sigma \in G(B_{r,s})\}$ contains a basis $\mathfrak{B}(V^{r,s})$, where one needs to choose two non-zero vectors $v_i \in V_i^{r,s}$.

Based on the latter discussions we restrict ourselves at bases $\mathfrak{B}(V^{r,s})$ consisting of non-null vectors and make the following definition.

Definition 3.2. *Fix an orthonormal basis $B_{r,s}$ of $\mathbb{R}^{r,s}$. An orthonormal basis $\mathfrak{B}(V^{r,s})$ of a minimal admissible module $V^{r,s}$ is called invariant basis if it is invariant under the action of $G(B_{r,s})$; that is for any $v_i \in \mathfrak{B}(V^{r,s})$ and $z_j \in B_{r,s}$, there exists $v_k \in \mathfrak{B}(V^{r,s})$ such that $J_{z_j} v_i = v_k$ or $J_{z_j} v_i = -v_k$.*

Definition 3.2 requires that the maps J_{z_j}, $z_j \in B_{r,s}$ act on an invariant basis $\mathfrak{B}(V^{r,s})$ by permutations up to the sign $\pm$.

Remark 3.1. *We emphasise that we require bases $\mathfrak{B}(V^{r,s})$ to be both orthonormal and invariant.*

EXAMPLE A. *Consider the Heisenberg Lie algebra $\mathfrak{n}_{1,0}(V)$ with the normalized basis $B_{1,0} = \{z\}$ for the centre and $V^{1,0} \cong \mathbb{R}^2$. Set $v_1 \in V^{1,0} \cong \mathbb{R}^{2,0}$, and $v_2 = J_z v_1$. Consider also*

$$u_1 = Av_1, \quad u_2 = Av_2,$$

where A is an orthogonal transformation of $V^{1,0}$. Then the basis $(V^{1,0}) = \{u_1, u_2\}$ is orthonormal. The basis $(V^{1,0}) = \{u_1, u_2\}$ will be invariant under the action of $G(B_{1,0})$ if and only if J_z commutes with A. Thus we see that a basis $\mathfrak{B}(V^{1,0})$ can be orthonormal, but not invariant under the action of $G(B_{1,0})$.

EXAMPLE B. *Consider the Lie algebra $\mathfrak{n}_{0,3}(V)$ with an orthonormal basis $B_{0,3} = \{z_1, z_2, z_3\}$ for the centre and a minimal admissible module $V^{0,3} \cong \mathbb{R}^{4,4}$ of the Clifford algebra $Cl_{0,3}$. We take $v \in V^{0,3}$, such that $\langle v, v\rangle_{V^{0,3}} = 1$. The eight vectors*

$$v,\ J_{z_1}v,\ J_{z_2}v,\ J_{z_3}v,\ J_{z_1}J_{z_2}v,\ J_{z_1}J_{z_3}v,\ J_{z_2}J_{z_3}v,\ J_{z_1}J_{z_2}J_{z_3}v \tag{3.4}$$

are linearly independent, have square of the norm equal to ± 1, and invariant under the action of $G(B_{0,3})$. Note that the value $\langle v, J_{z_1}J_{z_2}J_{z_3}v\rangle_{V^{0,3}} = \alpha$ is arbitrary and basis (3.4) is orthogonal if and only if $\alpha = 0$. Nevertheless, the vector $v \in V^{0,3}$ always can be chosen to make $\alpha = 0$, see [FM14, Lemmas 2.8, 2.9]. *This is an example, when the basis $\mathfrak{B}(V^{0,3})$ can be invariant, but not necessary orthonormal.*

Proposition 3.3. *Let $\mathfrak{B}(V^{r,s})$ be an invariant basis. Then it is an integral basis.*

Proof. We claim that for any $v \in V^{r,s}$ with $\langle v, v\rangle_{V^{r,s}} \neq 0$ we have:

$$J_{z_i}v = \pm J_{z_j}v, \quad \Longrightarrow \quad z_i = z_j. \tag{3.5}$$

Indeed, (3.5) implies $J_{z_i}J_{z_j}v = \pm v$ and therefore $\big(J_{z_i}J_{z_j}\big)^2 v = v$. Assume by contrary that $z_i \neq z_j$. Suppose first that both z_i and z_j are positive or negative. Then $\big(J_{z_i}J_{z_j}\big)^2 = -J_{z_i}^2 J_{z_j}^2 = -\mathrm{Id}$, which is a contradiction. From the other side, if z_i and z_j are opposite, then

$$\langle \pm v, \pm v\rangle_{V^{r,s}} = \langle J_{z_i}J_{z_j}v, J_{z_i}J_{z_j}v\rangle_{V^{r,s}} = \langle z_i, z_i\rangle_{r,s}\langle z_j, z_j\rangle_{r,s}\langle v, v\rangle_{V^{r,s}} = -\langle v, v\rangle_{V^{r,s}}$$

by (2.3), and v must be a null vector, which is again a contradiction.

Assume now that $\mathfrak{B}(V^{r,s}) = \{v_j\}$ is an invariant basis for $V^{r,s}$ and that $J_{z_\ell}v_i = \pm v_k$. Then by definition of the Lie bracket (2.2) we obtain

$$\langle z_\ell, [v_i, v_j]\rangle_{r,s} = \langle J_{z_\ell}v_i, v_j\rangle_{V^{r,s}} = \langle \pm v_k, v_j\rangle_{V^{r,s}} = \pm\delta_{kj}.$$

If $k = j$, then the orthonormality of $B_{r,s}$ and $\langle z_\ell, [v_i, v_j]\rangle_{r,s} = \pm 1$ imply that $[v_i, v_j] = \pm z_\ell$, and the structure constants in (3.1) are such that $c_{ij}^{\ell} = \pm 1$. If $k \neq j$ then $c_{ij}^{\ell} = 0$. □

The definition of an invariant basis leads to the definition of an invariant integral structure on pseudo H-type Lie algebras and (invariant) integral uniform subgroup on the respective pseudo H-type Lie groups.

Definition 3.4. *Let $B_{r,s} = \{z_k\}_{k=1}^{r+s}$ be an orthonormal basis for $\mathbb{R}^{r,s}$ and $\mathfrak{B}(V^{r,s}) = \{v_i\}_{i=1}^{N}$ an invariant basis for a minimal admissible module $V^{r,s}$.*

An invariant integral structure on the pseudo H-type Lie algebra $\mathfrak{n}_{r,s}(V)$ is the vector space over $\mathbb{Z}$ given by

$$\operatorname{span}_{\mathbb{Z}}\{\mathfrak{B}(V^{r,s})\} \oplus \operatorname{span}_{\mathbb{Z}}\{B_{r,s}\} = \Big\{ \sum_{i=1}^{N} n_i v_i \oplus \sum_{k=1}^{r+s} m_k z_k \;\Big|\; n_i, m_k \in \mathbb{Z}\Big\}.$$

An (invariant) integral uniform subgroup on the pseudo H-type Lie group $\mathbb{N}_{r,s}(V) = \{(v,z) \mid v \in V^{r,s}, z \in \mathbb{R}^{r,s}\}$ is given by the coordinates

$$\Bigg(\Big(\sum_{i=1}^{N} n_i v_i \mid n_i \in \mathbb{Z}\Big), \quad \Big(\frac{1}{2}\sum_{k=1}^{r+s} m_k z_k \mid m_k \in \mathbb{Z}\Big)\Bigg).$$

The main goal of the present work is the classification of invariant integral structures on pseudo H-type Lie algebras that give rise to classification of integral uniform subgroups on the corresponding pseudo H-type Lie groups. Note that invariant integral structures is a subclass of integral (not necessary invariant and/or orthonormal) structures on pseudo H-type Lie algebras. In the present work we make a first step and classify only *invariant integral structures.* Classification of general integral structures and more general rational structures is postponed for the future works. In the article [GW86] the authors made a classification of rational uniform subgroups on the Heisenberg groups, where the starting point was a *unique* invariant integral basis of the Heisenberg algebra. Thus, in an essence, we make a first step towards the full classification of rational structures on two step nilpotent Lie algebras related to Clifford algebras.

Remark 3.2. *We remark that in the cases of $r+s \leq 2$, the invariant integral structures are unique. If $(r,s) \in \{(1,0),(0,1)\}$ and z_1 is a vector for $\mathbb{R}^{r,s}$ with $|\langle z_1, z_1\rangle_{r,s}| = 1$, then $\mathcal{B}(V^{r,s}) = \{v, J_{z_1}v\}$ is an invariant basis of the minimal admissible module $V^{r,s}$ for any choice of a vector $v \in V^{r,s}$ with $\langle v, v\rangle_{V^{r,s}} = 1$. Thus $\{z_1, v, J_{z_1}v\}$ gives rise to an invariant integral structure of $\mathfrak{n}_{r,s}(V^{r,s})$ as in Definition 3.4. The Lie algebras $\mathfrak{n}_{1,0}$ and $\mathfrak{n}_{0,1}$ are not isometric, but they are both isomorphic to the Heisenberg Lie algebra.*

If $(r,s) \in \{(2,0),(1,1),(0,2)\}$ and $B_{r,s} = \{z_1, z_2\}$ is an orthonormal basis of $\mathbb{R}^{r,s}$, then $\mathcal{B}(V^{r,s}) = \{v, J_{z_1}v, J_{z_2}v, J_{z_1}J_{z_2}v\}$ is an invariant basis of the minimal admissible module $V^{r,s}$ for any choice of $v \in V^{r,s}$, $\langle v, v\rangle_{V^{r,s}} = 1$. The bases $\{B_{r,s}, \mathcal{B}(V^{r,s})\}$ generate a unique invariant integral structure of the respective H-type Lie algebras. By uniqueness we mean that for any choice of orthonormal basis $B_{r,s}$ and any $v \in V^{r,s}$ as above the invariant integral structures of the pseudo H-type Lie algebras will give the isomorphic invariant uniform subgroups in the pseudo H-type Lie groups. The proof is a simplified version of Theorem 6.2.

3.2. **A subgroup $\mathcal{S} \subset G(B_{r,s})$ of positive involutions.** In the present section we study subgroups $\mathcal{S}$ of $G(B_{r,s}) \subset \mathrm{Cl}_{r,s}$ which will be a core for the construction of invariant bases $\mathfrak{B}(V^{r,s})$. Some of the properties of $\mathcal{S}$ can be learned from the definition of the subgroups $\mathcal{S}$, but some of them became clear by considering their action on minimal admissible modules $V^{r,s}$.

Recall that the group $\mathrm{Pin}(r,s)$ consists of elements of the Clifford algebra of the form

$$\sigma = x_{i_1}\cdots x_{i_k}, \quad \langle x_{i_j}, x_{i_j}\rangle_{r,s} = \pm 1. \tag{3.6}$$

The subgroup $\mathrm{Spin}(r,s) \subset \mathrm{Pin}(r,s)$ is generated by the even number of elements in (3.6). Thus the group $G(B_{r,s})$ is a finite subgroup of $\mathrm{Pin}(r,s)$.

Definition 3.5. *We denote by $\mathcal{S}$ a subgroup of $G(B_{r,s})$ satisfying the conditions*

(*S*1) $-\mathbf{1} \notin \mathcal{S}$;
(*S*2) $p \in \mathrm{Pin}(r,0)\times\mathrm{Spin}(0,s)$ *and*
(*S*3) $p^2 = \mathbf{1}$.

Elements $p \in \mathcal{S}$ are called positive involutions.

The name *positive involution* refers to the action of $p \in \mathcal{S}$ on $V^{r,s}$: if $\langle v,v\rangle_{V^{r,s}} > 0$ $\big(\langle v,v\rangle_{V^{r,s}} < 0\big)$ then $\langle J_pv, J_pv\rangle_{V^{r,s}} > 0$ $\big(\langle J_pv, J_pv\rangle_{V^{r,s}} < 0\big)$. We denote by $\mathbb{S}_{r,s}$ (or just $\mathbb{S}$), the set of all subgroups of $G(B_{r,s})$ satisfying Definition 3.5. This set is a partially ordered set with respect to the inclusion relation among subsets.

Remark 3.3. *The groups $\mathcal{S}\in\mathbb{S}_{r,s}$ are necessarily commutative.*

Example 3.1. *Consider $G(B_{4,0})$. Then the example of possible subgroups $\mathcal{S}$ are*

$$\mathcal{S}_1 = \{\mathbf{1}, z_1z_2z_3\},\ \mathcal{S}_2 = \{\mathbf{1}, z_1z_2z_4\},\ \mathcal{S}_3 = \{\mathbf{1}, z_1z_3z_4\},\ \mathcal{S}_4 = \{\mathbf{1}, -z_1z_2z_4\}$$

and

$$\mathcal{S}_5 = \{\mathbf{1}, z_1z_2z_3z_4\}.$$

The first four groups are isomorphic under the action of the orthogonal group $\mathrm{O}(4)$. *A map $C \in \mathrm{O}(4)$ permutes the basis vectors $\{z_i\}$, $i = 1,2,3,4$ or change their sign. All five groups are isomorphic as abelian groups of order 2. However, the roles of the first four and the last one are different in construction of an invariant basis for $\mathfrak{B}(V^{4,0})$.*

To avoid the ambiguity occurring with the very similar groups $\mathcal{S}_2$ and $\mathcal{S}_4$, we define a bigger group.

Definition 3.6. *Let $\mathcal{S}$ be a group from Definition 3.5. We denote by $\widehat{\mathcal{S}} \subset G(B_{r,s})$ the extended group*

$$\widehat{\mathcal{S}} = \mathcal{S} \cup \{-\sigma :\ \sigma \in \mathcal{S}\}.$$

In Example 3.1 we have $\mathcal{S}_2, \mathcal{S}_4$ subgroups of $G(B_{4,0})$, where we fix the basis $\{z_1, z_2, z_3, z_4\}$. The subgroups $\mathcal{S}_2, \mathcal{S}_4$ are different, nevertheless

$$\widehat{\mathcal{S}_4} = \widehat{\mathcal{S}_2} = \{\pm\mathbf{1}, \pm z_1z_2z_4\}.$$

3.3. **Generators for a group $\mathcal{S}$ of positive involutions.** In this section, we study groups $\mathcal{S} \in \mathbb{S}$ by describing their generating sets.

Definition 3.7. *We denote by* $PI = \{\, p_i \,\}_{i=1}^{\ell}$, $\ell = \#[PI]$ *is the cardinality of the set* PI*, a subset in* $G(B_{r,s})$ *satisfying the conditions:*

($PI1$) $\mathbf{1} \notin PI$, $p_i p_j = p_j p_i$ *for* $i \neq j$*, and* $p_i \in PI$ *satisfy* $(S2) - (S3)$ *in Definition 3.5 for all* $i = 1, \dots, \ell$.

($PI2$) *The vectors*

$$\{\, \mathbf{1}, p_1, \ \dots, \ p_\ell, \ \ p_{i_1} \cdots p_{i_k} \mid \ 1 \leq i_1 < \ldots < i_k \leq \ell, \ k = 1, \dots, \ell \} \tag{3.7}$$

are linearly independent in the vector space $\mathrm{Cl}_{r,s}$.

Proposition 3.8. *The condition* $(PI2)$ *is equivalent to*

$(PI2)'$ *non of the products* $p_{i_1} \cdots p_{i_k}$, $1 \leq i_1 < \cdots < i_k \leq \ell = \#[PI]$, $k = 1, \dots, \ell$, *is equal to* $\pm\mathbf{1}$.

Proof. Recall that the elements

$$\{\epsilon_0 \mathbf{1}, \ \epsilon_{i_1,\dots,i_k} z_{i_1} \cdots z_{i_k}\} \ \subset \mathrm{Cl}_{r,s}, \tag{3.8}$$

$1 \leq i_1 < \cdots < i_k \leq r + s$, $k = 1, \dots, r + s$, where ϵ_0 and $\epsilon_{i_1,\dots,i_k}$ can be chosen to be "+" or "−", form a basis for $\mathrm{Cl}_{r,s}$.

It is obvious that $(PI2)$ implies $(PI2)'$. Assume that the condition $(PI2)'$ is fulfild. Then the collection in $(PI2)'$ is a reduced collection of linearly independent basis vectors from (3.8), and therefore they are linearly independent. □

As an example of a set PI we present the minimal length positive involutions, which can be classified in the following types:

$$T_1 \begin{cases} p = z_{i_1} z_{i_2} z_{i_3} z_{i_4}, \ \text{where all } z_{i_k} \text{ are positive basis vectors;} \\ p = z_{i_1} z_{i_2} z_{i_3} z_{i_4}, \ \text{where all } z_{i_k} \text{ are negative basis vectors;} \\ p = z_{i_1} z_{i_2} z_{i_3} z_{i_4}, \ \text{where two } z_{i_k} \text{ are positive and two } z_{i_l} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{are negative basis vectors;} \end{cases}$$

$$T_2 \begin{cases} q = z_{i_1} z_{i_2} z_{i_3}, \ \text{where all } z_{i_k} \text{ are positive basis vectors;} \\ q = z_{i_1} z_{i_2} z_{i_3}, \ \text{where one } z_{i_k} \text{ is positive and two } z_{i_l} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{are negative basis vectors.} \end{cases}$$

An easy combinatorial computation shows that generally positive involutions can contain either $3 \bmod 4$ or $4 \bmod 4$ basis vectors. This observation inspires us to make a more general definition.

