Ribbonness on classical link

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ABSTRACT

It is shown that if a link in 3-space bounds a proper oriented surface (without closed component) in the upper half 4-space, then the link bounds a proper oriented ribbon surface in the upper half 4-space which is a renewal embedding of the original surface. In particular, every slice knot is a ribbon knot, answering an old question by R. H. Fox affirmatively.

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1. Introduction

For a long time, the author has considered the (2,1)-cable of the figure-eight knot, which is not ribbon but rationally slice, as a candidate for a non-ribbon knot which might be slice (see [4, 5]). However, in [1], I. Dai, S. Kang, A. Mallick, J. Park and M. Stoffregen showed that it is not a slice knot. In this paper, the author comes back to elementary research beginning point of [10] on the difference between a slice knot and a ribbon knot. Then it is concluded that every slice knot is a ribbon knot. More generally, it is shown that if a link in 3-space bounds a proper oriented surface (without closed component) in the upper half 4-space, then the link bounds a proper oriented ribbon surface in the upper half 4-space which is a renewal embedding of the original surface.

This detailed explanation is done as follows. For a set A in the 3-space $\mathbf{R}^3 = \{(x,y,z)| -\infty < x,y,z < +\infty\}$ and an interval $J \subset \mathbf{R}$, let

$$AJ = \{(x,y,z,t)|\,(x,y,z) \in A,\ t \in J\}.$$

The *upper-half 4-space* $\mathbf{R}^4_+$ is denoted by $\mathbf{R}^3[0,+\infty)$. Let k be a link in the 3-space $\mathbf{R}^3$, and F a proper oriented surface in the upper-half 4-space $\mathbf{R}^4_+$ with $\partial F = k$. Let $b_j\,(j = 1,2,\dots,m)$ be finitely many disjoint oriented bands spanning the link k in $\mathbf{R}^3$, which are regarded as framed arcs spanning k in $\mathbf{R}^3$. Let k' be a link in $\mathbf{R}^3$ obtained from k by surgery along these bands. Then this band surgery operation is denoted by $k \to k'$. Let k have r knot components. If the link k' has $r-m$ components, then the band surgery operation $k \to k'$ is called a *fusion*. If the link k' has $r+m$ components, then the band surgery operation $k \to k'$ is called a *fission*. These terminologies are used in [10].

A *band sum* $k\#_b o$ of a link k and a trivial link o of components $o_i\,(i = 1,2,\dots,r)$ is a special fusion of the split sum $k + o$ along a disjoint band system $b_i\,(i = 1,2,\dots,r)$ spanning k and o_i for every i. For the knot components $k_i\,(i = 1,2,\dots,n)$ of k, assume that the band surgery operation $k \to k'$ induces the band surgery operation $k_i \to k'_i$ for all i. Then if the link k'_i is a knot for all i, then the band surgery operation $k \to k'$ is called a *genus addition*.

Every band surgery operation $k \to k'$ along a band system b is realized as a proper surface F_s^u in $\mathbf{R}^3[s,u]$ for any interval $[s,u]$, as follows (see [10]):

$$F_s^u \cap \mathbf{R}^3[t] = \begin{cases} k'[t], & \text{for } \frac{s+u}{2} < t \leq u, \\ (k \cup b)[t], & \text{for } t = \frac{s+u}{2}, \\ k[t], & \text{for } s \leq t < \frac{s+u}{2}. \end{cases}$$