Definition 3.9. *A positive involution containing* $4 \bmod 4$ *basis vectors is called a type* T_1 *involution. A positive involution containing* $3 \bmod 4$ *basis vectors is called a type* T_2 *involution.*

Notation 3.1. *For an element $\sigma = \pm z_{i_1} \cdots z_{i_k} \in G(B_{r,s})$, we denote by $\mathfrak{b}(\sigma) = \{z_{i_1}, \dots, z_{i_k}\}$ the set of the vectors in the product σ, and by $|\mathfrak{b}(\sigma)|$ we denote the number of the vectors in $\mathfrak{b}(\sigma)$. Analogously, $\mathfrak{b}^+(\sigma)$ $\big(\mathfrak{b}^-(\sigma)\big)$ is the set of positive (negative) vectors in σ and $|\mathfrak{b}^+(\sigma)|$ $(|\mathfrak{b}^-(\sigma)|)$ is the cardinality of the respective sets.*

Proposition 3.10. *The following properties can be easily verified*

(A) *Two type T_1 involutions p_1 and p_2 commute if the number $|\mathfrak{b}(p_1) \cap \mathfrak{b}(p_2)|$ is even. The product p_1p_2 is an involution of type T_1.*
(B) *A type T_1 involution p and a type T_2 involution q commute if the number $|\mathfrak{b}(p) \cap \mathfrak{b}(q)|$ is even. The product pq is an involution of type T_2.*
(C) *Two type T_2 involutions q_1 and q_2 commute if the number $|\mathfrak{b}(q_1) \cap \mathfrak{b}(q_2)|$ is odd. The product q_1q_2 is an involution of type T_1.*

Proof. The proof is based on the Clifford algebra property

$$z_1z_2 + z_2z_1 = -2\langle z_1, z_2\rangle_{r,s}\mathbf{1}, \quad z_1, z_2 \in \mathbb{R}^{r,s},$$

which for orthogonal vectors z_1 and z_2 leads to $z_1z_2 = -z_2z_1$. □

Notation 3.2. *We denote by $\mathbb{PI}_{r,s}$ the collection of sets PI satisfying Definition 3.7. The set $\mathbb{PI}_{r,s}$ is partially ordered by the inclusion relation similar to $\mathbb{S}_{r,s}$. If $PI \in \mathbb{PI}_{r,s}$, then we denote by $\mathcal{S}(PI)$ a group generated by the set PI.*

Proposition 3.11. (1) *Let $PI \in \mathbb{PI}$. Then*

$$\begin{aligned} \mathcal{S}(PI) &= \{\mathbf{1},\ p_1, \dots, p_\ell,\ p_{i_1}\cdots p_{i_k} |\ 1 \le i_1 < \cdots < i_k \le \ell \\ &\qquad 1 \le i_1 < \cdots < i_k \le \ell = \#[PI]\} \end{aligned} \tag{3.9}$$

is a group of order $\#[\,\mathcal{S}(PI)\,] = 2^{\#[PI]}$ in $G(B_{r,s})$ and $\mathcal{S}(PI) \in \mathbb{S}$.
(2) *Conversely, let $\mathcal{S} \in \mathbb{S}$. Then there is a (non unique) set $PI \in \mathbb{PI}$ such that $\mathcal{S}(PI) = \mathcal{S}$.*
(3) *Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_\ell)$ be a tuple consisting of ±1, and $PI = \{p_i\}_{i=1}^{\ell} \in \mathbb{PI}_{r,s}$. Then $\varepsilon \cdot PI = \{\varepsilon_1 p_1, \dots, \varepsilon_\ell p_\ell\} \in \mathbb{PI}_{r,s}$ and $\widehat{\mathcal{S}(PI)} = \widehat{\mathcal{S}(\varepsilon \cdot PI)}$.*

Proof. Set in (3.7) is linearly independent and coincides with $\mathcal{S}(PI)$ in (3.9), therefore $\#[\mathcal{S}(PI)] = 2^{\#[PI]}$. If p is in the set (3.7), then $-p$ is not in the set (3.7), which implies that $-\mathbf{1} \notin \mathcal{S}(PI)$. Any $p \in \mathcal{S}(PI)$ is a positive involution by definition of the set PI. We showed (1).

The second property will be proved by induction arguments with respect to the order of the group $\mathcal{S}$. Let $\mathcal{S} \in \mathbb{S}_{r,s}$ be given. Assume $p_1 \in \mathcal{S}$ and if there are no elements in $\mathcal{S}$ other than $\mathbf{1}, p_1$, then we can put $PI = \{p_1\}$ and $\mathcal{S}(PI) = \mathcal{S}$.

Assume now that there is a set $PI' = \{p_1, \dots, p_\ell\}_{\ell \ge 2}$ satisfying Definition 3.7. If

$$\mathcal{S}(PI') = \{\mathbf{1},\ p_1, \dots, p_\ell,\ \ p_{i_1}\cdots p_{i_k}\ |\ \ 1 \le i_1 < \cdots < i_k \le \ell,\ k = 1, \dots, \ell\},$$

is a proper subset of $\mathcal{S}$, then there is a positive involution $q \in \mathcal{S}$ such that $q \notin \mathcal{S}(PI')$, and $q \neq \pm\mathbf{1}$. Consider the set of commuting involutions

$$\mathcal{S}(PI')\cdot q = \{q,\ p_1q, \dots, p_\ell q,\ \ p_{i_1}\cdots p_{i_k}q \mid\ 1 \leq i_1 < \cdots < i_k \leq \ell,\ k = 1, \dots, \ell\}.$$

If $p_{i_1}\cdots p_{i_m} = p_{j_1}\cdots p_{j_{m'}}q$, then $q \in \mathcal{S}(PI')$, as a product of involutions $p_{j_1}\cdots p_{j_{m'}}$ and $p_{i_1}\cdots p_{i_m}$ from $\mathcal{S}(PI')$. Thus non of the elements in $\mathcal{S}(PI')$ can be written in the form $p_{j_1}\cdots p_{j_{m'}}q$ for $p_{j_1}\cdots p_{j_{m'}} \in \mathcal{S}(PI')$. If

$$p_{i_1}\cdots p_{i_k} \neq p_{j_1}\cdots p_{j_{k'}} \quad \text{for} \quad p_{i_1}\cdots p_{i_k},\ p_{j_1}\cdots p_{j_{k'}} \in \mathcal{S}(PI'),$$

then $p_{i_1}\cdots p_{i_k}q \neq p_{j_1}\cdots p_{j_{k'}}q$. So the set $PI'' = PI' \cup \{q\}$ satisfies Definition 3.7.

Continuing the procedure, we find in finitely many steps a set PI satisfying Definition 3.7 such that $\mathcal{S}(PI) = \mathcal{S}$.

The proof of the last assertion is easily follows from Definition 3.7. □

3.4. **Relation of $\mathcal{S}$ and an isotropy subgroup $\mathcal{S}_v$.** Now we relate a group $\mathcal{S}$ with the isotropy subgroup $\mathcal{S}_v$ for some $v \in V^{r,s}$ and show that they are in a close relation.

Proposition 3.12. *Let $v \in V^{r,s}$ be a non-null vector and let $\mathcal{S}_v$ denote the isotropy subgroup in $G(B_{r,s})$ of the vector v:*

$$\mathcal{S}_v = \{\sigma \in G(B_{r,s}) \mid J_\sigma v = v\}.$$

Then $\mathcal{S}_v$ satisfies Definition 3.5.

Proof. It is clear that $-\mathbf{1} \notin \mathcal{S}_v$. To check the second property we take $\sigma \in \mathcal{S}_v \subset G(B_{r,s})$ and assume by contrary that σ is a product containing an odd number of negative basis vectors from $B_{r,s}$. Then for $v \in V^{r,s}$ with $\langle v, v\rangle_{V^{r,s}} > 0$ we obtain

$$0 < \langle v, v\rangle_{V^{r,s}} = \langle J_\sigma v, J_\sigma v\rangle_{V^{r,s}} < 0$$

by (2.3), which is a contradiction. Similar argument is applied for a vector $v \in V^{r,s}$ with $\langle v, v\rangle_{V^{r,s}} < 0$. Hence $\sigma \in \mathrm{Pin}(r, 0) \times \mathrm{Spin}(0, s)$.

The square of every element in $G(B_{r,s})$ equal either $\mathbf{1}$ or $-\mathbf{1}$. If $\sigma \in \mathcal{S}_v$, then $J_\sigma^2 = \mathrm{Id}_{V^{r,s}}$. Hence $\sigma^2 = \mathbf{1}$. □

The relation of an arbitrary $\mathcal{S}$ to an isotropy group $\mathcal{S}_v$ for some $v \in V^{r,s}$ is given in the following statement.

Proposition 3.13. *Let $\mathcal{S} \in \mathbb{S}_{r,s}$ and $PI = \{p_1, \dots, p_\ell\} \in \mathbb{PI}_{r,s}$ be such that $\mathcal{S}(PI) = \mathcal{S}$. Let $E^{+1}(p_k) = \{u \in V^{r,s} \mid J_{p_k}u = u\}$. Then the intersection $\bigcap_{k=1}^{\ell} E^{+1}(p_k)$ contains a non-null vector v. Moreover, the group $\mathcal{S}(PI)$ is the isotropy subgroup $\mathcal{S}_v$ of the vector v, and $\#[\mathcal{S}] = \#[\mathcal{S}_v] = 2^{\#[PI]}$.*

If $r - s = 3 \bmod 4$, and there is $p_i \in PI$ such that J_{p_i} acts as $-\mathrm{Id}$ on the minimal admissible module $V^{r,s}$, then the change p_i to $-p_i$ leads to the above statement.

Proof. Let $r - s \neq 3 \bmod 4$ and let $E^{+1}(p_k)$, $E^{-1}(p_k)$ be the eigenspaces of an involution J_{p_k} with eigenvalue 1 and -1, respectively. If one of the spaces $E^{\pm 1}(p_k)$ is trivial, then the symmetric bi-linear form $\langle . , . \rangle_{V^{r,s}}$ on the non-trivial subspace is non-degenerate. If both of $E^{\pm 1}(p_k)$ are non-trivial spaces, then they are orthogonal with respect to $\langle . , . \rangle_{V^{r,s}}$ and the restriction of $\langle . , . \rangle_{V^{r,s}}$ onto $E^{\pm 1}(p_k)$ is non-degenerate too.

Assume $E^{+1}(p_1) \neq \{0\}$. Then the space $E^{+1}(p_1)$ is invariant under the action of the involution J_{p_2}. Therefore, $E^{+1}(p_1) \bigcap E^{+1}(p_2) \neq \{0\}$. By repeating the procedures we get that $E = \bigcap_{k=1}^{\ell} E^{+1}(p_k) \neq \{0\}$ and the restriction of $\langle . , . \rangle_{V^{r,s}}$ onto E is non-degenerate. Thus there is a non-null vector $v \in E$ such that $J_{p_k} v = v$ for all $k = 1, \ldots, \ell$. Hence $\mathcal{S}(PI) = \mathcal{S}_v$.

If $r - s = 3 \bmod 4$, then without loss of generality we can assume that J_{p_1} acts as $-\mathrm{Id}$. We change p_1 to $-p_1$ to get $E^{+1}(p_1) = \{u \in V^{r,s} \mid J_{p_1} u = u\}$ and continue the proof as above. □

Corollary 3.14. *Let $\mathcal{S} \in \mathbb{S}_{r,s}$, and let $\mathcal{S}_v = \mathcal{S}$ be an isotropy subgroup of v as in Proposition 3.13. The orbit $O_v = G(B_{r,s}).v$, defined in* (3.3)*, contains an invariant basis $\mathfrak{B}(V^{r,s})$ of the minimal admissible module $V^{r,s}$. There is no canonical way to prescribe the direction u or $-u$ for a basis vector in $\mathfrak{B}(V^{r,s})$. Therefore O_v is a set of basis vectors counted with signes $\pm$. Hence $G(B_{r,s})/S_v \cong G(B_{r,s}).v$ and $\dim(V^{r,s}) = \frac{1}{2}\#[G(B_{r,s}).v]$.*

Proof. If the group $\mathcal{S}_v$ is an isotropy subgroup of an invariant basis, then

$$\#[\mathcal{S}_v] \cdot \#[G(B_{r,s}).v] = 2^{r+s+1} = \#[G(B_{r,s})]. \tag{3.10}$$

Since the module is minimal admissible and the basis vectors are counted twice (with plus and minus signs), we conclude $\#[G(B_{r,s}).v] = 2\dim(V^{r,s})$. □

Remark 3.4. *We denote by $\mathbb{S}^M_{r,s}$ the subset in $\mathbb{S}_{r,s}$ consisting of subgroups $\mathcal{S} = \mathcal{S}(PI)$ satisfying* (3.10)*. Furthermore $\mathbb{PI}^M_{r,s}$ denotes the maximal set of PI: that is $\mathcal{S}(PI) \in \mathbb{S}^M_{r,s}$ if and only if $PI \in \mathbb{PI}^M_{r,s}$, see Proposition 3.11. Note that the correspondence from $\mathbb{PI}_{r,s}$ to $\mathbb{S}_{r,s}$, assigning $PI \mapsto \mathcal{S}(PI)$ is surjective but not necessarily injective.*

In Proposition 3.13, if $\mathcal{S}(PI) \in \mathbb{S}^M_{r,s}$, then $\mathcal{S}(PI) = \mathcal{S}_v \in \mathbb{S}^M_{r,s}$. Indeed, since $PI \in \mathbb{PI}^M_{r,s}$ if and only if $\mathcal{S} = \mathcal{S}(PI) \in \mathbb{S}^M_{r,s}$, we obtain $\mathcal{S}(PI) = \mathcal{S}_v \in \mathbb{S}^M_{r,s}$.

Notation 3.3. *We denote by $\ell(r,s)$ the maximal number of involutions in a set $PI_{r,s} \in \mathbb{PI}^M_{r,s}$. The value $\ell(r,s)$ depends only on the signature (r,s) and it satisfies $2^{\ell(r,s)} = \frac{2^{r+s}}{\dim(V^{r,s})}$ by Corollary 3.14.*

The orbit $O_v = G(B_{r,s}).v$ gives the invariant basis for $V^{r,s}$ up to a sign. Since the elements in $G(B_{r,s})$ either commute or anti-commute with elements in $\mathcal{S}_v$, we can more precisely describe the construction of an invariant basis for a minimal admissible module $V^{r,s}$.

Theorem 3.15. *Let $v \in V^{r,s}$ be a unit vector from Proposition 3.13. There is a set $\Sigma \subset G(B_{r,s})$ such that the family $\{J_\sigma v\}_{\sigma\in\Sigma}$ is an invariant basis of $V^{r,s}$.*

Proof. Let $\mathcal{S}_v \in \mathbb{S}^M_{r,s}$. We fix a maximal set $PI_{r,s} = \{p_i\}_{i=1}^{\ell(r,s)}$ such that $\mathcal{S}(PI_{r,s}) = \mathcal{S}_v$ and write $E^{\varepsilon_i}(p_i) = \{v \in V^{r,s} \mid J_{p_i}v = \varepsilon_i v\}$, where ε_i is either $+1$ or -1. We denote $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{\ell(r,s)})$ and define

$$E = \bigcap_{i=1}^{\ell(r,s)} E^{+1}(p_i), \quad E^{\varepsilon_1,\dots,\varepsilon_{\ell(r,s)}} = \bigcap_{i=1}^{\ell(r,s)} E^{\varepsilon_i}(p_i). \tag{3.11}$$

Before we continue the proof we note that $\dim(E) \in \{1, 2, 4, 8\}$, and either $\dim(V^{r,s}) = \dim(E) \times 2^{\ell(r,s)}$ or $\dim(V^{r,s}) = \dim(E) \times 2^{\ell(r,s)-1}$. In the latter case, one involution J_{p_i} acts as Id or $-$Id on $V^{r,s}$, which happens if $r - s = 3 \mod 4$, see details in [FM21]. Thus

$$\dim(E) = 2^{r+s-2\ell(r,s)} \quad \text{or} \quad \dim(E) = 2^{r+s-2(\ell(r,s)-1)}.$$

Let $\mathbf{C}_{G(B_{r,s})}(\mathcal{S}(PI_{r,s}))$ be the centralizer of the subgroup $\mathcal{S}(PI_{r,s})$ in $G(B_{r,s})$. Then by choosing a unit vector $v \in E$, we can find representatives $\{\sigma_i\}_{i=1}^{\dim(E)} \in \mathbf{C}_{G(B_{r,s})}\big(\mathcal{S}(PI_{r,s})\big)/\mathcal{S}(\widehat{PI_{r,s}})$, and $\{\tau_j\}_{j=1}^{2^{\ell(r,s)}} \in G(B_{r,s})/\mathbf{C}_{G(B_{r,s})}\big(\mathcal{S}(PI_{r,s})\big)$ such that

the vectors $\{J_{\sigma_i}v\}_{i=1}^{\dim(E)}$ form an orthonormal basis for E,

the vectors $\{J_{\tau_j}J_{\sigma_i}v\}_{i=1}^{\dim(E)}{}_{j=1}^{2^{\ell(r,s)}}$ form an orthonormal basis for $V^{r,s}$.

These $\{\sigma_i\}_{i=1}^{\dim(E)}$ and $\{\tau_j\}_{j=1}^{2^{\ell(r,s)}}$ form the set Σ. □

Proposition 3.16. *Fix the group $\mathcal{S}(PI_{r,s})$ and the representatives*

$$\{\sigma_i\}_{i=1}^{\dim(E)} \in \mathbf{C}_{G(B_{r,s})}\big(\mathcal{S}(PI_{r,s})\big)/\mathcal{S}(\widehat{PI_{r,s}}),$$
$$\{\tau_j\}_{j=1}^{2^{\ell(r,s)}} \in G(B_{r,s})/\mathbf{C}_{G(B_{r,s})}\big(\mathcal{S}(PI_{r,s})\big).$$

Assume that $v_1, v_2 \in E$ generate two sets of invariant bases

$$\mathfrak{B}_{v_k}(V^{r,s}) = \{v_k,\ J_{\sigma_i}v_k,\ J_{\tau_j}v_k,\ J_{\tau_j}J_{\sigma_i}v_k\}_{i=1}^{\dim(E)}{}_{j=1}^{2^{\ell(r,s)}}, \quad k = 1, 2,$$

as in Theorem 3.15. Then the invariant integral structures

$$\begin{aligned} &\operatorname{span}_{\mathbb{Z}}\{\mathfrak{B}_{v_1}(V^{r,s})\} \oplus \operatorname{span}_{\mathbb{Z}}\{B_{r,s}\} \\ &\operatorname{span}_{\mathbb{Z}}\{\mathfrak{B}_{v_2}(V^{r,s})\} \oplus \operatorname{span}_{\mathbb{Z}}\{B_{r,s}\} \end{aligned} \tag{3.12}$$

are isomorphic.

Proof. We define the correspondence $A\colon \mathfrak{B}_{v_1}(V^{r,s}) \to \mathfrak{B}_{v_2}(V^{r,s})$ by

$$\begin{aligned} v_1 &\mapsto v_2, & J_{\sigma_i}v_1 &\mapsto J_{\sigma_i}v_2, \\ J_{\tau_j}v_1 &\mapsto J_{\tau_j}v_2, & J_{\tau_j}J_{\sigma_i}v_1 &\mapsto J_{\tau_j}J_{\sigma_i}v_2, \end{aligned} \tag{3.13}$$

and extend it by linearity over $\mathbb{Z}$. Then the map $A \oplus \mathrm{Id}$ is an automorphism of invariant integral structures (3.12). To show that $A \oplus \mathrm{Id}$ is an isomoprhism, we denote the basis vectors from $\mathfrak{B}_{v_1}(V^{r,s})$ by $\{u_\alpha\}_{\alpha=1}^{\dim(V^{r,s})}$ and the basis vectors from $\mathfrak{B}_{v_2}(V^{r,s})$ by $\{w_\alpha\}_{\alpha=1}^{\dim(V^{r,s})}$, where $w_\alpha = Au_\alpha$. Then we note that the bases $\mathfrak{B}_{v_1}(V^{r,s})$ and $\mathfrak{B}_{v_2}(V^{r,s})$ are invariant, which means that for any $u_\alpha \in \mathfrak{B}_{v_1}(V^{r,s})$ and any $z_k \in B_{r,s}$ there is $u_\beta \in \mathfrak{B}_{v_1}(V^{r,s})$ such that

$$J_{z_k}u_\alpha = \pm u_\beta = \pm J_\varkappa v_1, \quad \text{for some} \quad \varkappa \in \Sigma = \{\sigma_i, \tau_j, \tau_j\sigma_i\} \tag{3.14}$$

The correspondence (3.13) and (3.14) imply that for chosen $u_\alpha \in \mathfrak{B}_{v_1}(V^{r,s})$ and $z_k \in B_{r,s}$ we have

$$J_{z_k}Au_\alpha = J_{z_k}w_\alpha = \pm w_\beta = \pm J_\varkappa v_2 = \pm AJ_\varkappa v_1 = AJ_{z_k}u_\alpha.$$

Note also that $A^\tau A = \mathrm{Id}_{V^{r,s}}$ since it maps an othonormal basis to an orthonormal basis. Then we have

$$\begin{aligned} \langle [Au_\alpha, Au_\beta], z_k\rangle_{r,s} &= \langle J_{z_k}Au_\alpha, Au_\beta\rangle_{V^{r,s}} = \langle AJ_{z_k}u_\alpha, Au_\beta\rangle_{V^{r,s}} \\ &= \langle A^\tau AJ_{z_k}u_\alpha, u_\beta\rangle_{V^{r,s}} = \langle J_{z_k}u_\alpha, u_\beta\rangle_{V^{r,s}} \\ &= \langle [u_\alpha, u_\beta], z_k\rangle_{r,s}. \end{aligned} \tag{3.15}$$

□

4. Equivalence and connectedness of groups $\mathcal{S}$

We define an equivalence relation between groups $\mathcal{S} \subset G(B_{r,s})$ that will descend to the equivalence of their generating sets $PI_{r,s}$. We also introduce parameters to distinguish sets $PI_{r,s}$ for a fixed value (r, s). Different sets of parameters will lead to non-equivalent generating sets and the groups. Our aim is to show that equivalent groups $\mathcal{S}$ lead to the isomorphic invariant integral structures on $\mathfrak{n}_{r,s}$.

4.1. Equivalence of groups $\mathcal{S}$.

We recall Notation 3.1 and extend it to the sets PI.

Notation 4.1. *Let $PI \in \mathbb{PI}_{r,s}$. We denote*

$$\mathfrak{b}^+(PI) = \{z_i \mid \ z_i \textit{ is a positive vector in some } p_i \in PI\},$$
$$\mathfrak{b}^-(PI) = \{z_i \mid \ z_i \textit{ is a negative vector in some } p_i \in PI\}.$$

We set also $|\mathfrak{b}^+(PI)|$, $|\mathfrak{b}^-(PI)|$ for the cardinality of the respective set, and $|\mathfrak{b}(PI)| = |\mathfrak{b}^+(PI)| + |\mathfrak{b}^-(PI)|$.

Definition 4.1. *A set PI consisting only of the involutions of type T_1 will be called $(T1)$-type set. A set PI consisting of the involutions of type T_1 and having at least one involution of type T_2 will be called $(T2)$-type set.*

Proposition 4.2. *Any $(T2)$-type set can be reduced to $(T2)$-type set containing at most one involution of type T_2 and the rest of involutions will be of type T_1.*

Proof. The proof follows directly from Proposition 3.10. □

Notation 4.2. *If $C \in \mathrm{O}(r,s)$, then we denote by the same letter C its natural extension $C\colon Cl^*_{r,s} \to Cl^*_{r,s}$ to the action on the group of invertible elements $Cl^*_{r,s} \subset Cl_{r,s}$.*

Let $B_{r,s}$ be a basis as in (3.2). Let $C \in \mathrm{O}(r,s)$. Then C is a signed permutation matrix for $B_{r,s}$ having only one nonzero component "± 1" in each column. We call such a map *(signed) re-ordering* of $B_{r,s}$. If $\sigma = z_{i_1} \cdots z_{i_k} \in G(B_{r,s})$, then C defines an element $C(\sigma) := C(z_{i_1}) \cdots C(z_{i_k}) \in G(B_{r,s})$. Since a re-ordering matrix C maps positive basis vectors to positive vectors and negative basis vectors to negative basis vectors, it induces a map $C\colon \mathbb{PI}_{r,s} \to \mathbb{PI}_{r,s}$. For the particular case (r,r) the map C can be chosen also to map positive basis vectors to negative vectors and vice versa. The changes for (r,r) will be discussed separately in a forthcoming paper.

Definition 4.3. *We say that the groups $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent, writing $\mathcal{S}_1 \sim \mathcal{S}_2$, if there is a map $C \in \mathrm{O}(r,s)$ such that its natural extention to $Cl^*_{r,s} \subset Cl_{r,s}$ gives the isomorphism between the extended groups $\widehat{\mathcal{S}_1}$ and $\widehat{\mathcal{S}_2}$; that is $C(\widehat{\mathcal{S}_1}) = \widehat{\mathcal{S}_2}$.*

Definition 4.4. *Let PI_1 and PI_2 be two sets of involutions. Then we say that PI_1 and PI_2 are equivalent, writing $PI_1 \sim PI_2$, if $\mathcal{S}(PI_1)$ is equivalent to $\mathcal{S}(PI_2)$ in the sense of Definition 4.3.*

Example 4.1. *Recall Example 3.1 and consider $G(B_{4,0})$. Let $PI_1 = \{z_1z_2z_3\}$ and $PI_2 = \{z_1z_2z_4\}$. Then $PI_1 \sim PI_2$, since the groups*

$$\widehat{\mathcal{S}(PI_1)} = \{\pm\mathbf{1},\ \pm z_1z_2z_3\} \quad \textit{and} \quad \widehat{\mathcal{S}(PI_2)} = \{\pm\mathbf{1},\ \pm z_1z_2z_4\}$$

in $Cl_{4,0}$ are isomorphic under $\mathrm{O}(4,0)$ which permutes the basis vectors z_3 and z_4, fixing z_1 and z_2. Nevertheless, PI_1 is not equivalent to $PI_3 = \{z_1z_2z_3z_4\}$, since there is no extention of $C \in \mathrm{O}(4,0)$ to $Cl^*_{r,s}$ which maps $\widehat{\mathcal{S}(PI_1)}$ to $\widehat{\mathcal{S}(PI_3)} = \{\pm\mathbf{1},\ \pm z_1z_2z_3z_4\} \subset Cl^*_{4,0}$.

Example 4.2. *In this example we present a construction of a sequence of subgroups that will be important in Section 5. We call these subgroups standard. Let $B_{r,s}$ be an orthonormal basis of $\mathbb{R}^{r,s}$. We form a set of mutually different pairs*

$$\pi_{i,j} = z_iz_j, \quad i<j, \quad i,j \in \begin{cases} \{1,\dots,r\} & \text{if } r \text{ is even} \\ \{1,\dots,r-1\} & \text{if } r \text{ is odd} \end{cases}, \tag{4.1}$$

$$\nu_{k,l} = z_kz_l, \quad k<l, \quad k,l \in \begin{cases} \{r+1,\dots,s\} & \text{if } s \text{ is even} \\ \{r+1,\dots,s-1\} & \text{if } s \text{ is odd} \end{cases}, \tag{4.2}$$

and

$$\mathfrak{b}(\pi_{i_1,j_1}) \cap \mathfrak{b}(\pi_{i_2,j_2}) = \emptyset, \quad \mathfrak{b}(\nu_{k_1,l_1}) \cap \mathfrak{b}(\nu_{k_2,l_2}) = \emptyset,$$

The cardinalities of the sets of pairs are

$$\mathbf{p} = \#\{\pi_{i,j}\} = \begin{cases} \frac{r}{2} & \text{if } r \text{ is even} \\ \frac{r-1}{2} & \text{if } r \text{ is odd} \end{cases}, \quad \mathbf{n} = \#\{\nu_{kl}\} = \begin{cases} \frac{s}{2} & \text{if } s \text{ is even} \\ \frac{s-1}{2} & \text{if } s \text{ is odd} \end{cases}.$$

Now we form a set of involutions of type T_1, which from now on will be denoted always by p_i. For any positive integers $\bar{p} \in \{1, \dots, \mathbf{p}\}$ and $\bar{n} \in \{1, \dots, \mathbf{n}\}$ we make a product of pairs:

(4.3)

$$\pi_{i_\alpha,j_\alpha}\pi_{i_\beta,j_\beta}, \quad \pi_{i_\alpha,j_\alpha}\nu_{k_\gamma,l_\gamma}, \quad \nu_{k_\gamma,l_\gamma}\nu_{k_\delta,l_\delta}, \quad \alpha,\beta \in \{1,\dots,\bar{p}\},\ \gamma,\delta \in \{1,\dots,\bar{n}\}.$$

We denote by $\mathcal{S}^{\bar{p},\bar{n}}$ the group generated by involutions (4.3)*.*

Proposition 4.5. *In the notation above the groups $\mathcal{S}^{\bar{p},\bar{n}}$ have the following properties.*

- (i) *$\mathcal{S}^{\bar{p},\bar{n}}$ is a subgroup of $G(B_{r,s})$ for any $\bar{p} \in \{1, \dots, \mathbf{p}\}$ and $\bar{n} \in \{1, \dots, \mathbf{n}\}$;*
- (ii) *$\mathcal{S}^{\bar{p}-k,\bar{n}}$ is a subgroup of $\mathcal{S}^{\bar{p},\bar{n}}$ for any $k = 0, 1, \dots, \bar{p}$;*
- (iii) *$\mathcal{S}^{\bar{p},\bar{n}-k}$ is a subgroup of $\mathcal{S}^{\bar{p},\bar{n}}$ for any $k = 0, 1, \dots, \bar{n}$;*
- (iv) *$\mathcal{S}^{\bar{p}-k_1,\bar{n}-k_2}$ is a subgroup of $\mathcal{S}^{\bar{p},\bar{n}}$ for any $k_1 = 0, 1, \dots, \bar{p}$ and $k_2 = 0, 1, \dots, \bar{n}$;*
- (v) *The standard groups $\mathcal{S}^{\bar{p},\bar{n}}$ are equivalent for fixed $(\bar{p}, \bar{n})$ in the sense of Definition 4.3;*
- (vi) *Any set $PI_{r,s}$ satisfying Definition 3.7 and such that $\mathcal{S}^{\mathbf{p},\mathbf{n}} = \mathcal{S}(PI_{r,s})$ will be equivalent in the sense of Definition 4.4;*
- (vii) *Pairs $\pi_{i,j}$ and $\nu_{k,l}$ commute with all elements in $\mathcal{S}^{\mathbf{p},\mathbf{n}}$;*
- (viii) *Let $\theta = z_{i_1} \cdots z_{i_{\mathbf{p}+\mathbf{n}}}$ be a product such that each z_{i_t}, $t = 1, \dots, \mathbf{p}+\mathbf{n}$ belongs only to one pair from* (4.1) *or* (4.2)*. Then θ commutes with all elements in $\mathcal{S}^{\mathbf{p},\mathbf{n}}$.*

Proof. Properties (i)-(iv) are obvious. Statements (v) and (vi) follows from the fact the pairs can be chosen up to a sign permutation of the basis in $\mathbb{R}^{r,s}$. Properties (vii) and (viii) are the consequence of the facts that pairs $\pi_{i,j}$, $\nu_{k,l}$, and the product θ will have even number of common elements and that the number of vectors z_i in any element of the group $\mathcal{S}^{\mathbf{p},\mathbf{n}} \subset G(B_{r,s})$ is also even. □

Example 4.3. *Consider $\mathbb{R}^{6,3}$ with the basis $B_{6,3} = \{z_1, \dots, z_9\}$. The first six elements of the basis are positive and the last three are negative. We can choose the pairs*

$$\pi_{1,2} = z_1z_2, \quad \pi_{3,4} = z_3z_4, \quad \pi_{5,6} = z_5z_6, \quad \nu_{78} = z_7z_8, \tag{4.4}$$

up to the sign permutation. They generate a group $\mathcal{S}^{\mathbf{3},\mathbf{1}} \subset G(B_{6,3})$ of cardinality $\#\mathcal{S}^{\mathbf{3},\mathbf{1}} = 8$. A possible choice of $(T1)$-type set of involutions PI generating $\mathcal{S}^{\mathbf{3},\mathbf{1}}$ is

$$PI_{6,3} = \{p_1 = \pi_{1,2}\pi_{3,4}, \quad p_2 = \pi_{1,2}\pi_{5,6}, \quad p_3 = \pi_{1,2}\pi_{7,8}\}. \tag{4.5}$$

Any pair from (4.4) *will commute with involutions in* (4.5) *and therefore with all elements in the group* $\mathcal{S}^{\mathbf{3,1}} \subset G(B_{6,3})$. *Furthermore,* $\theta = z_1z_3z_5z_7$, *which is chosen up to a sign permutation, commutes with elements in the group* $\mathcal{S}^{\mathbf{3,1}} \subset G(B_{6,3})$ *as well. The pairs*

$$\pi_{1,2}, \quad \pi_{3,4}, \quad \pi_{5,6} \quad \textit{generates the subgroup } \mathcal{S}^{\mathbf{3,0}} \subset \mathcal{S}^{\mathbf{3,1}}.$$

Likewise the pairs

$$\pi_{1,2}, \quad \pi_{3,4}, \quad \pi_{7,8} \quad \textit{generates the subgroup } \mathcal{S}^{\mathbf{2,1}} \subset \mathcal{S}^{\mathbf{3,1}}.$$

Each of the subgroups $\mathcal{S}^{\mathbf{3,0}}$ *and* $\mathcal{S}^{\mathbf{2,1}}$ *is a representative in its class of equivalence. Nevertheless, the groups* $\mathcal{S}^{\mathbf{3,0}}$ *and* $\mathcal{S}^{\mathbf{2,1}}$ *are not equivalent.*

4.2. **Connectivity of groups $\mathcal{S}$.** Here we introduce another tool of detecting non-equivalent subgroups $\mathcal{S} \subset G(B_{r,s})$, that we call "connectedness" for $\mathcal{S} = \mathcal{S}(PI_{r,s})$.

Definition 4.6. *A group* $\mathcal{S} \in \mathbb{S}_{r,s}$ *is called connected if there is no two subgroups* $\mathcal{S}_{(1)}, \mathcal{S}_{(2)} \subset \mathcal{S}$, *such that* $\mathcal{S}$ *is isomorphic to* $\mathcal{S}_{(1)} \times \mathcal{S}_{(2)}$ *with* $\mathfrak{b}(\mathcal{S}_{(1)}) \cap \mathfrak{b}(\mathcal{S}_{(2)}) = \emptyset$. *We write in this case* $\pi_0(\mathcal{S}) = 1$.

If a group $\mathcal{S} \in \mathbb{S}_{r,s}$ *admits the decomposition into subgroups* $\mathcal{S} = \mathcal{S}_{(1)} \times \ldots \times \mathcal{S}_{(k)}$ *with* $\pi_0(\mathcal{S}_{(i)}) = 1$ *and* $\mathfrak{b}(\mathcal{S}_{(i)}) \cap \mathfrak{b}(\mathcal{S}_{(j)}) = \emptyset$ *for any* $i \neq j$, *then we say that* $\mathcal{S}$ *has* k *connected components and we write* $\pi_0(\mathcal{S}) = k$.

Lemma 4.7. *Let* $PI = \{p_i\}_{i=1}^{\ell(r,s)} \in \mathbb{PI}_{r,s}^M$, *and* $|\mathfrak{b}(PI)| = r + s$. *Assume that there is* $z_\alpha \in G(B_{r,s})$ *such that* $z_\alpha \in \bigcap_{i=1}^{\ell(r,s)} \mathfrak{b}(p_i)$, *and moreover, there is no* $\sigma \in \mathcal{S}(PI)$ *such that* $\mathfrak{b}(\sigma) \subset \mathfrak{b}(p_i)$ *for any* $p_i \in PI$. *Then* $\pi_0(\mathcal{S}(PI)) = 1$.

Proof. Note that any product $\prod_j^{2k+1} p_j$ of odd number contains z_α. Let us assume that $\mathcal{S} = \mathcal{S}_{(1)} \times \mathcal{S}_{(2)}$ is a non-trivial decomposition.

If both subgroups include a product of odd number of involutions $\prod_j^{2l+1} p_j$, $p_j \in PI$, then $z_\alpha \in \mathfrak{b}(\mathcal{S}_{(1)}) \bigcap \mathfrak{b}(\mathcal{S}_{(2)})$. Therefore $\mathcal{S}$ should be connected.

Assume the subgroup $\mathcal{S}_{(1)}$ consists of only even products $\eta = \prod_j^{2k} p_j$ of involutions in PI. We write one of these products in the form $\eta = p_{i_0} \cdot \sigma \in \mathcal{S}_{(1)}$, where p_{i_0} is one of the generators from the set PI and σ is a product of odd number of some involutions in PI. It implies that $\sigma \in \mathcal{S}_{(2)}$. By the assumption $\mathfrak{b}(\sigma) \not\subset \mathfrak{b}(p_i)$ for any $p_i \in PI$, there exists a basis vector $z_\beta \in \mathfrak{b}(\sigma)$ such that $z_\beta \notin \mathfrak{b}(p_{i_0})$. This implies that $z_\beta \in \mathfrak{b}(p_{i_0} \cdot \sigma)$ and therefore $z_\beta \in \mathfrak{b}(\sigma) \cap \mathfrak{b}(p_{i_0} \cdot \sigma) \subset \mathfrak{b}(\mathcal{S}_{(2)}) \bigcap \mathfrak{b}(\mathcal{S}_{(1)})$. This shows that the group $\mathcal{S}$ is connected. □

Example 4.4. *The standard subgroups* $\mathcal{S}^{\mathbf{p,0}} \in \mathbb{S}_{r,0}$ *constructed in Example 4.2 are connected for any* $r \geq 0$.

Proposition 4.8. *Let* $PI_1, PI_2 \in \mathbb{PI}_{r,s}^M$ *be two generating sets. If* $PI_1 \sim PI_2$, *then* $\pi_0(PI_1) = \pi_0(PI_2)$.