For every band surgery sequence $k_1 \to k_2 \to \cdots \to k_{n-1} \to k_n$, the *realizing surface* F_s^u in $\mathbf{R}^3[s,t]$ is given by the union

$$F_{s_0}^{s_1} \cup F_{s_1}^{s_2} \cup \cdots \cup F_{s_{m-2}}^{s_{m-1}} \cup F_{s_{m-1}}^{s_m}$$

for any division

$$s = s_0 < s_1 < s_2 < \cdots < s_{m-1} < s_m = u$$

of the interval $[s,u]$. Note that the realizing surface F_s^u in $\mathbf{R}^3[s,t]$ is uniquely determined up to smooth isotopies of $\mathbf{R}^3[s,t]$ keeping $\mathbf{R}^3[s] \cup \mathbf{R}^3[t]$ fixed. For a band surgery sequence $k_1 \to k_2 \to \cdots \to k_{n-1} \to k_n$ where k_1 is a split sum $k'_1 + o$ for a link k'_1 and a trivial link o and k_n is a trivial link o', a *semi-closed realizing surface* $\mathrm{scl}(F_s^u)$ in $\mathbf{R}^3[s,t]$ bounded by the link k'_1 in $\mathbf{R}^3$ is constructed as follows.

$$\mathrm{scl}(F_s^u) = F_s^u \cup d[s] \cup d'[u]$$

for disk systems d, d' in $\mathbf{R}^3$ with $\partial d = o$ and $\partial d' = o'$. A *modified semi-closed realizing surface* $\mathrm{scl}(F_s^u)^+$ of the band surgery sequence $k_1 = k'_1 + o \to k_2 \to \cdots \to k_{n-1} \to k_n = o'$ is a proper surface in $\mathbf{R}^3[s,+\infty)$ bounded by the link k'_1 obtained from $\mathrm{scl}(F_s^u)$ by raising the level s of the disk d into the level $d+\varepsilon$ for a sufficiently small $\varepsilon > 0$.

Let F be an r-component proper surface without closed component in the upper-half 4-space $\mathbf{R}^4_+$ which bounds a link k in $\mathbf{R}^3$. By [10], the proper surface F in $\mathbf{R}^4_+$ is equivalent to a modified semi-closed realizing surface $\mathrm{scl}(F_0^1)^+$ of a band surgery $k + o \to o'$ in $\mathbf{R}^4_+$. Since the band system used for $k + o \to o'$ is made disjoint, the modified semi-closed realizing surface $\mathrm{scl}(F_0^1)^+$ is further equivalent to a modified semi-closed realizing surface $\mathrm{scl}(F_0^1)^+$ of a band surgery sequence

$$(*) \qquad k + o \to k_1 \cup o \to k_2 \cup o \to k_3 \to o_4 = o',$$

where

(0) k_1 is a link of r components and the operation $k + o \to k_1 \cup o$ is a fusion fixing o,

(1) the operation $k_1 \cup o \to k_2 \cup o$ is a genus addition fixing o,

(2) the operation $k_2 \cup o \to k_3$ is a fusion along a band system connecting every component of o to k_2 so that k_3 is a link with r components,

(3) the operation $k_3 \to o_4 = o'$ is a fission (cf. [10]).

In particular, in the band surgery sequence (*) above, if the trivial link o is taken the empty set $\emptyset$, then the step (2) is omitted and we have $k_2 = k_3$. A proper surface F in $\mathbf{R}^4_+$ is said to be *ribbon* if it is equivalent to a semi-closed realizing surface of a band surgery sequence (*) with $o = \emptyset$.

The purpose of this paper is to show the following theorem.

Theorem 1.1. Assume that a link k in the 3-space $\mathbf{R}^3$ bounds a proper oriented surface F without closed component in the upper-half 4-space $\mathbf{R}^4_+$. Then the link k in $\mathbf{R}^3$ bounds a ribbon surface F' in $\mathbf{R}^4_+$ which is a renewal embedding of F.

For a link k in $\mathbf{R}^3$, let $g^*(k)$ be the minimal genus of a smoothly embedded connected proper surface in $\mathbf{R}^4_+$ bounded by k, and $g^*_r(k)$ the minimal genus of a connected ribbon surface in $\mathbf{R}^4_+$ bounded by k. The following corollary is a direct consequence of Theorem 1.1.

Corollary 1.2. $g^*(k) = g^*_r(k)$ for every link k.