Proof. We write $PI_1 = \{p_k\}_{k=1}^{\ell(r,s)}$, $PI_2 = \{q_m\}_{m=1}^{\ell(r,s)}$ and $|\mathfrak{b}(PI_k)| = t$. By the assumption there exists a re-ordering map C of the basis $B_{r,s}$ such that $C\Big(\mathcal{S}\big(\widehat{\{p_k\}_{k=1}^{\ell(r,s)}}\big)\Big) = \mathcal{S}\big(\widehat{\{q_m\}_{m=1}^{\ell(r,s)}}\big)$. If

$$\mathcal{S}(PI_1) = \mathcal{S}_{(1)} \times \mathcal{S}_{(2)} = \mathcal{S}_{(1)}(PI_{11}) \times \mathcal{S}_{(2)}(PI_{12}),$$

with

$$PI_{11} = \{p_{i_k}\}_{k=1}^{a}, \quad |\mathfrak{b}(\{p_{i_k}\}_{k=1}^{a})| = \beta,$$

and

$$PI_{12} = \{p_{j_k}\}_{k=a+1}^{\ell(r,s)}, \quad |\mathfrak{b}(\{p_{j_k}\}_{k=a+1}^{\ell(r,s)})| = t-\beta,$$

then $\mathfrak{b}(\{p_{i_k}\}_{k=1}^{a}) \cap \mathfrak{b}(\{p_{j_k}\}_{k=a+1}^{\ell(r,s)}) = \emptyset$. The re-ordering map C will map the non intersecting sets $\mathfrak{b}(\{p_{i_k}\}_{k=1}^{a})$ and $\mathfrak{b}(\{p_{j_k}\}_{k=a+1}^{\ell(r,s)})$ onto non intersecting sets $\mathcal{Z}_1 = \{z_{i_1}, \dots, z_{i_\beta}\}$ and $\mathcal{Z}_2 = \{z_{j_{\beta+1}}, \dots, z_{i_t}\}$. The set $\mathcal{Z}_1$ (with possible change of signs) will form the set $PI_{21} = \{q_{i_k}\}_{k=1}^{a}$ and the set $\mathcal{Z}_2$ (again with possible change of signs) will form the set $PI_{22} = \{q_{j_k}\}_{k=a+1}^{t}$. Thus we obtain $\mathcal{S}(PI_2) = \mathcal{S}(PI_{21}) \times \mathcal{S}(PI_{22})$. □

We describe how the $\mathbb{Z}^2$ graded product of Clifford algebras can lead to the construction of disconnected subgroups $\mathcal{S} \subset G(B_{r,s})$. Consider the following decompositions of an orthonormal basis $B_{r,s} = \{z_1, \dots, z_r, z_{r+1}, \dots, z_{r+s}\}$:

$$\underbrace{z_1, \dots, z_{r_1}}_{\text{positive}}, \underbrace{z_{r+1}, \dots, z_{r+s_1}}_{\text{negative}}, \quad \text{and} \quad \underbrace{z_{r_1+1}, \dots, z_r}_{\text{positive}}, \underbrace{z_{r+s_1+1}, \dots, z_{r+s}}_{\text{negative}}.$$

We put $r_2 = r - r_1$ and $s_2 = s - s_1$ and consider the decomposition $\mathbb{R}^{r,s} \cong \mathbb{R}^{r_1,s_1} \oplus \mathbb{R}^{r_2,s_2}$, where we assume $r_1 + s_1 \geq r - r_1 + s - s_1 = r_2 + s_2$. This decomposition leads to the isomorphism $\mathrm{Cl}_{r_1,s_1}\widehat{\otimes}\mathrm{Cl}_{r_2,s_2} \cong \mathrm{Cl}_{r_1+r_2,s_1+s_2} = \mathrm{Cl}_{r,s}$, where $\widehat{\otimes}$ denotes the $\mathbb{Z}^2$-graded tensor product of Clifford algebras, see [LM89, Proposition 1.5]. For each of the Clifford algebras Cl_{r_k,s_k}, $k = 1, 2$, we consider the minimal admissible modules V^{r_k,s_k} and the corresponding sets PI_{r_k,s_k}. For $r = r_1 + r_2$ and $s = s_1 + s_2$, we have $\ell(r_1, s_1) \leq \ell(r, s)$. Let $PI_{r_1,s_1} \in \mathbb{PI}^M_{r_1,s_1}$ and $PI_{r_2,s_2} \in \mathbb{PI}^M_{r_2,s_2}$ satisfy

$$|\mathfrak{b}^+(PI_{r_1,s_1})| = r_1, \quad |\mathfrak{b}^-(PI_{r_1,s_1})| = s_1,$$
$$|\mathfrak{b}^+(PI_{r_2,s_2})| = r_2, \quad |\mathfrak{b}^-(PI_{r_2,s_2})| = s_2,$$

and $PI_{r_1,s_1} \bigcap PI_{r_2,s_2} = \emptyset$. We assume also that each set contains at most one type T_2 involution $q_k \in PI_{r_k,s_k}$, $k = 1, 2$. Then by non-commutativity of q_1 and q_2 it is easy to see the following properties:

If one of the sets PI_{r_1,s_1} or PI_{r_2,s_2} is $(T1)$-type set, then

$$PI_{r_1,s_1} \bigcup PI_{r_2,s_2} \in \mathbb{PI}_{r,s}.$$

This implies

$$\ell(r_1, s_1) + \ell(r_2, s_2) \leq \ell(r, s). \tag{4.6}$$

If both PI_{r_1,s_1} and PI_{r_2,s_2} are $(T2)$-type sets, containing type T_2 involutions $q_1 \in PI_{r_1,s_1}$ and $q_2 \in PI_{r_2,s_2}$, then

$$\big(PI_{r_1,s_1}\backslash\{q_1\}\big)\bigcup PI_{r_2,s_2} \in \mathbb{PI}_{r,s} \quad \text{and} \quad PI_{r_1,s_1}\bigcup\big(PI_{r_2,s_2}\backslash\{q_2\}\big) \in \mathbb{PI}_{r,s}.$$

This implies

$$\ell(r_1,s_1)+\ell(r_2,s_2)-1 \le \ell(r,s). \tag{4.7}$$

One can state similar properties for any number of components in a decomposition $PI = \cup_k PI_{r_k,s_k}$.

Remark 4.1. *If the equalities in* (4.6) *or* (4.7) *hold, then non-connected subgroups* $\mathcal{S}(PI_{r_1,s_1})$ *and* $\mathcal{S}(PI_{r_2,s_2})$ *can be constructed from lower dimensions and*

$$\mathcal{S}(PI_{r,s}) = \mathcal{S}(PI_{r_1,s_1}) \times \mathcal{S}(PI_{r_2,s_2}).$$

Particularly, if $r \le 9$ and $s \in \{0,1\}$, then all the groups are connected. It follows by showing that the inequalities (4.6) *and* (4.7) *are always strict.*

Proposition 4.9. *The number $\ell(r,s)$ has three periodicities:*

$$\begin{aligned}\ell(r+8,s) &= \ell(r+4,s+4)=\ell(r,s+8)=\ell(r,s)+4\\ &= \ell(r,s)+\ell(8,0)=\ell(r,s)+\ell(0,8)=\ell(r,s)+\ell(4,4).\end{aligned}$$

Proof. The number $\ell(r,s)$ is determined by $2^{\ell(r,s)}\cdot\dim(V^{r,s}) = 2^{r+s}$, see Notation 3.3. Hence,

$$2^{\ell(r+8,s)}\cdot\dim(V^{r+8,s}) = 2^{r+8+s} = 2^{r+s}2^8 = 2^{\ell(r,s)}\cdot\dim(V^{r,s})\cdot 2^8.$$

We know that $\dim(V^{r+8,s}) = 2^4\dim(V^{r,s})$ [FM17, Section 4.1]. Hence it holds $\ell(r+8,s)=\ell(r,s)+4$.

Other equalities hold by the same reason. □

5. Construction of subgroups in $\mathbb{S}^M_{r,s}$, $r \in \{3,\dots,16\}$, $s \in \{0,1\}$

5.1. **General method of the construction.** In this section we apply the previous theory for the classification of groups $\mathcal{S} \subset G(B_{r,s})$ and perform the exact construction of non-equivalent subgroups. We restrict ourself to $r \in \{1,\dots,16\}$ and $s = 0,1$ because we want to illustrate the main features that appears in classification without diving into technical details. The classification for arbitrary $\mathcal{S} \subset G(B_{r,s})$ is postponed for the forthcoming paper.

We start from $s=0$ and the classification for $s=1$ will be the strait forward generalisation. We classify groups $\mathcal{S} \subset \mathbb{S}^M_{r,0}$ according to parameters: $\pi_0(\mathcal{S})$, $|\mathfrak{b}(PI_{r,0})|$, and the type $(T1)$ or $(T2)$ of the set PI generating the group $\mathcal{S} \in \mathbb{S}^M_{r,s}$. We use the standard groups and notations introduced in Example 4.2. For the standard group we will add from none to two additional involutions, see Step 1 below for details. To distinguish the groups, where all previous parameters coincide, we assign the following signature about (TI)-type sets, $I = 1, 2$:

$$(5.1)\quad \begin{cases} \text{(i) We use the signature } (TI, \pi) \text{ if an additional involution} \\ \quad\quad \text{is related to product } \pi_{1,2}; \\ \text{(ii) We use the signature } (TI, \theta) \text{ if an additional involution} \\ \quad\quad \text{is related to product } \theta; \\ \text{(iii) We use the signature } (TI, \pi, \theta) \text{ if there are two additional} \\ \quad\quad \text{involutions, which are related to both products } \pi_{1,2} \text{ and } \theta; \\ \text{(iv) Finally we just write } (TI) \text{ if there is no involutions,} \\ \quad\quad \text{except of standards;} \end{cases}$$

We formulate the results in 15 theorems following the dimension r and illustrate each case by a table. We list the set of generators PI for each group. The group itself and the set of generators will be given up to a sign permutation. The word *unique* is understood in the sense of equivalence relation of Definition 4.3 or Definition 4.4.

5.1.1. *Main steps of the construction of $\mathcal{S} \in \mathbb{S}^M_{r,0}$ for a fixed $r > 0$.* We divide the construction into three steps.

Step 1. We start from a group satisfying $\pi_0(\mathcal{S}) = 1$ and $|\mathfrak{b}(PI_{r,0})| = r$. First we find standard subgroup $\mathcal{S}^{\mathbf{p},\mathbf{0}} \subset \mathcal{S}$ and complement it (if necessary) by involutions to reach the maximal number $\ell(r, 0)$ of involutions in $PI_{r,0}$ generating $\mathcal{S} \in \mathbb{S}^M_{r,0}$. The additional involutions will be formed by checking whether the product of $\pi_{1,2}$ and/or θ by z_r are involutions commuting with $\mathcal{S}^{\mathbf{p},\mathbf{0}}$. Then we consider a smaller standard subgroup $\mathcal{S}^{\mathbf{p-1},\mathbf{0}} \subset \mathcal{S}^{\mathbf{p},\mathbf{0}}$ and complement it by a *careful choice of* involutions to reach the maximal number $\ell(r, 0)$ for $\mathcal{S}(PI_{r,0})$, checking whether the connectivity $\pi_0(\mathcal{S}) = 1$ is not violated. We can repeat the last step several times if the condition $\pi_0(\mathcal{S}) = 1$ still holds.

Step 2. We continue to look on $\pi_0(\mathcal{S}) = 1$ and $|\mathfrak{b}(PI_{r,0})| = r - 1$. In most cases it will be a simple step back from $(r, 0)$ to $(r - 1, 0)$ as, for example, for reduction from $r = 4$ to $r = 3$.

Step 3. Next we check $\pi_0(\mathcal{S}) = 2$ and $\mathcal{S} = \mathcal{S}_{(1)} \times \mathcal{S}_{(2)}$. This step is reduced to combinations of the previous 2 steps. If needs, we can proceed to higher number of connected components.

The equivalence of the groups constructed in the previous three steps is summarised in the following proposition.

Proposition 5.1. *Let $\mathcal{S} = \mathcal{S}(PI_{r,0}) \in \mathbb{S}^M_{r,0}$, with $|\mathfrak{b}(PI_{r,0})| = r$ and $\pi_0(\mathcal{S}) = 1$. Then, the maximal standard subgroups, included in a given group $\mathcal{S} \in \mathbb{S}^M_{r,0}$, are equivalent modulo reordering by induction arguments with respect to the dimension $(r, 0)$, see also Proposition 4.5, item (v).*

Moreover, once we fix a standard group with its generators of form (4.3)*, the maximally complemented sets PI obtained by adding involutions as in Step 1,* *will be equivalent in the sense of Definition 4.4 if they have the same signature described in* (5.1) *and* $\pi_0(\mathcal{S}(PI)) = 1$*.*

Lemma 5.2. *If* $r = 3{+}8k, 5{+}8k, 6{+}8k, 7{+}8k$ *for* $k \geq 0$*, then sets* $PI_{r,0} \in \mathbb{PI}^M_{r,0}$ *satisfying* $\pi_0(\mathcal{S}(PI_{r,0})) = 1$ *and* $|\mathfrak{b}(PI_{r,0})| = r$ *are always of* $(T2)$*-type.*

Proof. We start from $r = 3 + 8k$. For the case $r = 3$ there is only one type T_2 involution. Let $k \geq 1$ and assume, by contrary, that there is a $(T1)$-type set $PI_{r,0} \in \mathbb{PI}^M_{r,0}$. We have $\ell(r, 0) = \ell(3 + 8k, 0) = 1 + 4k$. The standard subgroup $\mathcal{S}^{\mathbf{p},\mathbf{0}} \subset \mathcal{S}(PI_{r,0})$, $\mathbf{p} = 1{+}4k$, does not contain z_r, since r is odd. Let $p_1, \dots, p_{4k}$ will be involutions generating $\mathcal{S}^{\mathbf{p},\mathbf{0}}$, then $z_r \in \mathfrak{b}(p_{1+4k})$. It implies

$$\{p_1, \dots, p_{4k},\ z_r \cdot p_{1+4k}\} \in \mathbb{PI}^M_{r-1,0}.$$

This contradicts to $\ell(r - 1, 0) = \ell(2 + 8k, 0) = \ell(3 + 8k, 0) - 1 = \ell(r, 0) - 1$.

The arguments for the cases $r = 5 + 8k$, and $r = 7 + 8k$ are similar to the case $r = 3 + 8k$.

Let $r = 6 + 8k$. We assume that there is a $(T1)$-type set $PI_{r,0} \in \mathbb{PI}^M_{r,0}$. We have $\ell(r, 0) = \ell(6 + 8k, 0) = 3 + 4k$. The standard subgroup $\mathcal{S}^{\mathbf{p},\mathbf{0}} \subset \mathcal{S}(PI_{r,0})$, $\mathbf{p} = 3 + 4k$, contains z_r. Let $p_1, \dots, p_{2+4k}$ be involutions generating $\mathcal{S}^{\mathbf{p},\mathbf{0}}$, where we can assume that $z_r \in \mathfrak{b}(p_{2+4k})$ and $p_{3+4k} \in PI_{6+8k,0}$ is the last type T_1 involution.

1. If $z_r \notin \mathfrak{b}(p_{3+4k})$, then
$$\{p_1, \dots, p_{1+4k},\ z_r \cdot p_{2+4k},\ p_{3+4k}\} \in \mathbb{PI}^M_{r-1,0}.$$
This contradicts to $\ell(r - 1, 0) = \ell(5 + 8k, 0) = \ell(6 + 8k, 0) - 1 = \ell(r, 0) - 1$.
2. If $z_r \in \mathfrak{b}(p_{3+4k})$, then we replace $p_{3+4k} \in PI_{r,0}$ by another type T_1 involution $\tilde{p}_{3+4k} = p_{2+4k}p_{3+4k} \in \widetilde{PI_{r,0}}$. In this case $z_r \notin \mathfrak{b}(\hat{p}_{3+4k})$ and the situation is reduced to the previous step (1). Note that the group $\mathcal{S}(PI_{r,0})$ is equivalent $\mathcal{S}(\widetilde{PI_{r,0}})$.

We also note that for $r = 3 + 8k$ and $r = 7 + 8k$ the volume forms $\Omega_r = \prod_{i=1}^r z_i$ which are type T_2 involutions can be included to $PI_{r,0}$. It justifies the $(T2)$-type set of PIs in cases $r = 3 + 8k$ and $r = 7 + 8k$. □

5.2. Constructions of groups $\mathcal{S} \in \mathbb{S}^M_{r,0}$ for $r \in \{3, \dots, 16\}$.

Theorem 5.3. *There is a unique group* $\mathcal{S} \subset \mathbb{S}^M_{3,0}$*. It is generated by type* T_2 *involution* $p = z_1z_2z_3$*. Thus we have*

$$\pi_0(\mathcal{S}) = 1, \quad |\mathfrak{b}(PI_{3,0})| = 3, \quad \mathcal{S} = \{\mathbf{1}, p = z_1z_2z_3 = \pi_{1,2}z_3\}.$$

Proof. The group $\mathcal{S}$ is unique up to reordering. □

Theorem 5.4. *There are two non-equivalent groups in* $\mathbb{S}^M_{4,0}$*.*

Proof. The proof is obvious. □

Table 2. Groups for $r = 4$

	$\ell(4,0)$	$\pi_0(\mathcal{S})$	$\lvert\mathfrak{b}(PI)\rvert$	*Signature*	PI
$\mathcal{S}^{(1)}$	1	1	4	$(T1)$	$p = \pi_{1,2}\pi_{3,4}$
$\mathcal{S}^{(2)}$	1	1	3	$(T2, \pi)$	$q = \pi_{1,2}z_3$

Notation 5.1. *From now on we write* $\theta_{\overline{i,j}}$ *to indicate that product in* θ *starts from* z_i *and ends with* z_j *containing all* z_k *for odd* k *between* i *and* j. *We have*

$$|\mathfrak{b}(\theta_{\overline{i,j}})| = \frac{j-i}{2} + 1.$$

Theorem 5.5. *There is unique group in* $\mathbb{S}^M_{5,0}$.

Table 3. Groups for $r = 5$

	$\ell(5,0)$	$\pi_0(\mathcal{S})$	$\lvert\mathfrak{b}(PI)\rvert$	*Signature*	PI
$\mathcal{S}$	2	1	5	$(T2, \theta)$	$p = \pi_{1,2}\pi_{3,4}$ $q = \theta_{\overline{1,3}}z_5 = z_1z_3z_5$

Proof. We start from the standard subgroup $\mathcal{S}^{\mathbf{2,0}} = \{\mathbf{1}, p = \pi_{1,2}\pi_{3,4}\}$ of the maximal group $\mathcal{S} \subset \mathbb{S}^M_{5,0}$. The products $\pi_{1,2} = z_1z_2$ and $\theta = z_1z_3$ commute with the involution p. To complete the standard subgroup $\mathcal{S}^{\mathbf{2,0}}$ to the maximal group $\mathcal{S} \subset \mathbb{S}^M_{5,0}$ we add a type T_2 involutions

$$\text{either} \quad q_1 = \pi_{1,2}z_5 = z_1z_2z_5 \quad \text{or} \quad q_2 = \theta z_5 = z_1z_3z_5.$$

Both choices lead to the equivalent subgroups

$$\{\mathbf{1}, p = \pi_{1,2}\pi_{3,4}, q_1 = z_1z_2z_5\} \quad \text{and} \quad \{\mathbf{1}, p = \pi_{1,2}\pi_{3,4}, q_2 = z_1z_3z_5\}$$

by permutation $z_2 \leftrightarrow z_3$. □

Theorem 5.6. *There is unique group in* $\mathbb{S}^M_{6,0}$.

Table 4. Groups for $r = 6$

	$\ell(6,0)$	$\pi_0(\mathcal{S})$	$\lvert\mathfrak{b}(PI)\rvert$	*Signature*	PI
$\mathcal{S}$	3	1	6	$(T2, \theta)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $q = \theta_{\overline{1,5}}$

Proof. The standard subgroup $\mathcal{S}^{\mathbf{3,0}}$ is generated by the involutions

$$p_1 = \pi_{1,2}\pi_{3,4}, \quad p_2 = \pi_{1,2}\pi_{5,6}. \tag{5.2}$$

We need to add one involution since $\ell(6,0) = 3$. We observe that $\pi_{1,2}z_j$, $j = 1, \dots, 6$, does not commute with generators (5.2), but $\theta = \theta_{\overline{1,5}} = z_1z_3z_5$ is an involution itself commuting with generators (5.2). Thus we add $\theta_{\overline{1,5}}$ to make PI complete. It finishes the proof. □

Theorem 5.7. *There is unique group in* $\mathbb{S}^M_{7,0}$.

TABLE 5. Groups for $r = 7$

	$\ell(7,0)$	$\pi_0(\mathcal{S})$	$\lvert\mathfrak{b}(PI)\rvert$	*Signature*	PI
$\mathcal{S}$	4	1	7	$(T2,\pi,\theta)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \theta_{\overline{1,5}}z_7$ $q = \pi_{1,2}z_7$

Proof. The standard subgroup $\mathcal{S}^{\mathbf{3,0}} \subset \mathcal{S}$ is generated by involutions (5.2). We need to add two involutions since $\ell(7,0) = 4$, at least one of which must contain z_7. We observe that the products $\pi_{1,2}z_7$ and $\theta_{\overline{1,5}}z_7 = z_1z_3z_5z_7$ are both involutions commuting with generators (5.2) with each other. We append them both to reach $\ell(7,0) = 4$. The reductions to $|\mathfrak{b}(PI_{7,0})| = 6$ is not possible due to $\ell(6,0) < \ell(7,0)$. We finish the proof. □

Theorem 5.8. *There are two non-equivalent groups in* $\mathbb{S}^M_{8,0}$.

TABLE 6. Groups for $r = 8$

	$\ell(8,0)$	$\pi_0(\mathcal{S})$	$\lvert\mathfrak{b}(PI)\rvert$	*Sinature*	PI
$\mathcal{S}^{(1)}$	4	1	8	$(T1,\theta)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \pi_{1,2}\pi_{7,8}$ $p_4 = \theta_{\overline{1,7}}$
$\mathcal{S}^{(2)}$	4	1	7	$(T2,\pi,\theta)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \theta_{\overline{1,5}}z_7$ $q = \pi_{1,2}z_7$

Proof. The standard subgroup $\mathcal{S}^{\mathbf{4,0}} \subset \mathcal{S}^{(1)}$ is generated by involutions

$$p_1 = \pi_{1,2}\pi_{3,4}, \quad p_2 = \pi_{1,2}\pi_{5,6}, \quad p_3 = \pi_{1,2}\pi_{7,8}. \tag{5.3}$$

We need to add one involution since $\ell(8,0) = 4$. It is easy to see that only $\theta_{\overline{1,7}} = z_1z_3z_5z_7$ commutes with generators (5.3).

Consider standard subgroup $\mathcal{S}^{\mathbf{3,0}} \subset \mathcal{S}^{(2)}$ generated by (5.2). This case is reduced to $r = 7$ and it is indicated in Table 6. We finish the proof. □

Theorem 5.9. *There are three non-equivalent groups in* $\mathbb{S}^M_{9,0}$.

Proof. The standard subgroup $\mathcal{S}^{\mathbf{4,0}} \subset \mathcal{S}^{(1)}$ is generated by involutions in (5.3). We need to add one involution containing z_9 since $\ell(9,0) = 4$ and $|\mathfrak{b}(PI_{9,0})| = 9$. We add $q = \pi_{1,2}z_9$.

TABLE 7. Groups for $r = 9$

	$\ell(9,0)$	$\pi_0(\mathcal{S})$	$\lvert\mathfrak{b}(PI)\rvert$	*Signature*	PI
$\mathcal{S}^{(1)}$	4	1	9	$(T2,\pi)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \pi_{1,2}\pi_{7,8}$ $p_4 = \pi_{1,2}z_9$
$\mathcal{S}^{(2)}$	4	1	8	$(T1,\theta)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \pi_{1,2}\pi_{7,8}$ $p_4 = \theta_{\overline{1,7}}$
$\mathcal{S}^{(3)}$	4	1	7	$(T2,\pi,\theta)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \theta_{\overline{1,7}}$ $q = \pi_{1,2}z_7$

We release $|\mathfrak{b}(PI_{9,0})| = 9$ and consider $|\mathfrak{b}(PI_{9,0})| = 8$. It is easy to see that $\mathcal{S}^{(2)}$ is isomorphic to $\mathcal{S}^{(1)} \in \mathbb{S}^M_{8,0}$.

Consider standard subgroup $\mathcal{S}^{\mathbf{3,0}} \subset \mathcal{S}^{(3)}$ generated by (5.2). This case is reduced to $r = 7$ and it is indicated in the table. We finish the proof. □

Theorem 5.10. *There are four connected non-equivalent and two disconnected non-equivalent groups in* $\mathbb{S}^M_{10,0}$.

TABLE 8. Groups for $r = 10$

	$\ell(10,0)$	$\pi_0(\mathcal{S})$	$\lvert\mathfrak{b}(PI)\rvert$	*Signature*	PI
$\mathcal{S}^{(1)}$	4	1	10	$(T1,\pi)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \pi_{1,2}\pi_{7,8}$ $p_4 = \pi_{1,2}\pi_{9,10}$
$\mathcal{S}^{(2)}$	4	1	9	$(T2,\pi)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \pi_{1,2}\pi_{7,8}$ $p_4 = \pi_{1,2}z_9$
$\mathcal{S}^{(3)}$	4	1	8	$(T1,\theta)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \pi_{1,2}\pi_{7,8}$ $p_4 = \theta_{\overline{1,7}}$
$\mathcal{S}^{(4)}$	4	1	7	$(T2,\pi,\theta)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \theta_{\overline{1,5}}z_7$ $q = \pi_{1,2}z_7$
$\mathcal{S}^{(5)}$	4	2	7	$(T1,\theta)\times(T2,\pi)$ $r = 7+3$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(q)_{(2)} = \pi_{8,9}z_{10}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(p_3)_{(1)} = \theta_{\overline{1,7}}$,
$\mathcal{S}^{(6)}$	4	2	7	$(T2,\theta)\times(T1)$ $r = 6+4$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(p_1)_{(2)} = \pi_{7,8}\pi_{9,10}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(q)_{(1)} = \theta_{\overline{1,5}}$,

Proof. The standard subgroup $\mathcal{S}^{\mathbf{5,0}} \subset \mathcal{S}^{(1)}$ is generated by involutions

$$p_1 = \pi_{1,2}\pi_{3,4}, \quad p_2 = \pi_{1,2}\pi_{5,6}, \quad p_3 = \pi_{1,2}\pi_{7,8}, \quad p_4 = \pi_{1,2}\pi_{9,10}. \tag{5.4}$$

We do not need to add any involution, since $\ell(10,0) = 4$.

The rest of the connected groups comes from lower dimensions.

To construct the disconnected subgroup $\mathcal{S}^{(5)} = \mathcal{S}^{(5)}_{(1)} \times \mathcal{S}^{(5)}_{(2)}$ corresponding to the $\mathbb{Z}_2$-graded tensor product of the Clifford algebras $\mathrm{Cl}_{10,0} \cong \mathrm{Cl}_{7,0}\hat{\otimes}\mathrm{Cl}_{3,0}$ we consider standard subgroup $\mathcal{S}^{\mathbf{3,0}}_{(1)} \subset \mathcal{S}^{(5)}_{(1)}$ generated by (5.2) and add type T_1 involution $\theta_{\overline{1,7}}$. Then $\mathcal{S}^{(5)}_{(2)} = \{\mathbf{1}, \pi_{8,9}z_{10}\}$.

To obtain $\mathcal{S}^{(6)} = \mathcal{S}^{(6)}_{(1)} \times \mathcal{S}^{(6)}_{(2)}$ corresponding to the $\mathbb{Z}_2$-graded tensor product of the Clifford algebras $\mathrm{Cl}_{10,0} \cong \mathrm{Cl}_{6,0}\hat{\otimes}\mathrm{Cl}_{4,0}$ we take standard subgroup $\mathcal{S}^{\mathbf{3,0}}_{(1)} \subset \mathcal{S}^{(6)}_{(1)}$ generated by (5.2) and add type T_2 involution $\theta_{\overline{1,5}}$. Then $\mathcal{S}^{(6)}_{(2)} = \{\mathbf{1}, \pi_{7,8}\pi_{9,10}\}$.

□

Theorem 5.11. *There are one connected and two disconnected non-equivalent subgroups in* $\mathbb{S}^M_{11,0}$.

TABLE 9. Groups for $r = 11$

	$\ell(11,0)$	$\pi_0(\mathcal{S})$	$\lvert\mathfrak{b}(PI)\rvert$	*Signature*	PI
$\mathcal{S}^{(1)}$	5	1	11	$(T2,\pi)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \pi_{1,2}\pi_{7,8}$ $p_4 = \pi_{1,2}\pi_{9,10}$ $q = \pi_{1,2}z_{11}$
$\mathcal{S}^{(2)}$	5	2	11	$(T1,\theta) \times (T2,\pi)$ $r = 8+3$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(q)_{(2)} = \pi_{9,10}z_{11}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(p_3)_{(1)} = \pi_{1,2}\pi_{7,8}$, $(p_4)_{(1)} = \theta_{\overline{1,7}}$,
$\mathcal{S}^{(3)}$	5	2	11	$(T2,\pi,\theta) \times (T1)$ $r = 7+4$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(p_1)_{(2)} = \pi_{8,9}\pi_{10,11}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(p_3)_{(1)} = \theta_{\overline{1,7}}$, $(q)_{(1)} = \pi_{1,2}z_7$,

Proof. The standard subgroup $\mathcal{S}^{\mathbf{5,0}} \subset \mathcal{S}^{(1)}$ is generated by involutions (5.4). We need to add one involution, since $\ell(11,0) = 5$. We add $q = \pi_{1,2}z_{11}$. A reduction to the cases $\lvert\mathfrak{b}(PI_{11,0})\rvert = 10$ is not possible due to $\ell(10,0) < \ell(11,0)$.

To construct the disconnected subgroup $\mathcal{S}^{(2)} = \mathcal{S}^{(2)}_{(1)} \times \mathcal{S}^{(2)}_{(2)}$ corresponding to the $\mathbb{Z}_2$-graded tensor product of the Clifford algebras $\mathrm{Cl}_{11,0} \cong \mathrm{Cl}_{8,0}\hat{\otimes}\mathrm{Cl}_{3,0}$ we start from the standard subgroup $\mathcal{S}^{\mathbf{4,0}}_{(1)} \subset \mathcal{S}^{(2)}_{(1)}$ generated by (5.3) and add type T_1 involution $\theta_{\overline{1,7}}$. Then $\mathcal{S}^{(2)}_{(2)} = \{\mathbf{1}, \pi_{9,10}z_{11}\}$.

To obtain $\mathcal{S}^{(3)} = \mathcal{S}^{(3)}_{(1)} \times \mathcal{S}^{(3)}_{(2)}$ corresponding to the $\mathbb{Z}_2$-graded tensor product of the Clifford algebras $\mathrm{Cl}_{11,0} \cong \mathrm{Cl}_{7,0}\hat{\otimes}\mathrm{Cl}_{4,0}$ we consider standard subgroup

$\mathcal{S}^{\mathbf{3,0}}_{(1)} \subset \mathcal{S}^{(3)}_{(1)}$ generated by (5.2) and add type T_1 involution $\theta_{\overline{1,7}}$ and type T_2 involution $\pi_{1,2}z_7$. Then $\mathcal{S}^{(3)}_{(2)} = \{\mathbf{1}, \pi_{8,9}\pi_{10,11}\}$. □

Theorem 5.12. *There are three connected non-equivalent and five disconnected non-equivalent subgroups in* $\mathbb{S}^M_{12,0}$.

TABLE 10. Groups for $r = 12$

	$\ell(12,0)$	$\pi_0(\mathcal{S})$	$\lvert\mathfrak{b}(PI)\rvert$	*Signature*	PI
$\mathcal{S}^{(1)}$	5	1	12	$(T1,\pi)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \pi_{1,2}\pi_{7,8}$ $p_4 = \pi_{1,2}\pi_{9,10}$ $p_5 = \pi_{1,2}\pi_{11,12}$
$\mathcal{S}^{(2)}$	5	1	12	$(T2,\theta)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \pi_{1,2}\pi_{7,8}$ $p_4 = \pi_{1,2}\pi_{9,10}$ $q = \theta_{\overline{1,9}}z_{11}z_{12}$
$\mathcal{S}^{(3)}$	5	1	11	$(T2,\pi)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \pi_{1,2}\pi_{7,8}$ $p_4 = \pi_{1,2}\pi_{9,10}$ $q = \pi_{1,2}z_{11}$
$\mathcal{S}^{(4)}$	5	2	12	$(T1,\theta)\times(T1)$ $r = 8+4$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(p_1)_{(2)} = \pi_{9,10}\pi_{11,12}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(p_3)_{(1)} = \pi_{1,2}\pi_{7,8}$, $(p_4)_{(1)} = \theta_{\overline{1,7}}$,
$\mathcal{S}^{(5)}$	5	2	12	$(T1,\theta)\times(T2,\pi)$ $r = 7+5$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(p_1)_{(2)} = \pi_{8,9}\pi_{10,11}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(q)_{(2)} = \pi_{8,9}z_{12}$ $(p_3)_{(1)} = \theta_{\overline{1,7}}$,
$\mathcal{S}^{(6)}$	5	2	12	$(T2,\pi)\times(T1)$ $r = 6+6$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(p_1)_{(2)} = \pi_{7,8}\pi_{9,10}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(p_2)_{(2)} = \pi_{7,8}\pi_{11,12}$ $(q)_{(1)} = \theta_{\overline{1,5}}$,
$\mathcal{S}^{(7)}$	5	2	11	$(T1,\theta)\times(T2,\pi)$ $r = 8+3$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(q)_{(1)} = \pi_{9,10}z_{11}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(p_3)_{(1)} = \pi_{1,2}\pi_{7,8}$, $(p_4)_{(1)} = \theta_{\overline{1,7}}$,
$\mathcal{S}^{(8)}$	5	2	11	$(T2,\pi,\theta)\times(T1)$ $r = 7+4$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4}$, $(p_1)_{(2)} = \pi_{8,9}\pi_{10,11}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6}$, $(p_3)_{(1)} = \theta_{\overline{1,7}}$, $(q)_{(1)} = \pi_{1,2}z_7$,

Proof. The standard subgroup $\mathcal{S}^{\mathbf{6,0}} \subseteq \mathcal{S}^{(1)}$ is generated by involutions

$$p_1 = \pi_{1,2}\pi_{3,4}, \quad p_2 = \pi_{1,2}\pi_{5,6}, \quad p_3 = \pi_{1,2}\pi_{7,8}, \quad p_4 = \pi_{1,2}\pi_{9,10}, \quad p_5 = \pi_{1,2}\pi_{11,12}. \tag{5.5}$$

and it coincides with $\mathcal{S}^{(1)} \in \mathbb{S}^M_{12,0}$.

Consider the standard subgroup $\mathcal{S}^{\mathbf{5,0}} \subseteq \mathcal{S}^{(2)}$ generated by involutions (5.4). We need to add one involution containing z_{11} and z_{12}. We see that $q =$

$\theta_{\overline{1,9}} z_{11} z_{12}$ commutes with all involutions in (5.4). Adding q as the type T_2 involution will finish the construction of the maximal group $\mathcal{S}^{(2)}$, see Table 10.

To construct the disconnected subgroup $\mathcal{S}^{(3)} = \mathcal{S}^{(3)}_{(1)} \times \mathcal{S}^{(3)}_{(2)}$ corresponding to the $\mathbb{Z}_2$-graded tensor product of the Clifford algebras $\mathrm{Cl}_{12,0} \cong \mathrm{Cl}_{8,0} \hat{\otimes} \mathrm{Cl}_{4,0}$ we consider standard subgroup $\mathcal{S}^{\mathbf{4,0}}_{(1)} \subset \mathcal{S}^{(3)}_{(1)}$ generated by (5.3). We add the involution $p_4 = \theta_{\overline{1,7}}$ to the set of generators for $\mathcal{S}^{\mathbf{4,0}}_{(1)}$ and generate the first component $\mathcal{S}^{(3)}_{(1)}$ in the product $\mathcal{S}^{(3)} = \mathcal{S}^{(3)}_{(1)} \times \mathcal{S}^{(3)}_{(2)}$. Then we set $\mathcal{S}^{(3)}_{(2)} = \{\mathbf{1}, \pi_{9,10}\pi_{11,12}\}$.