Since a slice knot in $\mathbf{R}^3$ is the boundary knot of a smoothly embedded proper disk in $\mathbf{R}^4_+$ and a ribbon knot in $\mathbf{R}^3$ is the boundary knot of a ribbon disk in $\mathbf{R}^4_+$, Corollary 1.2 contains an affirmative answer to Fox Problem 25 [2].

Corollary 1.3. Every slice knot is a ribbon knot.

2. Proof of Theorem 1.1

The following lemma is a starting point of the proof of Theorem 1.1.

Lemma 2.1. For a knot k in $\mathbf{R}^3$, assume that a band sum $o' = k\#_b o$ of k and a trivial link o is a trivial knot in $\mathbf{R}^3$. Then the knot k is a ribbon knot in $\mathbf{R}^3$.

Proof of Lemma 2.1. Let $-k^*$ be the reflected inverse knot of a knot k in $\mathbf{R}^3$. Then the connected sum $(-k^*)\#k$ is a ribbon knot in $\mathbf{R}^3$ (see [3]). Since the band sum $o' = k\#_b o$ is a trivial knot, the connected sum $(-k^*)\#(k\#_b o)$ obtained by locally tying $-k^*$ to a string of k in $k\#_b o$ is equivalent to the knot $(-k^*)\#o' = -k^*$. On the other hand, the knot $(-k^*)\#(k\#_b o)$ is a ribbon knot because it is a band sum of the ribbon knot $(-k^*)\#k$ and the trivial link o. Thus, the knot $-k^*$ is a ribbon knot. Since the reflected inverse knot of a ribbon knot is a ribbon knot, the knot k is a ribbon knot. This completes the proof of Lemma 2.1. □

Remark 2.2. A ribbon presentation of the connected sum $(-k^*)\#k$ for a knot k in $\mathbf{R}^3$ can be obtained from the chord diagram of any given diagram $D(k)$ of k by [6, 7, 8, 9]. In fact, by [9], let D be an inbound diagram of $D(k)$ (namely, an arc diagram obtained from $D(k)$ by removing an open arc not containing a crossing point) with the end points in the infinite region of the plane $\mathbf{R}^3$, and C a chord diagram of D. The diagram obtained from the based loop system of C by surgery along a band system thickening the chord system is a ribbon presentation of the connected sum $(-k^*)\#k$. This is because the connected sum $(-k^*)\#k$ is the middle cross-section of the spun knot $S(k)$ of k in $\mathbf{R}^4$ and the chord diagram C canonically represents the spun knot $S(k)$ as a ribbon S^2-knot (see [6, 9, 11]).

Lemma 2.1 is generalized as follows.

Lemma 2.3. For a link k of n knot components in $\mathbf{R}^3$, assume that a band sum

$k\#_b o$ of k and a trivial link o is a ribbon link in $\mathbf{R}^3$. Then the link k is a ribbon link in $\mathbf{R}^3$.

Proof of Lemma 2.3. For the components $k_i\,(i = 1, 2, \ldots, n)$ of k, the band sum $k' = k\#_b o$ is the union of band sums $k'_i = k_i\#_b o_i\,(i = 1, 2, \ldots, n)$. Let $o_{ij}\,(j = 1, 2, \ldots, n_i)$ be the components of the trivial link o_i, and b_{ij} the band spanning k_i and o_{ij} used for the band sum $k'_i = k_i\#_b o_i$ for all $j\,(j = 1, 2, \ldots, n_i)$. Since the link k' is a ribbon link with components $k'_i\,(i = 1, 2, \ldots, n)$, there is a fusion $o' \to k'$ with a trivial link o' consisting of fusions $o'_i \to k'_i\,(i = 1, 2, \ldots, n)$. Let $o'_{ih}\,(h = 1, 2, \ldots, m_i)$ be the components of o'_i, and $b'_{ih}\,(h = 1, 2, \ldots, m_i)$ the bands used for the fusion $o'_i \to k'_i$. By band slides and by regarding bands as framed arcs, the bands $b_{ij}(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n_i)$, $b'_{ih}\,(i = 1, 2, \ldots, n;\ h = 1, 2, \ldots, m_i)$ are made disjoint. Further, the bands $b_{ij}\,(j = 1, 2, \ldots, n_i)$ are taken to be attached only to the component o'_{i1}. Let $B'_{ij}\,(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m_i)$ be disjoint 3-balls in $\mathbf{R}^3$ containing the component o'_{ij} in the interior. Let $d_{ij}\,(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n_i)$ be a disjoint disk system bounded by the trivial loop system $o_{ij}\,(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n_i)$ in $\mathbf{R}^3$. Let $a'_{ih}\,(i = 1, 2, \ldots, n;\ h = 1, 2, \ldots, m_i)$ be a core arc system of the band system $b'_{ih}\,(i = 1, 2, \ldots, n;\ h = 1, 2, \ldots, m_i)$, and $a''_{ih}\,(i = 1, 2, \ldots, n;\ h = 1, 2, \ldots, m_i)$ an arc system obtained from $a'_{ih}\,(i = 1, 2, \ldots, n;\ h = 1, 2, \ldots, m_i)$ by deforming not to meet the disjoint disk system $d_{ij}\,(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n_i)$. The deformation should be taken so that the arc system $a''_{ih}\,(i = 1, 2, \ldots, n;\ h = 1, 2, \ldots, m_i)$ is isotopic to the arc system $a'_{ih}\,(i = 1, 2, \ldots, n;\ h = 1, 2, \ldots, m_i)$ when the disk system $d_{ij}\,(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n_i)$ is forgotten. Let $b''_{ih}\,(i = 1, 2, \ldots, n;\ h = 1, 2, \ldots, m_i)$ be the band system thickening the core arc system $a''_{ih}\,(i = 1, 2, \ldots, n;\ h = 1, 2, \ldots, m_i)$. Then the disjoint disk system $d_{ij}\,(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n_i)$ can be moved into B'_{i1} while keeping the band system $b''_{ih}\,(i = 1, 2, \ldots, n;\ h = 1, 2, \ldots, m_i)$ fixed. In this move, some parts of the band system $b_{ij}\,(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n_i)$ may be moved. Since o_{i1} and $d_{ij}\,(j = 1, 2, \ldots, n_i)$ are disjoint except for the meeting part of the band system $b_{ij}\,(j = 1, 2, \ldots, n_i)$, there is a knot k''_i such that the trivial knot o_{i1} is the band sum $k''_i\#_b o_i$ using the bands $b_{ij}\,(j = 1, 2, \ldots, n_i)$. By Lemma 2.1, the knot k''_i is a ribbon knot and thus there is a fusion $o''_i \to k''_i$ for a trivial link o''_i in $\mathbf{R}^3$. Note that the knot k''_i is disjoint from $B'_{ij}\,(i = 2, 3, \ldots, m_i)$, so that the trivial link o''_i is movable into B'_{i1} although some parts of the bands used for the fusion $o'' \to k''$ may not be in B'_{i1}. The link k is a fusion of the trivial link consisting of the split sum of o'_i, $(i =, 2, 3, \ldots, n)$; $o''_i\,(i = 1, 2, \ldots, n)$, meaning that the link k is a ribbon link. This completes the proof of Lemma 2.3. □

The proof of Theorem 1.1 is done as follows.