Analogously we construct the disconnected subgroups related to the decomposition of the Clifford algebras $\mathrm{Cl}_{12,0} \cong \mathrm{Cl}_{7,0} \hat{\otimes} \mathrm{Cl}_{5,0}$ and $\mathrm{Cl}_{12,0} \cong \mathrm{Cl}_{6,0} \hat{\otimes} \mathrm{Cl}_{6,0}$. In both of these cases we remove one of the type T_2 involutions and obtain 5 involutions in the total set $PI_{12,0}$. Note also that if in the decomposition $\mathrm{Cl}_{12,0} \cong \mathrm{Cl}_{7,0} \hat{\otimes} \mathrm{Cl}_{5,0}$ for $\mathcal{S}^{(5)} = \mathcal{S}^{(5)}_{(1)} \times \mathcal{S}^{(5)}_{(2)}$ we take the set $PI_{7,0}$ to be of $(T2)$-type set generating $\mathcal{S}^{(5)}_{(1)}$ and $PI_{4,0}$ for $\mathcal{S}^{(5)}_{(2)}$ to be of $(T1)$-type set, then we obtain a group isomorphic to $\mathcal{S}^{(8)}$.

If $|\mathfrak{b}(PI_{12,0})| = 11$, then the constructions reduce to the case of $\mathcal{S} \subset \mathbb{S}^M_{11,0}$. □

Theorem 5.13. *There are three connected non-equivalent and three disconnected non-equivalent subgroups in* $\mathbb{S}^M_{13,0}$.

Proof. The standard subgroup $\mathcal{S}^{\mathbf{6,0}} \subseteq \mathcal{S} \subset \mathbb{S}^M_{13,0}$ is generated by involutions (5.5). We add either $q = \pi_{1,2} z_{13}$ or $q = \theta_{\overline{1,13}}$ as type T_2 involutions. We obtain two connected groups $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$.

Consider the standard subgroup $\mathcal{S}^{\mathbf{5,0}} \subseteq \mathcal{S}^{(3)}$ generated by involutions (5.4). We need to add two involutions containing z_{11}, z_{12} and z_{13}. We see that type T_1 involution $p_5 = \theta_{\overline{1,9}} z_{11} z_{12} z_{13}$ commutes with all involutions in (5.4). Adding p_5 as the type T_1 involution and $q = \pi_{1,2} z_{13}$ as type T_2 involution, we obtain the maximal group $\mathcal{S}^{(3)}$, see Table 11.

To construct the disconnected subgroup $\mathcal{S}^{(4)} = \mathcal{S}^{(4)}_{(1)} \times \mathcal{S}^{(4)}_{(2)}$ corresponding to the $\mathbb{Z}_2$- graded tensor product of the Clifford algebras $\mathrm{Cl}_{13,0} \cong \mathrm{Cl}_{8,0} \hat{\otimes} \mathrm{Cl}_{5,0}$ we consider standard subgroup $\mathcal{S}^{\mathbf{4,0}}_{(1)} \subset \mathcal{S}^{(4)}_{(1)}$ generated by (5.3). We add the involutions $p_4 = \theta_{\overline{1,7}}$ to the set of generators for $\mathcal{S}^{\mathbf{4,0}}_{(1)}$ and generate the first component $\mathcal{S}^{(4)}_{(1)}$ in the product $\mathcal{S}^{(4)} = \mathcal{S}^{(4)}_{(1)} \times \mathcal{S}^{(4)}_{(2)}$. Then we set $\mathcal{S}^{(4)}_{(2)}$ generated by the set of $PI = \{(p_1)_{(2)} = \pi_{9,10}\pi_{11,12}, (q)_{(2)} = \pi_{9,10} z_{13}\}$.

Analogously we construct disconnected subgroups $\mathcal{S}^{(k)} = \mathcal{S}^{(k)}_{(1)} \times \mathcal{S}^{(k)}_{(2)}$, $k = 5, 6$, corresponding to the $\mathbb{Z}_2$-graded tensor product $\mathrm{Cl}_{13,0} \cong \mathrm{Cl}_{7,0} \hat{\otimes} \mathrm{Cl}_{6,0}$. For $k = 5$ we choose $PI_{7,0}$ for the group $\mathcal{S}^{(5)}_{(1)}$ to be $(T2)$-type set and two standard involutions in $PI_{6,0}$ for the groups $\mathcal{S}^{(5)}_{(2)}$ to be $(T1)$-type set. For $k = 6$ we change the type of the sets PI.

There are no groups with $|\mathfrak{b}(PI_{13,0})| = 12$ because $\ell(13, 0) > \ell(12, 0)$. □

TABLE 11. Groups for $r = 13$

	$\ell(13,0)$	$\pi_0(\mathcal{S})$	$\lvert\mathfrak{b}(PI)\rvert$	Signature	PI
$\mathcal{S}^{(1)}$	6	1	13	$(T2,\pi)$	$p_1=\pi_{1,2}\pi_{3,4}$ $p_2=\pi_{1,2}\pi_{5,6}$ $p_3=\pi_{1,2}\pi_{7,8}$ $p_4=\pi_{1,2}\pi_{9,10}$ $p_5=\pi_{1,2}\pi_{11,12}$ $q=\pi_{1,2}z_{13}$
$\mathcal{S}^{(2)}$	6	1	13	$(T2,\theta)$	$p_1=\pi_{1,2}\pi_{3,4}$ $p_2=\pi_{1,2}\pi_{5,6}$ $p_3=\pi_{1,2}\pi_{7,8}$ $p_4=\pi_{1,2}\pi_{9,10}$ $p_5=\pi_{1,2}\pi_{11,12}$ $q=\theta_{\overline{1,13}}$
$\mathcal{S}^{(3)}$	6	1	13	$(T2,\pi,\theta)$	$p_1=\pi_{1,2}\pi_{3,4}$ $p_2=\pi_{1,2}\pi_{5,6}$ $p_3=\pi_{1,2}\pi_{7,8}$ $p_4=\pi_{1,2}\pi_{9,10}$ $p_5=\theta_{\overline{1,9}}z_{11}z_{12}z_{13}$ $q=\pi_{1,2}z_{12}$
$\mathcal{S}^{(4)}$	6	2	13	$(T1)\times(T2)$ $r=8+5$	$(p_1)_{(1)}=\pi_{1,2}\pi_{3,4},\quad (p_1)_{(2)}=\pi_{9,10}\pi_{11,12}$ $(p_2)_{(1)}=\pi_{1,2}\pi_{5,6},\quad (q)_{(2)}=\pi_{9,10}z_{13}$ $(p_3)_{(1)}=\pi_{1,2}\pi_{7,8},$ $(p_4)_{(1)}=\theta_{\overline{1,7}},$
$\mathcal{S}^{(5)}$	6	2	13	$(T2,\pi,\theta)\times(T1)$ $r=7+6$	$(p_1)_{(1)}=\pi_{1,2}\pi_{3,4},\quad (p_1)_{(2)}=\pi_{8,9}\pi_{10,11}$ $(p_2)_{(1)}=\pi_{1,2}\pi_{5,6},\quad (p_2)_{(2)}=\pi_{8,9}\pi_{12,13}$ $(p_3)_{(1)}=\theta_{\overline{1,7}},$ $(q)_{(1)}=\pi_{1,2}z_7,$
$\mathcal{S}^{(6)}$	6	2	13	$(T1,\theta)\times(T2,\theta)$ $r=7+6$	$(p_1)_{(1)}=\pi_{1,2}\pi_{3,4},\quad (p_1)_{(2)}=\pi_{8,9}\pi_{10,11}$ $(p_2)_{(1)}=\pi_{1,2}\pi_{5,6},\quad (p_2)_{(2)}=\pi_{8,9}\pi_{12,13}$ $(p_3)_{(1)}=\theta_{\overline{1,7}},\quad (q)_{(2)}=z_8z_{10}z_{12}$

Theorem 5.14. *There are two connected non-equivalent and two disconnected non-equivalent subgroups in $\mathbb{S}^M_{14,0}$.*

Proof. The standard subgroup $\mathcal{S}^{\mathbf{7,0}} \subseteq \mathcal{S}^{(1)} \subset \mathbb{S}^M_{14,0}$ is generated by involutions

$$
\begin{aligned}
&p_1=\pi_{1,2}\pi_{3,4},\quad p_2=\pi_{1,2}\pi_{5,6},\quad p_3=\pi_{1,2}\pi_{7,8},\quad p_4=\pi_{1,2}\pi_{9,10},\\
&p_5=\pi_{1,2}\pi_{11,12},\quad p_6=\pi_{1,2}\pi_{13,14}.
\end{aligned} \tag{5.6}
$$

We add type T_2 involution $q=\theta_{\overline{1,13}}$ and obtain the connected group $\mathcal{S}^{(1)}$.

Next we consider the standard subgroup $\mathcal{S}^{\mathbf{6,0}} \subseteq \mathcal{S}^{(2)}$ generated by involutions (5.5). We need to add two involutions containing z_{13} and z_{14}. We see that type T_1 involution $p_6=\theta_{\overline{1,11}}z_{13}z_{14}$ commutes with involutions in (5.5). Adding either $q_1=\pi_{1,2}z_{13}$ or $q_2=\pi_{1,2}z_{14}$ as type T_2 involution, we obtain the maximal group $\mathcal{S}^{(2)}$, see Table 12. Adding q_1 or q_2, we create equivalent groups $\mathcal{S}^{(2)}$.

To construct the disconnected subgroup $\mathcal{S}^{(3)}=\mathcal{S}^{(3)}_{(1)}\times\mathcal{S}^{(3)}_{(2)}$ corresponding to the $\mathbb{Z}_2$-graded tensor product of the Clifford algebras $\mathrm{Cl}_{14,0}\cong\mathrm{Cl}_{8,0}\hat{\otimes}\mathrm{Cl}_{6,0}$ we consider standard subgroup $\mathcal{S}^{\mathbf{4,0}}_{(1)}\subset\mathcal{S}^{(3)}_{(1)}$ generated by (5.3). We add the

TABLE 12. Groups for $r = 14$

	$\ell(14,0)$	$\pi_0(\mathcal{S})$	$\lvert\mathfrak{b}(PI)\rvert$	*Signature*	PI
$\mathcal{S}^{(1)}$	7	1	14	$(T2,\theta)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \pi_{1,2}\pi_{7,8}$ $p_4 = \pi_{1,2}\pi_{9,10}$ $p_5 = \pi_{1,2}\pi_{11,12}$ $p_6 = \pi_{1,2}\pi_{13,14}$ $q = \theta_{\overline{1,13}}$
$\mathcal{S}^{(2)}$	7	1	14	$(T2,\pi,\theta)$	$p_1 = \pi_{1,2}\pi_{3,4}$ $p_2 = \pi_{1,2}\pi_{5,6}$ $p_3 = \pi_{1,2}\pi_{7,8}$ $p_4 = \pi_{1,2}\pi_{9,10}$ $p_5 = \pi_{1,2}\pi_{11,12}$ $p_6 = \theta_{\overline{1,11}}z_{13}z_{14}$ $q = \pi_{1,2}z_{14}$
$\mathcal{S}^{(3)}$	7	2	14	$(T1,\theta)\times(T2,\theta)$ $r = 8+6$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4},\quad (p_1)_{(2)} = \pi_{9,10}\pi_{11,12}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6},\quad (p_2)_{(2)} = \pi_{9,10}\pi_{13,14}$ $(p_3)_{(1)} = \pi_{1,2}\pi_{7,8},\quad (q)_{(2)} = \theta_{\overline{9,13}}$ $(p_4)_{(1)} = \theta_{\overline{1,7}},$
$\mathcal{S}^{(4)}$	7	2	14	$(T2,\pi,\theta)\times(T1,\theta)$ $r = 7+7$	$(p_1)_{(1)} = \pi_{1,2}\pi_{3,4},\quad (p_1)_{(2)} = \pi_{8,9}\pi_{10,11}$ $(p_2)_{(1)} = \pi_{1,2}\pi_{5,6},\quad (p_2)_{(2)} = \pi_{8,9}\pi_{12,13}$ $(p_3)_{(1)} = \theta_{\overline{1,7}},\quad (p_3)_{(2)} = \theta_{\overline{9,13}}z_{14}$ $(q_4)_{(1)} = \pi_{1,2}z_7,$

involutions $p_4 = \theta_{\overline{1,7}}$ to the set of generators for $\mathcal{S}^{\mathbf{4,0}}_{(1)}$ and generate the first component $\mathcal{S}^{(3)}_{(1)}$ in the product $\mathcal{S}^{(3)} = \mathcal{S}^{(3)}_{(1)} \times \mathcal{S}^{(3)}_{(2)}$. Then we set $\mathcal{S}^{(3)}_{(2)}$ to be generated by

$$PI = \{(p_1)_{(2)} = \pi_{9,10}\pi_{11,12},\ (p_2)_{(2)} = \pi_{9,10}\pi_{13,14},\ (q)_{(2)} = \theta_{\overline{9,13}}\}.$$

The disconnected subgroup $\mathcal{S}^{(4)} = \mathcal{S}^{(4)}_{(1)} \times \mathcal{S}^{(4)}_{(2)}$ corresponding to the $\mathbb{Z}_2$-graded tensor product of the Clifford algebras $\mathrm{Cl}_{14,0} \cong \mathrm{Cl}_{7,0}\hat{\otimes}\mathrm{Cl}_{7,0}$, generated similarly. We remove the type T_2 involution from one of the sets $PI_{7,0}$ generating $\mathcal{S}^{(4)}_{(k)}$, $k = 1$ or $k = 2$ in order to get a commutative set for $\mathcal{S}^{(4)}$ with $\ell(14,0) = 7$.

There are no groups with $|\mathfrak{b}(PI_{14,0})| = 13$ because $\ell(14,0) > \ell(13,0)$. □

Theorem 5.15. *There are one connected and one disconnected subgroups in* $\mathbb{S}^M_{15,0}$.

Proof. The standard subgroup $\mathcal{S}^{\mathbf{7,0}} \subseteq \mathcal{S}^{(1)} \subset \mathbb{S}^M_{15,0}$ is generated by involutions (5.6) We add type T_1 involution $p_7 = \theta_{\overline{1,15}}$ and type T_2 involution $q = \pi_{1,2}z_{15}$. We obtain the connected group $\mathcal{S}^{(1)}$.

To construct the disconnected subgroup $\mathcal{S}^{(2)} = \mathcal{S}^{(2)}_{(1)} \times \mathcal{S}^{(2)}_{(2)}$ corresponding to the $\mathbb{Z}_2$-graded tensor product of the Clifford algebras $\mathrm{Cl}_{15,0} \cong \mathrm{Cl}_{8,0}\hat{\otimes}\mathrm{Cl}_{7,0}$ we

TABLE 13. Groups for $r = 15$

	$\ell(15,0)$	$\pi_0(\mathcal{S})$	$\lvert\mathfrak{b}(PI)\rvert$	*Signature*	PI
$\mathcal{S}^{(1)}$	8	1	15	$(T2,\pi)$	$p_1=\pi_{1,2}\pi_{3,4}$ $p_2=\pi_{1,2}\pi_{5,6}$ $p_3=\pi_{1,2}\pi_{7,8}$ $p_4=\pi_{1,2}\pi_{9,10}$ $p_5=\pi_{1,2}\pi_{11,12}$ $p_6=\pi_{1,2}\pi_{13,14}$ $p_7=\theta_{\overline{1,13}}z_{15}$ $q=\pi_{1,2}z_{15}$
$\mathcal{S}^{(2)}$	8	2	15	$(T1,\theta)\times(T2,\pi,\theta)$ $r=8+7$	$(p_1)_{(1)}=\pi_{1,2}\pi_{3,4}$, $(p_1)_{(2)}=\pi_{9,10}\pi_{11,12}$ $(p_2)_{(1)}=\pi_{1,2}\pi_{5,6}$, $(p_2)_{(2)}=\pi_{9,10}\pi_{13,14}$ $(p_3)_{(1)}=\pi_{1,2}\pi_{7,8}$, $(p_3)_{(2)}=\theta_{\overline{9,15}}$ $(p_4)_{(1)}=\theta_{\overline{1,7}}$, $(q)_{(2)}=\pi_{9,10}z_{15}$

proceed as in the case $r = 14$ for $\mathcal{S}^{(3)}_{(1)}$, and set $\mathcal{S}^{(3)}_{(2)}$ to be generated by

$$\begin{aligned} PI = \{\quad (p_1)_{(2)} &= \pi_{9,10}\pi_{11,12},\ (p_2)_{(2)} = \pi_{9,10}\pi_{13,14},\\ (p_3)_{(2)} &= \theta_{\overline{9,13}},\quad (q)_{(2)} = \pi_{9,10}z_{15}\}. \end{aligned}$$

There are no groups with $|\mathfrak{b}(PI_{15,0})| = 14$ because $\ell(15,0) > \ell(14,0)$. □

Theorem 5.16. *There are two connected non-equivalent and two disconnected non-equivalent subgroups in* $\mathbb{S}^M_{16,0}$.

TABLE 14. Groups for $r = 16$

	$\ell(16,0)$	$\pi_0(\mathcal{S})$	$\lvert\mathfrak{b}(PI)\rvert$	*Signature*	PI
$\mathcal{S}^{(1)}$	8	1	16	$(T1,\pi,\theta)$	$p_1=\pi_{1,2}\pi_{3,4}$ $p_2=\pi_{1,2}\pi_{5,6}$ $p_3=\pi_{1,2}\pi_{7,8}$ $p_4=\pi_{1,2}\pi_{9,10}$ $p_5=\pi_{1,2}\pi_{11,12}$ $p_6=\pi_{1,2}\pi_{13,14}$ $p_7=\theta_{\overline{1,13}}z_{15}$ $p_8=\pi_{1,2}z_{15}z_{16}$
$\mathcal{S}^{(2)}$	8	1	15	$(T2,\pi,\theta)$	$p_1=\pi_{1,2}\pi_{3,4}$ $p_2=\pi_{1,2}\pi_{5,6}$ $p_3=\pi_{1,2}\pi_{7,8}$ $p_4=\pi_{1,2}\pi_{9,10}$ $p_5=\pi_{1,2}\pi_{11,12}$ $p_6=\pi_{1,2}\pi_{13,14}$ $p_7=\theta_{\overline{1,13}}z_{15}$ $q=\pi_{1,2}z_{15}$
$\mathcal{S}^{(3)}$	8	2	16	$(T1,\theta)\times(T1,\theta)$ $r=8+8$	$(p_1)_1=\pi_{1,2}\pi_{3,4}$, $(p_1)_2=\pi_{9,10}\pi_{11,12}$ $(p_2)_1=\pi_{1,2}\pi_{5,6}$, $(p_2)_2=\pi_{9,10}\pi_{13,14}$ $(p_3)_1=\pi_{1,2}\pi_{7,8}$, $(p_3)_2=\pi_{9,10}\pi_{15,16}$ $(p_4)_1=\theta_{\overline{1,7}}$, $(p_4)_2=\theta_{\overline{9,15}}$
$\mathcal{S}^{(4)}$	8	2	15	$(T1,\theta)\times(T2,\pi,\theta)$ $r=8+7$	$(p_1)_1=\pi_{1,2}\pi_{3,4}$, $(p_1)_2=\pi_{9,10}\pi_{11,12}$ $(p_2)_1=\pi_{1,2}\pi_{5,6}$, $(p_2)_2=\pi_{9,10}\pi_{13,14}$ $(p_3)_1=\pi_{1,2}\pi_{7,8}$, $(p_3)_2=\theta_{\overline{9,15}}$ $(p_4)_1=\theta_{\overline{1,7}}$, $(q)_2=\pi_{9,10}z_{15}$

Proof. The standard subgroup $\mathcal{S}^{\mathbf{8,0}} \subseteq \mathcal{S}^{(1)} \subset \mathbb{S}^{M}_{16,0}$ is generated by involutions

$$(5.7)\qquad \begin{array}{l} p_1 = \pi_{1,2}\pi_{3,4}, \quad p_2 = \pi_{1,2}\pi_{5,6}, \quad p_3 = \pi_{1,2}\pi_{7,8}, \quad p_4 = \pi_{1,2}\pi_{9,10} \\ p_5 = \pi_{1,2}\pi_{11,12}, \quad p_6 = \pi_{1,2}\pi_{13,14}, \quad p_7 = \pi_{1,2}\pi_{15,16}. \end{array}$$

We add type T_1 involution $p_7 = \theta_{\overline{1,15}}$ and obtain the connected group $\mathcal{S}^{(1)}$.