Proof of Theorem 1.1. Consider that a proper oriented surface F is given by the sequence

$$k + o \to k_1 \cup o \to k_2 \cup o \to k_3,$$

which are given by the band surgery operations that $k_3 \to k_2 \cup o$ is a fission, $k_2 \cup o \to k_1 \cup o$ is a genus addition fixing o and $k_1 \cup o \to k + o$ is a fission fixing o, forming the inverse sequence

$$k_3 \to k_2 \cup o \to k_1 \cup o \to k + o$$

of the sequence $k + o \to k_1 \cup o \to k_2 \cup o \to k_3$. By band slides, note that the link $k_2 \cup o$ can be the split link $k_2 + o$. Replace the bands used for the genus addition $k_2 \cup o \to k_1 \cup o$ and the fission $k_1 \cup o \to k + o$ by bands such that

(i) every band does not change the attaching parts, and

(ii) every band does not pass the trivial link o, and

(iii) every band is deformable into the original band if the trivial link o is forgotten.

Then the genus addition $k_2 \cup o \to k_1 \cup o$ changes into a genus addition $k_2 + o \to k_1 + o$ fixing o and the fission $k_1 \cup o \to k + o$ changes into a fission $k_1 + o \to k + o$ fixing o, respectively, so that the sequence

$$k + o \to k_1 \cup o \to k_2 \cup o \to k_3$$

changes into a sequence

$$k + o \to k_1 + o \to k_2 + o \to k_3,$$

where the operation $k + o \to k_1 + o$ is a fusion fixing o, the operation $k_1 + o \to k_2 + o$ is a genus addition fixing o, and the operation $k_2 + o \to k_3$ is a fusion meaning that k_3 is a band sum $k_2 \#_b o$ of k_2 and o. Since k_3 is a ribbon link, k_2 is a ribbon link by Lemma 2.3. Thus, there is a sequence

$$k \to k_1 \to k_2 \to o'_3,$$

where the operation $k \to k_1$ is a fusion, the operation $k_1 \to k_2$ is a genus addition and the operation $k_2 \to o'_3$ is a fission with o'_3 a trivial link. This means that the link k in $\mathbf{R}^3$ bounds a ribbon surface F' in $\mathbf{R}^4_+$ which is a renewal embedding of F. This completes the proof of Theorem 1.1. □

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references

[1] I. Dai, S. Kang, A. Mallick, J. Park and M. Stoffregen, The $(2,1)$-cable of the figure eight knot is not slice. arXiv:2207.14187

[2] R. H. Fox, Some problems in knot theory, Topology of 3-manifolds and related topics, (1962), 168-176, Prentice-hall, Inc., Engelwood Cliffs, N. J. USA.

[3] R. H. Fox and J. W. Milnor, Singularities of 2-spheres in 4-space and cobordism of knots, Osaka J. Math. 3 (1966) 257-267.

[4] A. Kawauchi, The $(2,1)$-cable of the figure eight knot is rationally slice, (in a handwritten manuscript)(1980) on the web: Available from: https://sites.google.com/view/ kawauchiwriting

[5] A. Kawauchi, Rational-slice knots via strongly negative-amphicheiral knots, Communications in Mathematical Research 25 (2009),177-192.

[6] A. Kawauchi, A chord diagram of a ribbon surface-link, Journal of Knot Theory and Its Ramifications 24 (2015), 1540002 (24 pages).

[7] A. Kawauchi, Supplement to a chord diagram of a ribbon surface-link, Journal of Knot Theory and Its Ramifications, 26 (2017), 1750033 (5 pages).

[8] A. Kawauchi, A chord graph constructed from a ribbon surface-link, Contemporary Mathematics (AMS), 689 (2017), 125-136. USA.

[9] A. Kawauchi, Knotting probability of an arc diagram, Journal of Knot Theory Ramifications 29 (2020), 2042004 (22 pages).

[10] A. Kawauchi, T. Shibuya, S. Suzuki, Descriptions on surfaces in four-space I : Normal forms, Mathematics Seminar Notes, Kobe University, 10 (1982), 75-125; II: Singularities and cross-sectional links, Mathematics Seminar Notes, Kobe University 11 (1983), 31-69. Available from: https://sites.google.com/view/ kawauchiwriting

[11] T. Yajima, On the fundamental groups of knotted 2-manifolds in the 4-space, J. Math. Osaka City Univ. 13 (1962), 63–71.