To construct the disconnected subgroups $\mathcal{S}^{(2)} = \mathcal{S}^{(2)}_{(1)} \times \mathcal{S}^{(2)}_{(2)}$ corresponding to the $\mathbb{Z}_2$-graded tensor product of the Clifford algebras $\mathrm{Cl}_{16,0} \cong \mathrm{Cl}_{8,0}\hat{\otimes}\mathrm{Cl}_{8,0}$ and $\mathrm{Cl}_{16,0} \cong \mathrm{Cl}_{8,0}\hat{\otimes}\mathrm{Cl}_{7,0}$ we proceed as in the previous cases.

The group with $|\mathfrak{b}(PI_{16,0})| = 15$ coincides with the group $\mathcal{S}^{(1)} \in \mathbb{S}^{M}_{15,0}$. □

Theorem 5.17. *Theorems 5.3 – 5.16 are true for H-type Lie algebras $\mathfrak{n}_{r,1}$, $r \in \{3, \dots, 16\}$.*

Proof. For $s = 1$, the negative basis vector plays no role in forming the involutions, see Definition 3.5. □

TABLE 15. Number of non-equivalent groups

r	1	2	3	4	5	6	7	8
$\pi_0(\mathcal{S}) = 1$	0	0	1	2	1	1	1	2
$\pi_0(\mathcal{S}) = 2$	0	0	0	0	0	0	0	0
r	9	10	11	12	13	14	15	16
$\pi_0(\mathcal{S}) = 1$	3	4	1	3	3	2	1	2
$\pi_0(\mathcal{S}) = 2$	0	2	2	5	3	2	1	2

6. ISOMORPHISM OF INVARIANT INTEGRAL STRUCTURES

Theorem 6.1. *If*

$$(6.1)\qquad (r, s) \in \{(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (2, 1), (0, 2)\},$$

then for any orthonormal basis $B_{r,s} = \{z_j\}$ and $v \in V^{r,s}$, with $\langle v, v\rangle_{V^{1,0}} = \pm 1$ the invariant orthonormal structures spanned by bases as in Table 16 are isomorphic.

TABLE 16. Invariant integral structures for (r, s) in Theorem 6.1

2	$\{v, J_{z_1}v, J_{z_2}v, J_{z_1}J_{z_2}v, z_1, z_2\}$		
1	$\{v, J_{z_1}v, z_1\}$	$\{v, J_{z_1}v, J_{z_2}v, J_{z_1}J_{z_2}v, z_1, z_2\}$	$\{v, J_{z_1}v, J_{z_2}v, J_{z_1}J_{z_2}v, z_1, z_2\}$
0	v	$\{v, J_{z_1}v, z_1\}$	$\{v, J_{z_1}v, J_{z_2}v, J_{z_1}J_{z_2}v, z_1, z_2\}$
s/r	0	1	2

Proof. There are only trivial groups $\mathcal{S} \subset \mathbb{S}^{M}_{r,s}$ for (r, s) as in (6.1) since there are no involutions. The proof of uniqueness is literally repeats the proof of Proposition 3.16. See also discussions in Remark 3.2. □

6.1. **Isomorphic invariant integral structures.** We fix an orthonormal basis $B_{r,s} = \{z_1, \dots, z_{r+s}\}$ and a group $\mathcal{S} = \mathcal{S}(PI_{r,s})$. Recall the construction of an invariant basis $\mathcal{B}_v(V^{r,s})$ on the minimal admissible module $V^{r,s}$ from Theorem 3.15, which used the centraliser of the isotropy group $\mathcal{S} = \mathcal{S}(PI_{r,s}) = \mathcal{S}_v$ of a unit vector $v \in V^{r,s}$. The invariant integral structure on the Lie algebra $\mathfrak{n}_{r,s}(V^{r,s})$ given by $\mathcal{S}$ will be denoted by

$$\mathcal{L}(\mathcal{S}) = \operatorname{span}_{\mathbb{Z}}\{\mathcal{B}_v(V^{r,s})\} \oplus \operatorname{span}_{\mathbb{Z}}\{B_{r,s}\}.$$

Theorem 6.2. *If two groups $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent; that is there exists a map $C \in O(r,s)$ such that $C(\widehat{S}_1)) = \widehat{S}_2$, then the invariant integral structures $\mathcal{L}(\mathcal{S}_1)$ and $\mathcal{L}(\mathcal{S}_2)$ are isomorphic under a map $A \oplus C$, where $A \colon V^{r,s} \to V^{r,s}$ is an orthogonal map with respect to $\langle .\,,.\rangle_{V^{r,s}}$; that is $A^{\tau}A = \mathrm{Id}_{V^{r,s}}$.*

Proof. The proof is a light generalisation of Proposition 3.16. Let $\mathcal{S}_1 = \mathcal{S}(PI_1)$ and $\mathcal{S}_2 = \mathcal{S}(PI_2)$ be equivalent groups. It imply that there is $C \in \mathrm{O}(r,s)$ such that $C(\widehat{S}_1) = \widehat{S}_2$ where we denoted by the same letter C the extention of the orthogonal map to the group $\mathrm{Cl}^*_{r,s} \subset \mathrm{Cl}_{r,s}$ of invertible elements of the Clifford algebra $\mathrm{Cl}_{r,s}$. Let

$$\mathcal{B}_v(V^{r,s}) = \Big\{v, J_{\sigma_i}(v), J_{\tau_j}(v), J_{\tau_j}J_{\sigma_i}(v) \mid \ \sigma_i, \tau_j, \sigma_i\tau_j \in \Sigma(\mathcal{S}_1)\Big\} \tag{6.2}$$

be the invariant basis, constructed in Theorem 3.15 by making use the eigenspaces of involutions from PI_1. The set PI_1 is equivalent to PI_2 under C. We use the method of Theorem 3.15 and obtain a basis

$$\begin{aligned}\mathcal{B}_w(V^{r,s}) \;\; &= \;\; \Big\{w, J_{C(\sigma_i)}(w), J_{C(\tau_j)}(w), J_{C(\tau_j)}J_{C(\sigma_i)}(w) \mid \\ &\qquad C(\sigma_i), C(\tau_j), C(\sigma_i)C(\tau_j) \in \Sigma(\mathcal{S}_2)\Big\},\end{aligned} \tag{6.3}$$

where $\mathcal{S}_2 \cong \mathcal{S}(PI_2) \cong \mathcal{S}(C(PI_1))$ and the set PI_2 was replaced by $C(PI_1)$. Note that since $C(B_{r,s}) = B_{r,s}$ we also have $G(B_{r,s}) = G\big(C(B_{r,s})\big)$.

We construct a correspondence $A \colon \mathcal{B}_v(V^{r,s}) \to \mathcal{B}_w(V^{r,s})$ by

$$\begin{aligned}&v \longmapsto w, \quad J_{\sigma_i}(v) \longmapsto J_{C(\sigma_i)}(w), \quad J_{\tau_j}(v) \longmapsto J_{C(\tau_j)}(w),\\ &J_{\tau_j}(v)J_{\sigma_i}(v) \longmapsto J_{C(\tau_j)}(w)J_{C(\sigma_i)}(w),\end{aligned}$$

and $C \colon z_k \longmapsto C(z_k)$. The correspondence $A \oplus C$ extended to a linear map over $\mathbb{R}$ or $\mathbb{Z}$ is an orthogonal map on $V^{r,s}$ since it maps orthonormal basis (6.2) to orthonormal basis (6.3). To show that the linear map $A \oplus C$ is an isomorphism of invariant integral structures, we argue as in Proposition 3.16. By the invariance of the bases $\mathcal{B}_v(V^{r,s})$ and $\mathcal{B}_w(V^{r,s})$ we have

$$J_{C(z_k)}Au_{\alpha} = \pm J_{C(\kappa)}v_2 = \pm AJ_{\varkappa}v_1 = AJ_{z_k}u_{\alpha}$$

for any $u_\alpha \in \mathcal{B}_v(V^{r,s})$, $z_k \in B_{r,s}$, and for some $\varkappa \in \Sigma = \{\sigma_i, \tau_j, \tau_j\sigma_i\}$. It implies

$$\begin{aligned}\langle [Au_\alpha, Au_\beta], C(z_k)\rangle_{r,s} &= \langle J_{C(z_k)}Au_\alpha, Au_\beta\rangle_{V^{r,s}} = \langle AJ_{z_k}u_\alpha, Au_\beta\rangle_{V^{r,s}} \\ &= \langle A^\tau AJ_{z_k}u_\alpha, u_\beta\rangle_{V^{r,s}} = \langle J_{z_k}u_\alpha, u_\beta\rangle_{V^{r,s}} \\ &= \langle [u_\alpha, u_\beta], z_k\rangle_{r,s}.\end{aligned}$$

for any $u_\alpha, u_\beta \in \mathcal{B}_v(V^{r,s})$ and $z_k \in B_{r,s}$. □

Theorem 6.3. *Let $\mathcal{S}_1, \mathcal{S}_2 \in \mathbb{S}^M$ and $\mathcal{L}(\mathcal{S}_1)$, $\mathcal{L}(\mathcal{S}_2)$ be the corresponding invariant integral structures. If there is an isomorphism*

$$A \oplus C\colon \mathcal{L}(\mathcal{S}_1) \to \mathcal{L}(\mathcal{S}_2) \tag{6.4}$$

with $A\colon V^{r,s} \to V^{r,s}$ such that $A^\tau A = \mathrm{Id}_{V^{r,s}}$, then $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent in the sense of Definition 4.3.

Proof. Let

$$\mathcal{L}(\mathcal{S}_1) = \mathrm{span}_{\mathbb{Z}}\{\mathcal{B}_v(V^{r,s})\} \oplus \mathrm{span}_{\mathbb{Z}}\{B_{r,s}\} = L_1 \oplus \mathrm{span}_{\mathbb{Z}}\{B_{r,s}\}$$
$$\mathcal{L}(\mathcal{S}_2) = \mathrm{span}_{\mathbb{Z}}\{\mathcal{B}_u(V^{r,s})\} \oplus \mathrm{span}_{\mathbb{Z}}\{B_{r,s}\} = L_2 \oplus \mathrm{span}_{\mathbb{Z}}\{B_{r,s}\}$$

be the invariant integral srtuctures generated by the groups $\mathcal{S}_1$ and $\mathcal{S}_2$. Here we also assume that $\mathcal{S}_1 = \mathcal{S}_v$ is the isotropy subgroup of a unit vector $v \in V^{r,s}$ and $\mathcal{S}_2 = \mathcal{S}_u$ is the isotropy subgroup of a unit vector $u \in V^{r,s}$. Since $A \oplus C$ is an isomorphism, we obtain $A(L_1) = L_2$. By noting that $A^{-1}(L_2) = A^\tau(L_2) = L_1$, we deduce that $A^\tau A(L_1) = L_1$.

We denote by the same letter $A \oplus C \in \mathrm{Aut}(\mathfrak{n}_{r,s})$ the automorphism of $\mathfrak{n}_{r,s}(V^{r,s})$ which restriction to $\mathcal{L}(\mathcal{S}_1)$ gives map (6.4). The properties $A^\tau A = \mathrm{Id}_{V^{r,s}}$ and $A^\tau J_{C(z)}A = J_z$ imply $AJ_zx = J_{C(z)}Ax$ for $x \in L_1$ and $C \in \mathrm{O}(r,s)$, the latter one being an orthogonal transformation over $\mathbb{Z}$ as well. For $v \in \mathfrak{B}_v(V^{r,s})$ we find a basis vector $u_j \in \mathfrak{B}_u(V^{r,s})$ such that $Av = u_j$. If there holds $Av = -u_j$, then the proof is similar. By renumbering the basis vectors $\{u_j\}$ we can assume that $Av = u$. We have for the stationary group of Av

$$\begin{aligned}\mathcal{S}_{Av} &= \{\tilde\sigma \in G\big(C(B_{r,s})\big) \mid \ J_{\tilde\sigma}Av = Av\} \\ &= \{\tilde\sigma \in G\big(C(B_{r,s})\big) \mid \ J_{\tilde\sigma}u = u\} = \mathcal{S}_u\end{aligned} \tag{6.5}$$

Since $\tilde\sigma = C(z_{i_1})\dots C(z_{i_k})$, and $AJ_zx = J_{C(z)}Ax$, $x \in L_1$ we have

$$Av = J_{\tilde\sigma}Av = J_{C(z_{i_1})}\dots J_{C(z_{i_k})}Av = AJ_{z_{i_1}}\dots J_{z_{i_k}}v = AJ_\sigma v.$$

This implies $v = J_\sigma v$ for any $\sigma \in G(B_{r,s})$. Thus we conclude that if $\tilde\sigma \in \mathcal{S}_{Av}$, for $\tilde\sigma = C(z_{i_1})\dots C(z_{i_k}) \in G(C(B_{r,s}))$ then $\sigma = z_{i_1}\dots z_{i_k} \in \mathcal{S}_v$. Thus the groups $\mathcal{S}_{Av}$ and $\mathcal{S}_v$ are equivalent. The equalities (6.5) shows that $\mathcal{S}_2 = \mathcal{S}_u = \mathcal{S}_{Av}$ and $\mathcal{S}_1 = \mathcal{S}_v$ are equivalent. □

Table 17 shows the classical groups $\mathbb{A}$ such that the map $A \oplus \mathrm{Id}$ with $A \in \mathbb{A}$ is the automorphism of H-type Lie algebras $\mathfrak{n}_{r,s}(V^{r,s})$, see also [FM21, Table 3] for non-minimal admissible modules. The groups $\mathrm{Sp}(n), \mathrm{O}(n,\mathbb{C}), \mathrm{U}(n), \mathrm{O}^*(n)$ are subgroups of orthogonal transformations.

TABLE 17. Groups $\mathbb{A}$

8	$\mathrm{GL}(1,\mathbb{R})$								
7	$\mathrm{O}(1,\mathbb{R})$	$\mathrm{U}(1)$	$\mathrm{Sp}(1)$	$\mathrm{Sp}(1)\times\mathrm{Sp}(1)$					
6	$\mathrm{O}(2,\mathbb{C})$	$\mathrm{O}^*(2)$	$\mathrm{GL}(1,\mathbb{H})$	$\mathrm{Sp}(1)$					
5	$\mathrm{O}^*(4)$	$\mathrm{O}^*(2)\times\mathrm{O}^*(2)$	$\mathrm{O}^*(2)$	$\mathrm{U}(1)$					
4	$\mathrm{GL}(1,\mathbb{H})$	$\mathrm{O}^*(2)$	$\mathrm{O}(1,\mathbb{C})$	$\mathrm{O}(1\mathbb{R})$	$\mathrm{GL}(1,\mathbb{R})$				
3	$\mathrm{Sp}(1)$	$\mathrm{U}(1)$	$\mathrm{O}(1,\mathbb{R})$	$\mathrm{O}(1,\mathbb{R})\times\mathrm{O}(1,\mathbb{R})$	$\mathrm{O}(1)$	$\mathrm{U}(1)$	$\mathrm{Sp}(1)$	$\mathrm{Sp}(1)\times\mathrm{Sp}(1)$	
2	$\mathrm{Sp}(2,\mathbb{C})$	$\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{GL}(2,\mathbb{R})$	$\mathrm{O}(2\mathbb{R})$	$\mathrm{O}(2,\mathbb{C})$	$\mathrm{O}^*(2)$	$\mathrm{GL}(1,\mathbb{H})$	$\mathrm{Sp}(1)$	
1	$\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{Sp}(2,\mathbb{R})\times\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{Sp}(4,\mathbb{R})$	$\mathrm{U}(2)$	$\mathrm{O}^*(4)$	$\mathrm{O}^*(2)\times\mathrm{O}^*(2)$	$\mathrm{O}^*(1)$	$\mathrm{U}(1)$	
0		$\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{Sp}(2,\mathbb{C})$	$\mathrm{Sp}(1)$	$\mathrm{GL}(1\mathbb{H})$	$\mathrm{O}^*(2)$	$\mathrm{O}(1,\mathbb{C})$	$\mathrm{O}(1,\mathbb{R})$	$\mathrm{GL}(1,\mathbb{R})$
	0	1	2	3	4	5	6	7	8

Theorem 6.4. *Let (r,s) be such that the groups $\mathbb{A}$ in Table 17 is a subgroup of orthogonal transformations. The groups $\mathcal{S}_1,\mathcal{S}_2\in\mathbb{S}^M_{r,s}$ are equivalent in sense of Definition (4.4), if and only if the corresponding invariant integral structures $\mathcal{L}(\mathcal{S}_1)$ and $\mathcal{L}(\mathcal{S}_2)$ are isomorphic.*

Proof. If (r,s) as in the statement of Theorem 6.4 then for an automorphism $\tilde{A}\oplus\mathrm{Id}$ of $\mathfrak{n}_{r,s}(V^{r,s})$ we have $\tilde{A}^\tau\tilde{A}=\mathrm{Id}_{V^{r,s}}$. It implies that the general automorphisms $A\oplus C$ of $\mathfrak{n}_{r,s}(V^{r,s})$ also satisfies $A^\tau A=\mathrm{Id}_{V^{r,s}}$, see [FM21, Section 3.2].

Thus if the invariant integral structures $\mathcal{L}(\mathcal{S}_1)$ and $\mathcal{L}(\mathcal{S}_2)$ are isomorphic, then they will be isomorphic under a map $A\oplus C$ with $A^\tau A=\mathrm{Id}_{V^{r,s}}$. It implies that the group $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent by Theorem 6.3.

Conversely, if we assume now that the groups $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent, then by Theorem 6.2 the corresponding invariant integral structures will be isomorphic. □

6.2. Non-isomorphic invariant integral structures.

Theorem 6.5. *Let $\mathcal{S}_1=\mathcal{S}(PI_1)\in\mathbb{S}^M_{r,s}$ and $\mathcal{S}_2=\mathcal{S}(PI_2)\in\mathbb{S}^M$ be non-equivalent groups such that there is a type T_1 involution in $p\in PI_1$ and an involution $q\in PI_2$ such that $p\cdot q=-q\cdot p$. Then the invariant integral structures $\mathcal{L}(\mathcal{S}_1)$ and $\mathcal{L}(\mathcal{S}_2)$ are not isomorphic.*

Proof. Let $\mathfrak{n}_{r,s}(V^{r,s})$ be a pseudo H-type Lie algebra and $p\in PI_1$, $q\in PI_2$ as in the statement of Theorem 6.5. We denote by $E(p)=\{x\in V^{r,s}\mid\ J_px=x\}$ the eigenspace of type T_1 involution $p\in PI_1$ and by

$$E_+(q)=\{x\in E(p)\mid\ J_qx=x\},\quad E_-(q)=\{x\in E(p)\mid\ J_qx=-x\}$$

the non-trivial eigen spaces of $q\in PI_2$. Then the subspaces in the direct sum $E(p)=E_+(q)\oplus E_-(q)$ are orthogonal.

Let us assume that there exists an isomorphism $A\oplus C\colon\mathcal{L}(\mathcal{S}_1)\to\mathcal{L}(\mathcal{S}_2)$ and write

$$F(p)=A\big(E(p)\big),\quad F_\pm=F_\pm\big(C(q)\big)=\{y\in F(p)\mid\ J_{C(q)}y=\pm y\}.$$

Note the following: since $AJ_p=J_pA$, we obtain that $A^\tau A\big(E(p)\big)=E(p)$. The map C, extended to the Clifford algebra $\mathrm{Cl}_{r,s}$, satisfies $C(p)C(q)=-C(q)C(p)$.

Therefore

$$F(p) = F_+ \oplus F_-, \tag{6.6}$$

where F_+, F_- are non-trivial orthogonal vector spaces.

Let $x \in E(p)$ and put $Ax = y_+(x) + y_-(x)$, where $y_+(x) \in F_+$ and $y_-(x) \in F_-$. We also have

$$Ax = AJ_p x = J_{C(p)}Ax = J_{C(p)}(y_+(x) + y_-(x)) = J_{C(p)}y_+(x) + J_{C(p)}y_-(x).$$

Since

$$J_{C(p)}\colon F_+ \to F_-, \quad \text{and} \quad J_{C(p)}y_+(x) \in F_-, \quad J_{C(p)}y_-(x) \in F_+$$

we obtain $y_+(x) = J_{C(p)}y_-(x)$ and $y_-(x) = J_{C(p)}y_+(x)$ by the uniqueness of the decomposition into a direct sum of vector spaces. We conclude

$$Ax = y_+(x) + J_{C(p)}y_+(x).$$

Let $\{v_i\}$ be an orthonormal basis of the space $E(p)$, which is a part of the invariant basis on $V^{r,s}$ defined by the $\mathcal{S}_1$. The matrix components a_{ij} of the operator $A^\tau A\colon E(p) \to E(p)$ with respect to the basis $\{v_i\}$ have the form

$$\begin{aligned} a_{ij} &= \langle A^\tau Av_i, v_j\rangle_{V^{r,s}} = \langle Av_i, Av_j\rangle_{V^{r,s}} \\ &= \langle y_+(v_i) + J_{C(p)}(y_+(v_i)), y_+(v_j) + J_{C(p)}y_+(v_j)\rangle_{V^{r,s}} \\ &= 2\langle y_+(v_i), y_+(v_j)\rangle_{V^{r,s}}, \end{aligned}$$

where we used the orthogonality of the vector spaces F_+ and F_- in (6.6).

Hence the non-vanishing components of the matrix $A^\tau A$ are always even numbers, so that A can not be invertible in $SL(n,\mathbb{Z})$. It implies that there are no an isomorphism $A \oplus C$ between the invariant integral structures $\mathcal{L}(\mathcal{S}_1)$ and $\mathcal{L}(\mathcal{S}_2)$. □

Corollary 6.6. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be in $\mathbb{S}^M_{r,s}$ and assume*

(1) *$\mathcal{S}_1 = \mathcal{S}_1(PI_1)$ and $\mathcal{S}_2 = \mathcal{S}_2(PI_2)$ are not equivalent in the sense of the Definition 4.4,*
(2) *one of the sets PI_k, $k = 1,2$ is of $(T1)$-type.*

Then Theorem 6.5 holds.

Proof. Since a generating set PI_1 of $\mathcal{S}_1$ consists only of involutions of type T_1, the non-existence of an involution $q \in PI_2$ such that $pq = -qp$ for any $p \in PI_1$ requires that $PI_1 \subset \mathcal{S}_2$ by the maximality of the groups $\mathcal{S}_1$ and $\mathcal{S}_2$. But then $\mathcal{S}_1 = \mathcal{S}_2$ which is a contradiction. □

There are 3 pairs consisting of non-equivalent groups for $(r, 0)$, which does not satisfies the conditions of Theorem 6.5 For $r = 12$ we have two non-equivalent groups $\mathcal{S}^{(5)}$ and $\mathcal{S}^{(8)}$ violating the conditions of Theorem 6.5, see

Table 10. The generating set is presented here

$$PI^{(5)} = \{ \quad p_1 = z_1z_2z_3z_4, p_2 = z_1z_2z_5z_6, p_3 = z_1z_3z_5z_7, p_4 = z_8z_9z_{10}z_{11}, \varkappa_1 = z_8z_9z_{12}\},$$

$$PI^{(8)} = \{ \quad p_1 = z_1z_2z_3z_4, p_2 = z_1z_2z_5z_6, p_3 = z_1z_3z_5z_7, p_4 = z_8z_9z_{10}z_{11}, \varkappa_2 = z_1z_2z_7\}$$

For $r = 13$ there are two sets of pairs of non-equivalent groups violating the conditions of Theorem 6.5, see Table 11. The first collection contains the groups $\mathcal{S}^{(k)}$, $k = 1, 2$ which are all connected. The second collection contains the groups $\mathcal{S}^{(k)}$, $k = 5, 6$ which are products of two smaller subgroups. The generating sets for the first collection are:

$$PI^{(1)} = \{ \quad p_1 = z_1z_2z_3z_4, p_2 = z_1z_2z_5z_6, p_3 = z_1z_2z_7z_8, p_4 = z_1z_2z_9z_{10}, p_5 = z_1z_2z_{11}z_{12}, \rho_1 = z_1z_2z_{13}\},$$

$$PI^{(2)} = \{ \quad p_1 = z_1z_2z_3z_4, p_2 = z_1z_2z_5z_6, p_3 = z_1z_2z_7z_8, p_4 = z_1z_2z_9z_{10}, p_5 = z_1z_2z_{11}z_{12}, \rho_2 = z_1z_3z_5z_7z_9z_{11}z_{13}\},$$

The generating sets for the second collection are:

$$PI^{(5)} = \{ \quad p_1 = z_1z_2z_3z_4, p_2 = z_1z_2z_5z_6, p_3 = z_1z_3z_5z_7, p_4 = z_8z_9z_{10}z_{11}, p_5 = z_8z_9z_{12}z_{13}, \tau_1 = z_1z_2z_7\},$$

$$PI^{(6)} = \{ \quad p_1 = z_1z_2z_3z_4, p_2 = z_1z_2z_5z_6, p_3 = z_1z_3z_5z_7, p_4 = z_8z_9z_{10}z_{11}, p_5 = z_8z_9z_{12}z_{13}, \tau_2 = z_8z_{10}z_{12}\},$$

We formulate three theorems and prove them. The method is essentially the same and differs only by a choice of a convenient basis for the space E invariant under the action of type T_1 involutions. We start from $r = 13$ since the dimension of E is equal to four and the calculations are more transparent.

Theorem 6.7. *Let $r = 13$. The invariant orthogonal lattices $\mathcal{L}(\mathcal{S}^{(5)})$ and $\mathcal{L}(\mathcal{S}^{(6)})$ defined by non-equivalent groups $\mathcal{S}^{(5)} = \mathcal{S}(PI^{(5)})$ and $\mathcal{S}^{(6)} = \mathcal{S}(PI^{(6)})$ are not isomorphic.*

Proof. The minimal admissible module $V^{13,0}$ is isometric to $\mathbb{R}^{128,0}$. Let $E = \{x \in V^{13,0} \mid \; J_{p_i}(x) = x, \; i = 1, 2, 3, 4, 5\}$ be the eigenspace of involutions of type T_1. Then $\dim(E) = 4$ and $E = E_+(\tau_1) \oplus E_-(\tau_1)$, there $E_\pm(\tau_1)$ are the eigenspaces of τ_1. Let $v \in E_+(\tau_1)$, $\langle v, v\rangle_{V^{13,0}} = 1$. The vectors

$$v_1 = v, \; v_2 = J_{z_8}J_{z_9}v, \; v_3 = J_{z_8}J_{z_{10}}J_{z_{12}}v = J_{\tau_2}v, \; v_4 = J_{z_9}J_{z_{10}}J_{z_{12}}v = J_{z_9}J_{\tau_2}v,$$

form an orthonormal basis of E. In fact,

$$\langle v, v_1\rangle_{V^{13,0}} = \langle v, J_{z_8}J_{z_9}v\rangle_{V^{13,0}} = -\langle z_8, z_9\rangle_{\mathbb{R}^{13,0}}\langle v, v\rangle_{V^{13,0}} = 0,$$

and analogously $\langle v_2, v_3\rangle_{V^{13,0}} = 0$. Furthermore, from one side

$$\langle v, v_3\rangle_{V^{13,0}} = \langle J_{\tau_1}v, J_{\tau_2}v\rangle_{V^{13,0}} = \langle v, J_{\tau_1}J_{\tau_2}v\rangle_{V^{13,0}} = -\langle v, J_{\tau_2}J_{\tau_1}v\rangle_{V^{13,0}}, \tag{6.7}$$

But from other side

$$\langle v, v_3\rangle_{V^{13,0}} = \langle J_{\tau_1}v, J_{\tau_2}v\rangle_{V^{13,0}} = \langle J_{\tau_2}J_{\tau_1}v, v\rangle_{V^{13,0}}. \tag{6.8}$$

The equalities (6.7) and(6.8) imply the orthogonality of v and v_3. The orthogonality of the rest of vectors are reduced to the calculations as in (6.7) and(6.8), where we only used that the skew symmetry of J_{z_k} with respect to product $\langle . , .\rangle_{V^{13,0}}$ and skew symmetry of the Clifford product $J_{z_k}J_{z_l} = -J_{z_l}J_{z_k}$.

Assume that there exists an isomorphism $A \oplus C\colon \mathfrak{n}_{13,0} \to \mathfrak{n}_{13,0}$ between the invariant orthogonal lattices $\mathcal{L}(\mathcal{S}^{(5)})$ to $\mathcal{L}(\mathcal{S}^{(6)})$.

We show that A is an orthogonal transformation. In fact, we have

$$\begin{aligned}\langle Av_1, Av_2\rangle_{V^{13,0}} &= \langle Av, J_{C(z_8)}J_{C(z_9)}Av\rangle_{V^{13,0}}\\ &= \langle C(z_8), C(z_9)\rangle_{\mathbb{R}^{13,0}}\langle Av, Av\rangle_{V^{13,0}} = 0.\end{aligned}$$

Furthermore, by making use of the fact that the product $J_{\tau_2}J_{\tau_1}$ contains 6 numbers of different J_{z_k}, we get

$$\begin{aligned}\langle Av_1, Av_3\rangle_{V^{13,0}} &= \langle Av, AJ_{\tau_2}v\rangle_{V^{13,0}} = \langle Av, AJ_{\tau_2}J_{\tau_1}v\rangle_{V^{13,0}}\\ &= \langle Av, J_{C(\tau_2)}J_{C(\tau_1)}Av\rangle_{V^{13,0}} = (-1)^{11}\langle J_{C(\tau_2)}J_{C(\tau_1)}Av, Av\rangle_{V^{13,0}}. \qquad (6.9)\end{aligned}$$

In the last step we used the skew symmetry of $J_{C(z_k)}$ with respect to $\langle . , .\rangle_{V^{13,0}}$ and skew symmetry $J_{C(z_k)}J_{C(z_l)} = -J_{C(z_l)}J_{C(z_k)}$. It shows Av_1 and Av_3 are orthogonal. Analogously we obtain $\langle Av_1, Av_4\rangle_{V^{13,0}} = 0$.

Next we show

$$\begin{aligned}\langle Av_2, Av_3\rangle V^{13,0} &= \langle AJ_{z_8}J_{z_9}v, AJ_{\tau_2}v\rangle_{V^{13,0}} = \langle J_{C(z_8)}J_{C(z_9)}Av, J_{C(\tau_2)}J_{C(\tau_1)}Av\rangle_{V^{13,0}}\\ &= -\langle Av, J_{C(z_9)}J_{C(z_{10})}J_{C(z_{12})}J_{C(\tau_1)}Av\rangle_{V^{13,0}}\\ &= (-1)^{12}\langle Av, J_{C(z_9)}J_{C(z_{10})}J_{C(z_{12})}J_{C(\tau_1)}Av\rangle_{V^{13,0}} = 0,\end{aligned}$$

by using the same arguments as in (6.9). The value $\langle Av_2, Av_4\rangle_{V^{13,0}} = 0$ is shown in the same way.

Finally,

$$\begin{aligned}\langle Av_3, Av_4\rangle_{V^{13,0}} &= \langle J_{C(\tau_2)}J_{C(\tau_1)}Av, J_{C(z_9)}J_{C(\tau_2)}J_{C(\tau_1)}Av, \rangle_{V^{13,0}}\\ &= \langle J_{C(z_8)}Av, J_{C(z_9)}Av\rangle_{V^{13,0}} = 0.\end{aligned}$$

This shows that $A^\tau A = \lambda = ||A(v)||_{V^{13,0}}Id_{V^{13,0}}$ and then $A^\tau A \in SL(4,\mathbb{Z})$ requires $||A(v)||_{V^{13,0}} = 1$. Hence by Theorem 6.3, the groups $\mathcal{S}^{(5)}$ and $\mathcal{S}^{(6)}$ are equivalent, that is a contradiction. □

Theorem 6.8. *Let $r = 13$. The invariant orthogonal lattices $\mathcal{L}(\mathcal{S}^{(1)})$ and $\mathcal{L}(\mathcal{S}^{(2)})$ defined by non-equivalent groups $\mathcal{S}^{(1)} = \mathcal{S}(PI^{(1)})$ and $\mathcal{S}^{(2)} = \mathcal{S}(PI^{(2)})$ are not isomorphic.*

Proof. As in Theorem 6.7 we define $E = \{x \in V^{13,0} \mid J_{p_i}(x) = x,\ i = 1,2,3,4,5\}$ and $E = E_+(\rho_1) \oplus E_-(\rho_1)$. Let $v \in E_+(\rho_1)$, $\langle v, v\rangle_{V^{13,0}} = 1$. Note that $\rho_1\rho_2 = -\rho_2\rho_1$ and the product $J_{\rho_1}J_{\rho_2}$ contains six different maps J_{z_k}. We show as in Theorem 6.7 that the vectors

$$v_1 = v, \quad v_2 = J_{z_1}J_{z_2}v, \quad v_3 = J_{\rho_2}v, \quad v_4 = J_{z_2}J_{\rho_2}v,$$

form an orthonormal basis of E. Assuming that there is an isomorphism $A \oplus C\colon \mathfrak{n}_{13,0} \to \mathfrak{n}_{13,0}$ mapping the invariant orthogonal lattices $\mathcal{L}(\mathcal{S}^{(1)})$ to $\mathcal{L}(\mathcal{S}^{(2)})$ we show that $A^{\tau}A = \mathrm{Id}_{V^{13,0}}$ and obtain the contradiction as in Theorem 6.7. □

Theorem 6.9. *Let $r = 12$. The invariant orthogonal lattices $\mathcal{L}(\mathcal{S}^{(5)})$ and $\mathcal{L}(\mathcal{S}^{(8)})$ defined by non-equivalent groups $\mathcal{S}^{(5)} = \mathcal{S}(PI^{(5)})$ and $\mathcal{S}^{(8)} = \mathcal{S}(PI^{(8)})$ are not isomorphic.*

Proof. The minimal admissible module $V^{12,0}$ is isometric to $\mathbb{R}^{128,0}$. Let $E = \{x \in V^{12,0} \mid J_{p_i}(x) = x,\ i = 1,2,3,4\}$ be the eigenspace of involutions of type T_1. Then $\dim(E) = 8$ and $E = E_+(\varkappa_2) \oplus E_-(\varkappa_2)$, there $E_\pm(\varkappa_2)$ are the eigenspaces of $J_{\varkappa_2} = J_{z_1}J_{z_2}J_{z_7}$. Let $v \in E_+(\varkappa_2)$, $\langle v, v\rangle_{V^{12,0}} = 1$. The vectors

$$\begin{aligned}
&v_1 = v, && v_2 = J_{z_8}J_{z_9}v = \mathbf{I}v,\\
&v_3 = J_{z_8}J_{z_{10}}v = \mathbf{J}v, && v_4 = J_{z_9}J_{z_{10}}v = \mathbf{K}v,\\
&v_5 = J_{\varkappa_1}v = J_{z_8}J_{z_9}J_{z_{12}}v, && v_6 = \mathbf{I}v_5 = J_{z_{12}}v,\\
&v_7 = \mathbf{J}v_5 = -J_{z_9}J_{z_{10}}J_{z_{12}}v, && v_8 = \mathbf{K}v_5 = J_{z_8}J_{z_{10}}J_{z_{12}}v,
\end{aligned}$$

form an orthonormal basis of E by making of calculations as in (6.7) and (6.8). Note that the space E is two dimensional quaternion space with the quaternion structure

$$\mathbf{I} = J_{z_8}J_{z_9}, \quad \mathbf{J} = J_{z_8}J_{z_{10}}, \quad \mathbf{K} = J_{z_9}J_{z_{10}}, \quad \mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = \mathbf{IJK} = -\mathbf{1}.$$

Then we continue the proof as in Theorem 6.7 and obtain a contradiction. □

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